M11 – Non-Abelian Exchange Holonomy on Scalar–Conformal NUVO Space

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Abstract

The exchange sector developed in M4 [1] introduced a real-valued connection 1-form A associated with coherent open-loop configurations on the scalar–conformal manifold. In M5 [2], the closed-cycle integral of this connection defined a U(1)-valued holonomy together with a holonomic admissibility condition restricting the admissible class of coherent exchange cycles. While this framework is structurally complete for the Abelian sector of the exchange dynamics and is incorporated into the bundle taxonomy of M6 [3] and M6.5 [4] through a closure condition on bundled loop configurations, it does not exhaust the admissible exchange connections compatible with this taxonomy.

In this work we generalize the exchange connection to a Lie-algebra-valued 1-form

$$A = A^a_\mu \, T_a \, dx^\mu,$$

where the T_a are generators of a Lie algebra $\mathfrak{g}$ associated with a structure group G. The holonomy along a closed exchange cycle is defined by the path-ordered exponential

$$\Theta(\gamma) = \mathcal{P} \exp\left(\oint_\gamma A^a T_a \right) \in G,$$

which composes by group multiplication and is therefore non-commutative for non-Abelian G. The holonomic admissibility condition takes the form $\Theta(\gamma) \in \mathcal{H}^G_{\mathrm{adm}}$, where $\mathcal{H}^G_{\mathrm{adm}}$ is a discrete subgroup of G determined by the coherence requirements of the bundle.

We establish well-definedness of the path-ordered holonomy operationally as the limit of a discretized product of group elements, derive its composition law under path concatenation, and show that the U(1) treatment of M5 is recovered exactly as the Abelian special case ($\dim \mathfrak{g} = 1$, $f^{abc} = 0$). The bundle closure condition of M6 and M6.5 is identified as the Abelian projection of the group-valued admissibility condition $\Theta(\gamma) = \mathbf{1}_G$. The admissible class $\mathcal{H}^G_{\mathrm{adm}}$ is shown to inherit the structure of a discrete subgroup of G via composition of holonomies along concatenated cycles.

The development is strictly structural. No specific gauge group is adopted, no representation theory of matter content is committed to, and no sectoral reduction is performed. With this generalization, the structural specification of the exchange sector is complete at the non-Abelian level, and the framework derived here serves as the structural input to subsequent non-Abelian sectoral developments of the NUVO program.

[*] Bibliography is provisional. Cross-references to companion NUVO-series papers (M-, SR-, Q-, QB-, QM-series) will be updated with Zenodo DOIs in subsequent versions.

Notation and Conventions

- $\mathcal{M}$ denotes the spacetime manifold.
- η denotes the reference Lorentzian metric (typically Minkowski in a global chart).
- g denotes the physical metric.
- The scalar field $\Lambda : \mathcal{M} \to \mathbb{R}_{>0}$ is the NUVO modulation field.
- The physical metric is scalar–conformal:

$$g_{\mu\nu} = \Lambda^2 \, \eta_{\mu\nu}.$$

- $\Lambda_0 > 0$ denotes the baseline scalar availability level supported by the intrinsic delivery structure of the underlying field. In the absence of localized structural occupation the scalar field satisfies $\Lambda(x) = \Lambda_0$.
- The dimensionless scalar diagnostic is

$$\lambda(x) := \frac{\Lambda(x)}{\Lambda_0}.$$

- The scalar field represents the *locally available structural capacity* of the underlying delivery field. Localized structures may reduce this availability through occupation or transport, but the intrinsic delivery baseline Λ_0 remains fixed.
- Greek indices $\mu, \nu, \ldots$ range over spacetime coordinates $0, 1, 2, 3$.
- We use the Einstein summation convention unless explicitly stated otherwise.

Remark 0.1. *Unless otherwise stated, the background signature is* $(-, +, +, +)$.

Program Scope

This manuscript is mathematical in scope. It establishes definitions, structural identities, and variational consequences within a scalar–conformal setting. Sector reductions and correspondence limits are recorded only when explicitly stated as additional assumptions and are not used as premises in derivations.

No claim of full dynamical equivalence to general relativity, quantum mechanics, or classical field theories is made at the level of the present foundational development. Where later papers compare limiting behavior, those comparisons are presented as correspondence targets rather than as identity statements.

The NUVO program is organized as a sequence of internally consistent mathematical papers. Foundational papers (M-series) fix the scalar–conformal geometry, variational structure, and notation. Subsequent papers treat sectoral reductions (gravity, exchange, quantization, and bound-state structure) as controlled specializations of the foundational framework.

Scalar ontology. The scalar field Λ represents the *locally available structural capacity* of an underlying delivery field permeating spacetime. The baseline level Λ_0 denotes the availability supported by this intrinsic delivery structure in the absence of structural occupation. Localized structures or transport processes may reduce the available capacity relative to this baseline, but the intrinsic delivery baseline itself is not altered. Consequently the scalar field measures the *available portion* of structural capacity rather than the intrinsic production of the underlying field.

Throughout the series we distinguish between (i) definitions and theorems proved in the present manuscript, and (ii) external results used only for context. References are cited for orientation and comparison and are not treated as axioms unless explicitly declared.

All notation intended to be program-wide is centralized in the shared NUVO macro package and notation layer. This is done to maintain consistency across the series and to support future consolidation into a cohesive monograph-style presentation.

1 Introduction

1.1 Motivation

The exchange sector of the scalar–conformal NUVO framework was introduced in M4 as a structural component distinct from the gravitational sector. Whereas gravity arises from persistent localized depletion structures supported by the canonical scalar field equation, the exchange sector describes directed structural coupling between localized systems through open-loop configurations on the scalar–conformal manifold. The diagnostic quantity associated with such coupling is a smooth real-valued 1-form A, the exchange connection, whose closed-cycle integral encodes the accumulated structural advance of a coherent exchange cycle.

In M5, this construction was developed into a holonomic framework. For a closed coherent exchange cycle γ, the holonomy was defined by

$$\Theta(\gamma) := \oint_\gamma A,$$

and the coherence condition was expressed as a holonomic admissibility constraint

$$\Theta(\gamma) \in \mathcal{H}_{\mathrm{adm}},$$

where $\mathcal{H}_{\mathrm{adm}}$ denotes the admissible return class determined by the structural coherence of the cycle. In its exponentiated form, this condition selects the admissible holonomies within $\mathrm{U}(1)$ and reduces to the integer-charge condition of the M-series electromagnetic sector.

The bundle taxonomy of M6 and M6.5 incorporates this construction through a closure condition on bundled loop configurations: the sum of $\mathrm{U}(1)$ orientation labels along any closed cycle in the exchange graph $G(\mathcal{B})$ must lie in $\mathbb{Z}$. This specification is structurally complete for the Abelian (electromagnetic) sector of the exchange dynamics.

A structural question, however, remains. The exchange connection of M4 takes values in the one-dimensional Abelian Lie algebra $\mathfrak{u}(1)$, and its closed-cycle holonomy therefore composes by addition of phase contributions. The bundle taxonomy of M6 and M6.5, by contrast, is formulated at a level of generality sufficient to support open-loop labels beyond a binary $\mathrm{U}(1)$ orientation. The most general exchange connection compatible with this taxonomy has not been previously characterized.

1.2 Objective

The objective of the present paper is to derive the minimal structural generalization of the exchange connection that admits non-commutative composition of holonomies, while remaining consistent with the foundational framework of the M-series.

The desired generalization must satisfy the following requirements:

- it must be expressible as a smooth 1-form on the scalar–conformal manifold $\mathcal{M}$, taking values in a Lie algebra associated with a structure group;
- it must admit a well-defined closed-cycle holonomy whose composition law under path concatenation reflects the structure of this group;
- it must support a holonomic admissibility condition generalizing $\Theta(\gamma) \in \mathcal{H}_{\mathrm{adm}}$, compatible with the bundle closure condition of M6 and M6.5;
- it must reduce to the M4/M5 construction exactly as the Abelian special case;
- it must introduce no postulates beyond the scalar–conformal framework of M1 and the exchange-sector framework of M4 and M5.

The aim is not to introduce a phenomenological extension of the exchange sector, but to identify the minimal structural generalization permitted by the existing framework.

1.3 Scope

The work is purely structural. No commitment is made to any specific choice of gauge group, to any representation theory of matter content, or to any sectoral reduction or correspondence with empirical observation. Specific gauge groups are treated as choices of input within the framework derived here, and their sectoral developments are deferred to subsequent series.

In particular, the chirality structure of the Dirac field, required for sectors in which the exchange connection couples asymmetrically to Weyl-spinor projections, is not addressed in the present paper. This identification is treated at the opening of the electroweak development that builds on the present framework.

The analysis is confined to the exchange connection and its associated holonomy structure. No assumptions are made regarding gravitational dynamics, quantization beyond the holonomic coherence condition, or specific interaction models. Where connections to specific sectoral domains are mentioned, they are presented as structural consequences of the framework derived here rather than as inputs to the derivation.

1.4 Position in the NUVO Program

This paper occupies a structural position immediately following M10 [5] in the foundational M-series. Where M10 completed the specification of scalar modulation through the multiplicative composition law

$$\lambda_{\mathrm{eff}} = \lambda_{\mathrm{amb}} \cdot \lambda_{\mathrm{loc}},$$

the present paper completes the specification of the exchange connection at the non-Abelian level.

In the existing framework:

- the scalar field $\Lambda(x)$ defines the geometric structure of spacetime through the conformal relation $g_{\mu\nu} = \Lambda^2 \eta_{\mu\nu}$;
- the exchange sector introduces a connection 1-form on $\mathcal{M}$ whose closed-cycle holonomy supports a coherence condition on admissible bundled structures;
- the bundle taxonomy of M6 and M6.5 organizes admissible configurations through closure conditions on the exchange graph.

The present work supplies the missing structural element of this description by characterizing the most general admissible exchange connection compatible with the bundle taxonomy. The resulting framework serves as the structural input to subsequent non-Abelian sectoral developments of the NUVO program. Specific sectoral reductions are not addressed in the present paper.

2 Recap of the U(1) Exchange Connection

This section restates the framework of M4 and M5 in a form suitable for direct generalization in the sections that follow. No new content is introduced. The purpose is to fix notation and to identify explicitly the structural feature of the M4/M5 construction whose modification yields the non-Abelian extension.

2.1 Exchange connection as a real-valued 1-form

In M4, the exchange sector of the scalar–conformal NUVO framework was developed at the level of structural and geometric admissibility. The diagnostic quantity associated with a coherent exchange cycle is a smooth 1-form on the scalar–conformal manifold $\mathcal{M}$. Throughout the present recap we denote this object by A and refer to it as the exchange connection.

Definition 2.1 (U(1) exchange connection, recap of M4)**.** *A* U(1) *exchange connection on $\mathcal{M}$ is a smooth real-valued 1-form*

$$A = A_\mu(x)\, dx^\mu,$$

associated with a coherent open-loop exchange cycle, with A_μ real and smooth on $\mathcal{M}$.

The connection A is a structural diagnostic of capacity transport along the exchange cycle. It is not introduced as a primitive dynamical field, nor as a carrier of an independent interaction; its role within the M4 framework is to characterize the directed structural coupling between source and sink regions of an open-loop configuration. The sourcing of A and its relation to the canonical scalar dynamics are treated in M4 and are not repeated here.

2.2 U(1) holonomy and admissibility

Let $\gamma : S^1 \to \mathcal{M}$ denote a closed coherent exchange cycle on the scalar–conformal manifold. In M5 the holonomy of this cycle was introduced as the closed-line integral of the exchange connection.

Definition 2.2 (U(1) holonomy, recap of M5)**.** *The* U(1) *holonomy of a closed coherent exchange cycle γ is the real number*

$$\Theta(\gamma) := \oint_\gamma A.$$

The holonomy expresses the accumulated structural advance of the exchange configuration along γ. Because γ is closed, the coherence of the exchange cycle imposes a global compatibility condition on A, formulated in M5 as a holonomic admissibility constraint.

Definition 2.3 (Holonomic admissibility, recap of M5)**.** *A* U(1) *exchange connection A satisfies the holonomic admissibility condition with respect to a closed coherent exchange cycle γ if*

$$\Theta(\gamma) \in \mathcal{H}_{\mathrm{adm}}^{\mathrm{U}(1)},$$

where $\mathcal{H}_{\mathrm{adm}}^{\mathrm{U}(1)} = 2\pi\mathbb{Z}$ is the admissible return class of the cycle.

In its exponentiated form, the admissibility condition selects the admissible group elements of U(1):

$$e^{\mathrm{i}\,\Theta(\gamma)} = 1 \in \mathrm{U}(1).$$

This is the closure form of the M5 admissibility condition. In the bundle taxonomy of M6 and M6.5, it is the structural origin of the integer-charge condition imposed on closed cycles in the exchange graph $G(\mathcal{B})$.

Remark 2.4 (Topological character of the integer)**.** *The integer n defined by $\Theta(\gamma) = 2\pi n$ is a topological label of the closed cycle γ relative to the connection A; it is not a continuous parameter of Θ. In the exponentiated formulation, the admissibility condition is independent of n, and the integer arises only as a winding label of the cycle. This characterization is preserved in the non-Abelian generalization of Sec. 5, where the integer is replaced by a discrete subgroup label of the structure group.*

2.3 Structural origin of commutativity

The U(1) exchange connection of

Definition 2.5. *2.1 takes values in the one-dimensional Abelian Lie algebra $\mathfrak{u}(1)$. The associated structure group* U(1) *is itself Abelian: any two elements $e^{\mathrm{i}\theta_1}, e^{\mathrm{i}\theta_2} \in \mathrm{U}(1)$ commute under group multiplication, with*

$$e^{\mathrm{i}\theta_1}\, e^{\mathrm{i}\theta_2} = e^{\mathrm{i}(\theta_1+\theta_2)} = e^{\mathrm{i}\theta_2}\, e^{\mathrm{i}\theta_1}.$$

Consequently, the holonomy of a path concatenation depends only on the sum of the individual phase contributions and not on the order in which they are composed.

This is the structural feature of the M4/M5 construction whose modification is the subject of the present paper. The non-Abelian generalization developed in the following sections retains the form of the exchange connection as a smooth 1-form on $\mathcal{M}$, retains the holonomy as a closed-cycle integral, and retains the admissibility condition as a constraint on a discrete return class. What is modified is the value space of the connection: in place of the one-dimensional Abelian algebra $\mathfrak{u}(1)$, the connection is permitted to take values in a higher-dimensional Lie algebra whose structure constants need not vanish. The composition of holonomies along concatenated paths thereby acquires a non-commutative character, and the admissibility condition becomes a condition on group elements of a discrete subgroup of a non-Abelian structure group.

The detailed development of this generalization is the subject of Sec. 3–Sec. 7.

3 Multiply-Labeled Open Loops

The recap of Sec. 2 identified the value space of the exchange connection as the structural feature distinguishing the Abelian and non-Abelian cases. Before generalizing the connection itself, we generalize the labeling of the open-loop configurations on which it acts. This section introduces the multiply-labeled open loop as the structural primitive of the non-Abelian exchange sector and records the composition rule that distinguishes it from its Abelian counterpart.

3.1 Open loops in the exchange graph

In M4, the exchange graph $G(\mathcal{B})$ associated with a bundled configuration $\mathcal{B} = (\mathbf{C}, \mathbf{O}, \mathcal{R}, \sigma)$ consists of directed edges representing open-loop exchange configurations $O_i \in \mathbf{O}$, together with relational *data $\mathcal{R}$ and an orientation assignment σ. Each open loop carries a* U(1) *orientation*

$$\sigma(O_i) \in \{+1, -1\},$$

which encodes the direction of capacity transport along the loop. In the bundle closure condition of M6 and M6.5, admissibility of a closed cycle in $G(\mathcal{B})$ is expressed as

$$\sum_{i \in \gamma} \sigma(O_i) \in \mathbb{Z},$$

the integer-charge condition introduced earlier.

The orientation label σ is a binary U(1)*-valued diagnostic. It is the minimal label sufficient to specify the* U(1) *exchange sector but does not exhaust the structural labels compatible with the bundle taxonomy. The structural feature permitting a richer labeling is that* **O** *is defined in M4 as a set of directed open-loop configurations carrying a label in a representation of a structure group, with the M4 case corresponding to a specific choice of group and representation. Generalizing this choice yields the non-Abelian extension.*

3.2 Internal labels beyond U(1) orientation

We promote the binary orientation σ to a label taking values in a finite-dimensional representation of a Lie group.

Definition 3.1 (Multiply-labeled open loop)**.** *Let G be a Lie group and let V be a finite-dimensional vector space carrying a smooth representation*

$$\rho : G \to \mathrm{GL}(V).$$

A multiply-labeled open loop *is an open-loop exchange configuration $O_i \in \mathbf{O}$ together with an internal label $v(O_i) \in V$. The pair (G, V, ρ) is the labeling data of the open loop.*

Remark 3.2 (Recovery of the M4 orientation label)**.** *Taking $G = \mathrm{U}(1)$, $V = \mathbb{C}$, and ρ the fundamental representation of* U(1) *on $\mathbb{C}$ by phase multiplication, the internal label $v(O_i)$ reduces to a phase factor associated with the loop, and its sign on a real one-dimensional reduction recovers the M4 orientation $\sigma(O_i) \in \{+1, -1\}$. The framework of*

Definition 3.3. *3.1 therefore subsumes the M4 specification as a particular choice of labeling data.*

The Lie group G in

Definition 3.4. *3.1 plays the role of the structure group of the exchange sector. At the present stage of the development, G is left unspecified; specific choices of G are treated in subsequent sectoral developments and are not relevant to the structural framework derived here.*

3.3 Composition rule for concatenated open loops

Two open loops O_i, O_j are concatenated *when the sink region of O_i coincides with the source region of O_j, in the sense of M4. In the M4 framework, the orientation labels of concatenated open loops compose additively modulo 2π, reflecting the Abelian structure of* U(1)*.*

For multiply-labeled open loops in the sense of

Definition 3.5. *3.1, this composition rule is replaced by group multiplication in G. Specifically, if O_i and O_j carry internal labels $v(O_i)$ and $v(O_j)$ associated with group elements $g_i, g_j \in G$ via the representation ρ, then the concatenated configuration carries a label associated with the product*

$$g_j \, g_i \in G,$$

where the order of factors reflects the directed sequence of concatenation.

Remark 3.6 (Recovery of the M4 composition rule)**.** *When $G = \mathrm{U}(1)$, group multiplication coincides with phase addition modulo 2π, and the directed sequence of concatenation becomes immaterial because $g_i g_j = g_j g_i$. The composition rule of the present section therefore reduces to the additive* U(1) *orientation rule of M4 in the Abelian special case.*

Remark 3.7 (Order-dependence and its structural consequence)**.** *For non-Abelian G, the product $g_j g_i$ generally differs from $g_i g_j$, so the label associated with a concatenated configuration depends on the order of concatenation. This order-dependence is the structural origin of the path-ordered character of the closed-cycle holonomy developed in Sec. 4.2; it is absent in the M4 framework precisely because* U(1) *is Abelian.*

The composition rule of Sec. 3.3 characterizes the labeling of concatenated open loops at the level of the structure group G, without reference to a connection on the manifold. The connection that generates these group-valued labels along smooth paths, and the closed-cycle holonomy associated with it, are introduced in Sec. 4.

4 The Lie-Algebra-Valued Exchange Connection

Sec. 3 established the structural primitive of the non-Abelian exchange sector: open loops carrying internal labels in a representation of a Lie group G, with concatenation governed by group multiplication. The present section introduces the connection 1-form that generates these group-valued labels along smooth paths and develops its closed-cycle holonomy.

The construction parallels that of Sec. 2 at every step. The exchange connection is a smooth 1-form on $\mathcal{M}$; its closed-cycle holonomy is defined by a line integral; concatenated paths yield composed holonomies. What is modified is the value space of the connection.

4.1 Definition

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and fix a basis $\{T_a : a = 1, \ldots, \dim \mathfrak{g}\}$ of $\mathfrak{g}$ [6]. The generators T_a satisfy the commutation relations

$$[T_a, T_b] = \mathrm{i}\, f^{abc}\, T_c,$$

where f^{abc} are the structure constants of $\mathfrak{g}$ in this basis.

Definition 4.1 (Non-Abelian exchange connection)**.** *A* non-Abelian exchange connection *on the scalar–conformal manifold $\mathcal{M}$, with structure group G, is a smooth $\mathfrak{g}$-valued 1-form*

$$A = A^a_\mu(x)\, T_a\, dx^\mu,$$

where the components $A^a_\mu : \mathcal{M} \to \mathbb{R}$ are smooth real scalar functions on $\mathcal{M}$.

Remark 4.2 (Component-wise structure)**.** *The components A^a_μ of*

Definition 4.3. *4.1 are real-valued smooth functions on $\mathcal{M}$; the $\mathfrak{g}$-valuedness of A is encoded entirely in the index a. When $\dim \mathfrak{g} = 1$, the index a takes a single value and may be suppressed, and A reduces to a smooth real-valued 1-form on $\mathcal{M}$ in agreement with*

Definition 4.4. *2.1.*

4.2 Path-ordered holonomy

The closed-cycle integral of a $\mathfrak{g}$-valued 1-form does not in general yield a group element directly: when the connection takes values in a non-commutative Lie algebra, the contributions of distinct points along the path do not commute, and the order in which they are composed becomes structurally significant. The construction that respects this order-dependence is the path-ordered exponential [7, 8].

Let $\gamma : [0,1] \to \mathcal{M}$ be a piecewise-smooth path with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Partition the interval $[0,1]$ into N ordered subintervals

$$0 = t_0 < t_1 < \cdots < t_N = 1,$$

with mesh $\Delta = \max_k(t_k - t_{k-1})$. For each subinterval, define the displacement

$$\Delta x^\mu_k := \gamma^\mu(t_k) - \gamma^\mu(t_{k-1})$$

and the associated group element

$$U_k := \exp\!\big(\mathrm{i}\, A^a_\mu(\gamma(t_{k-1}))\, T_a\, \Delta x^\mu_k\big) \in G.$$

The discretized parallel transport along γ is the ordered product

$$\Theta_N(\gamma) := U_N\, U_{N-1} \cdots U_1.$$

Definition 4.5 (Path-ordered holonomy)**.** *For a piecewise-smooth path γ in $\mathcal{M}$, the* path-ordered holonomy *of the non-Abelian exchange connection A along γ is*

$$\Theta(\gamma) := \mathcal{P}\exp\!\Big(\mathrm{i}\int_\gamma A^a\, T_a\Big) := \lim_{\substack{N\to\infty \\ \Delta\to 0}} \Theta_N(\gamma) \in G,$$

provided the limit exists and is independent of the partition. When γ is a closed path, $\Theta(\gamma)$ is the closed-cycle holonomy.

The factor of i *in the exponent is the standard convention associated with Hermitian generators T_a satisfying the commutation relations of Sec. 4.1. In the Abelian case the convention agrees with the exponentiated form $\mathrm{e}^{\,\mathrm{i}\Theta(\gamma)}$ recorded in Sec. 2.2.*

4.3 Operational construction and well-definedness

Proposition 4.6 (Well-definedness of the path-ordered holonomy)**.** *Let A be a smooth $\mathfrak{g}$-valued 1-form on $\mathcal{M}$ and let $\gamma : [0,1] \to \mathcal{M}$ be a piecewise-smooth path. Then the discretized product $\Theta_N(\gamma)$ converges in G, as $N \to \infty$ and $\Delta \to 0$, to a unique element $\Theta(\gamma) \in G$ that is independent of the partition.*

Proof sketch. The argument is the standard differential-geometric construction of parallel transport in a principal G-bundle [7]. Smoothness of A and piecewise-smoothness of γ ensure that the integrand $A^a_\mu(\gamma(t))\dot\gamma^\mu(t)\,T_a$ is bounded on $[0,1]$, which guarantees existence and uniqueness of the solution to the ordinary differential equation

$$\frac{dU(t)}{dt} = \mathrm{i}\,A^a_\mu(\gamma(t))\,\dot\gamma^\mu(t)\,T_a\,U(t), \qquad U(0) = \mathbf{1}_G,$$

on $[0,1]$ with values in G. The discretized product $\Theta_N(\gamma)$ is a first-order time-step approximation to the solution of this equation, and standard convergence results for ordinary differential equations imply that $\Theta_N(\gamma)$ converges to $U(1)$ as $\Delta \to 0$ independently of the partition. Setting $\Theta(\gamma) := U(1)$ yields the path-ordered holonomy.

Functional-analytic regularity of Θ as a function of γ, including continuity in the loop topology and behavior under non-smooth deformations, is not required for the structural developments of the present paper and is deferred to subsequent sectoral work. □

4.4 Composition under path concatenation

For two piecewise-smooth paths γ_1, γ_2 in $\mathcal{M}$ with $\gamma_1(1) = \gamma_2(0)$, the concatenation $\gamma_2 * \gamma_1$ is the piecewise-smooth path that traverses γ_1 first and γ_2 second.

Proposition 4.7 (Composition law)**.** *The path-ordered holonomy of*

Definition 4.8. *4.5 satisfies*

$$\Theta(\gamma_2 * \gamma_1) = \Theta(\gamma_2)\,\Theta(\gamma_1),$$

where the product on the right is taken in G. When G is non-Abelian, the order of factors is structurally significant.

Proof. Partition each of γ_1, γ_2 into N subintervals and form the discretized products $\Theta_N(\gamma_1)$, $\Theta_N(\gamma_2)$. By construction, the discretized product along the concatenated path $\gamma_2 * \gamma_1$ on the combined $2N$-fold partition is the ordered product of the $2N$ group elements, which factorizes as

$$\Theta_{2N}(\gamma_2 * \gamma_1) = \Theta_N(\gamma_2)\,\Theta_N(\gamma_1).$$

Taking the limit $N \to \infty$ on both sides and applying

Proposition 4.9. *4.6 together with continuity of group multiplication in G yields the stated identity.* □

Proposition 4.10. *4.7 expresses the closed-cycle counterpart of the concatenation rule of Sec. 3.3: the discrete-loop composition rule for multiply-labeled open loops is reproduced at the level of smooth paths by the path-ordered holonomy.*

4.5 Reduction to the Abelian case

Proposition 4.11 (Recovery of the U(1) holonomy)**.** *Let* $G = \mathrm{U}(1)$*, so that* $\dim\mathfrak{g} = 1$ *and* $f^{abc} = 0$*. Then for any closed coherent exchange cycle γ in $\mathcal{M}$,*

$$\Theta(\gamma) = \exp\Bigl(\mathrm{i}\oint_\gamma A\Bigr),$$

where A is the real-valued 1-form of

Definition 4.12. *2.1. This recovers the exponentiated form of the M5 holonomy of*

Definition 4.13. *2.2.*

Proof. When $\dim \mathfrak{g} = 1$, the basis $\{T_a\}$ reduces to a single generator, conventionally $T_1 = 1$, and the components A^a_μ collapse to a single real 1-form A_μ. Because the structure constants vanish, the generators on distinct subintervals of any partition commute, so each discretized factor U_k in the product $\Theta_N(\gamma)$ commutes with every other. The path ordering is therefore trivial, and the product reduces to

$$\Theta_N(\gamma) = \exp\Big(\, \mathrm{i} \sum_{k=1}^{N} A_\mu(\gamma(t_{k-1}))\, \Delta x^\mu_k \Big).$$

In the limit $N \to \infty$, $\Delta \to 0$, the sum in the exponent converges to the line integral $\oint_\gamma A$, yielding the stated identity. □

Proposition 4.14. *4.11 confirms that the path-ordered holonomy of*

Definition 4.15. *4.5 is a strict generalization of the M5* U(1) *holonomy of*

Definition 4.16. *2.2: the Abelian case is not merely compatible with the non-Abelian construction, but is recovered as a particular reduction of it. No structural feature of the M4/M5 framework is modified in the present extension; the value space of the connection is enlarged, and all subsequent results of M4 and M5 concerning the* U(1) *exchange sector are preserved without qualification.*

5 Generalized Admissibility Condition

The construction of Sec. 4 associates to each closed coherent exchange cycle a group element $\Theta(\gamma) \in G$. The structural constraint imposed by coherence of the cycle is now formulated as a condition on this group element, generalizing the U(1) *admissibility condition of*

Definition 5.1. *2.3.*

5.1 Statement

Definition 5.2 (Generalized holonomic admissibility)**.** *Let A be a non-Abelian exchange connection on $\mathcal{M}$ with structure group G, and let $\mathcal{B}$ be a bundled configuration with exchange graph $G(\mathcal{B})$. The connection A is said to satisfy the* generalized holonomic admissibility condition *with respect to $\mathcal{B}$ if for every closed coherent exchange cycle γ in $G(\mathcal{B})$,*

$$\Theta(\gamma) \in \mathcal{H}^G_{\mathrm{adm}},$$

where $\mathcal{H}^G_{\mathrm{adm}}$ is a discrete subgroup of G determined by the coherence requirements of $\mathcal{B}$.

Definition 5.3. *5.2 is the structural counterpart of*

Definition 5.4. *2.3: in both cases the closed-cycle holonomy is required to lie in a designated return class. What differs is the value space of the holonomy and, correspondingly, the structure of the admissible class.*

5.2 Discrete-subgroup structure

The admissible class $\mathcal{H}^G_{\mathrm{adm}}$ *in*

Definition 5.5. *5.2 is required to be discrete in G. This requirement is structural and follows from two considerations.*

First, the M5 admissibility condition selects a discrete return class $2\pi\mathbb{Z}$ *within the connected one-parameter group of phase advances. The exponentiated form of this condition,* $\mathrm{e}^{\,\mathrm{i}\,\Theta(\gamma)} = 1$*, selects the trivial subgroup* $\{\mathbf{1}\} \subset \mathrm{U}(1)$*, with the integer* n *in* $\Theta(\gamma) = 2\pi n$ *acting as a topological label of the cycle in the sense of*

Remark 5.6. *2.4. Continuity of the admissible class would dissolve this integer label and abolish the closure structure of M5, which is incompatible with the integer charge condition of M4 and the bundle taxonomy of M6 and M6.5. The non-Abelian admissibility condition must therefore preserve a discrete return class.*

Second, the bundle closure condition of M6 and M6.5 is a discrete algebraic constraint on the exchange graph: it requires the cumulative $\mathrm{U}(1)$ *orientation along any closed cycle to lie in* $\mathbb{Z}$*. Compatibility with this constraint, in any choice of structure group G, requires the admissible class of* $\Theta(\gamma)$ *to be discrete in G.*

The minimal discrete subgroup of G is the trivial subgroup

$$\mathcal{H}^G_{\mathrm{adm}} = \{\mathbf{1}_G\}.$$

This case generalizes the closure form of the M5 admissibility condition $\mathrm{e}^{\,\mathrm{i}\,\Theta(\gamma)} = 1$*. Larger discrete subgroups may correspond to coherence cycles whose closure tolerates a nontrivial central phase; specific choices of* $\mathcal{H}^G_{\mathrm{adm}}$ *depend on the structure of* $\mathcal{B}$ *and are not fixed at the structural level developed here. The framework of*

Definition 5.7. *5.2 accommodates any such choice.*

5.3 Compatibility with the M6/M6.5 bundle admissibility

The bundle taxonomy of M6 and M6.5 imposes a closure condition on admissible bundled configurations: the cumulative $\mathrm{U}(1)$ *orientation along any closed cycle in the exchange graph* $G(\mathcal{B})$ *must lie in* $\mathbb{Z}$*. Bundles whose open loops carry non-Abelian internal labels in the sense of*

Definition 5.8. *3.1 cannot satisfy this closure condition in its M4 form, since the orientation* $\sigma(O_i) \in \{+1, -1\}$ *does not capture the full structural content of the multiply-labeled configuration.*

The natural generalization of the bundle closure condition is the group-valued requirement

$$\Theta(\gamma) = \mathbf{1}_G$$

along any closed cycle γ *in* $G(\mathcal{B})$*, with the holonomy* $\Theta(\gamma)$ *defined by the path-ordered integral of*

Definition 5.9. *4.5 along the smooth realization of* γ *in* $\mathcal{M}$*. This is the case* $\mathcal{H}^G_{\mathrm{adm}} = \{\mathbf{1}_G\}$ *of*

Definition 5.10. *5.2. More general discrete admissible classes* $\mathcal{H}^G_{\mathrm{adm}} \supsetneq \{\mathbf{1}_G\}$ *extend this generalized closure condition to bundles whose coherence requirements tolerate a discrete return label.*

Theorem 5.11 (Reduction of the closure condition)**.** *Let* $\mathcal{B}$ *be a bundled configuration whose open loops carry* $\mathrm{U}(1)$ *orientation labels in the sense of M4, and let* γ *be any closed cycle in the exchange graph* $G(\mathcal{B})$*. Under the identification* $G = \mathrm{U}(1)$ *with the labeling data of*

Remark 5.12. *3.2, the M6/M6.5 bundle closure condition*

$$\sum_{i\in\gamma}\sigma(O_i)\in\mathbb{Z}$$

is equivalent to the generalized admissibility condition of

Definition 5.13. *5.2 with admissible class* $\mathcal{H}^G_{\mathrm{adm}}=\{\mathbf{1}_G\}$.

Proof. By

Remark 5.14. *3.2, the M4 orientation label* $\sigma(O_i)\in\{+1,-1\}$ *is the real one-dimensional reduction of the* $\mathrm{U}(1)$ *representation label on* O_i*. Composition of labels along the concatenated open loops of* γ *proceeds, by Sec. 3.3, by group multiplication in* $\mathrm{U}(1)$*, which in this representation acts as additive accumulation of orientation contributions. The cumulative orientation label associated with* γ *is therefore*

$$\sum_{i\in\gamma}\sigma(O_i),$$

and the M4/M6 closure condition requires this sum to lie in $\mathbb{Z}$.

By

Proposition 5.15. *4.11, the closed-cycle holonomy of the non-Abelian exchange connection in the case* $G=\mathrm{U}(1)$ *is*

$$\Theta(\gamma)=\exp\Big(\mathrm{i}\oint_\gamma A\Big),$$

where the line integral $\oint_\gamma A$ *accumulates additively the contributions of* A *along* γ*. In the discrete representation on* $G(\mathcal{B})$*, this line integral coincides with the cumulative orientation* $\sum_{i\in\gamma}\sigma(O_i)$ *up to the normalization* 2π *implied by the exponentiation, so that*

$$\Theta(\gamma)=\mathbf{1}_{\mathrm{U}(1)}\quad\Longleftrightarrow\quad\sum_{i\in\gamma}\sigma(O_i)\in\mathbb{Z}.$$

The right-hand side is the M6/M6.5 closure condition, and the left-hand side is the generalized admissibility condition of

Definition 5.16. *5.2 in the case* $\mathcal{H}^G_{\mathrm{adm}}=\{\mathbf{1}_G\}$*. The two conditions are therefore equivalent under the stated identification.* ☐

Theorem 5.17. *5.11 establishes that the bundle closure condition of M6 and M6.5 is the Abelian projection of the generalized holonomic admissibility condition. The structural content of the M6/M6.5 closure condition is preserved without modification under the non-Abelian generalization; what is added is the admission of bundle configurations whose coherence is expressed by group-valued holonomies in non-Abelian structure groups.*

Remark 5.18 (Compatibility with the bundle taxonomy)**.** *The framework of*

Definition 5.19. *5.2 extends the M6/M6.5 bundle taxonomy without modifying its structural content: bundled configurations whose open loops carry* $\mathrm{U}(1)$ *orientation labels remain admissible exactly as in M6 and M6.5, while bundled configurations with non-Abelian internal labels become admissible under the group-valued closure condition. No previous specification of bundled structure is altered, qualified, or replaced.*

6 Group Structure of the Admissible Class

Combining the closure properties of Secs. 6.1 and 6.2 yields the structural result of the present section, drawing on standard Lie-group machinery [6, 9].

Definition 5.2 introduces the admissible class $\mathcal{H}^G_{\mathrm{adm}}$ as a discrete subgroup of G determined by the coherence requirements of $\mathcal{B}$. At the level of that definition, the subgroup property is taken as input. The present section establishes that this is the natural choice: the set of holonomies attained by closed coherent exchange cycles in $G(\mathcal{B})$ is itself closed under group multiplication and inversion in G, and therefore inherits subgroup structure from G via composition of holonomies along concatenated cycles.

The two structural ingredients have already been established. The composition law of proposition 4.7 furnishes closure under concatenation of cycles, while the path-ordered construction of definition 4.5 furnishes a natural notion of cycle inversion. The present section assembles these ingredients.

6.1 Closure under concatenation

Let γ_1, γ_2 be closed coherent exchange cycles in $G(\mathcal{B})$ sharing a common basepoint $x_0 \in \mathcal{M}$. The concatenation $\gamma_2 * \gamma_1$ traverses γ_1 first and γ_2 second, returning to x_0, and is itself a closed coherent cycle in $G(\mathcal{B})$.

By proposition 4.7, the holonomy of the concatenated cycle is

$$\Theta(\gamma_2 * \gamma_1) = \Theta(\gamma_2)\,\Theta(\gamma_1),$$

with the product taken in G. If both $\Theta(\gamma_1)$ and $\Theta(\gamma_2)$ lie in the admissible class $\mathcal{H}^G_{\mathrm{adm}}$, their product also lies in $\mathcal{H}^G_{\mathrm{adm}}$ by closure of the subgroup under multiplication, and the concatenated cycle is therefore admissible. The admissibility condition of definition 5.2 is thus preserved under concatenation of closed cycles at a common basepoint.

6.2 Inverse cycles

For a closed coherent exchange cycle γ, the reverse-traversed cycle γ^{-1} is obtained by reparametrizing γ in the opposite direction.

Lemma 6.1 (Holonomy of the inverse cycle)**.** *For any piecewise-smooth closed path γ in $\mathcal{M}$,*

$$\Theta(\gamma^{-1}) = \Theta(\gamma)^{-1},$$

with the inverse taken in G.

Proof. Let $\{U_k : k = 1, \ldots, N\}$ be the discretized factors associated with a partition of γ as in Sec. 4.2, so that $\Theta_N(\gamma) = U_N\, U_{N-1} \cdots U_1$. Reverse traversal of γ inverts both the order of the partition and the sign of each displacement Δx^μ_k, so the discretized factors associated with γ^{-1} are

$$U_k^{-1} = \exp\!\big(-\,\mathrm{i}\, A^a_\mu(\gamma(t_{k-1}))\, T_a\, \Delta x^\mu_k\big),$$

taken in reverse order. The discretized product along γ^{-1} is therefore

$$\Theta_N(\gamma^{-1}) = U_1^{-1}\, U_2^{-1} \cdots U_N^{-1} = (U_N\, U_{N-1} \cdots U_1)^{-1} = \Theta_N(\gamma)^{-1},$$

where the second equality uses $(AB)^{-1} = B^{-1}A^{-1}$ in G. Taking the limit $N \to \infty$ and applying continuity of group inversion in G together with proposition 4.6 yields the stated identity. □

If $\Theta(\gamma)$ lies in $\mathcal{H}^G_{\mathrm{adm}}$, then $\Theta(\gamma)^{-1}$ also lies in $\mathcal{H}^G_{\mathrm{adm}}$ by closure of the subgroup under inversion. The admissibility condition of definition 5.2 is thus preserved under inversion of closed cycles.

6.3 Subgroup property of the holonomy image

Combining the closure properties of Secs. 6.1 and 6.2 yields the structural result of the present section.

Proposition 6.2 (Subgroup property of the holonomy image)**.** *Let $\mathcal{B}$ be a bundled configuration with exchange graph $G(\mathcal{B})$, fix a basepoint $x_0 \in \mathcal{M}$, and let*

$$\mathcal{I}(\mathcal{B};x_0) := \left\{ \Theta(\gamma) \ : \ \gamma \text{ a closed coherent cycle in } G(\mathcal{B}) \text{ based at } x_0 \right\} \subset G.$$

Then $\mathcal{I}(\mathcal{B};x_0)$ is a subgroup of G. If A satisfies the generalized holonomic admissibility condition of definition 5.2 with respect to $\mathcal{B}$, then

$$\mathcal{I}(\mathcal{B};x_0) \subseteq \mathcal{H}^G_{\mathrm{adm}}.$$

Proof. The trivial cycle at x_0 has holonomy $\mathbf{1}_G$, so $\mathbf{1}_G \in \mathcal{I}(\mathcal{B};x_0)$. Closure under multiplication follows from Sec. 6.1: for any $\Theta(\gamma_1), \Theta(\gamma_2) \in \mathcal{I}(\mathcal{B};x_0)$, the concatenated cycle $\gamma_2 * \gamma_1$ is itself a closed cycle in $G(\mathcal{B})$ based at x_0, and its holonomy $\Theta(\gamma_2)\,\Theta(\gamma_1)$ lies in $\mathcal{I}(\mathcal{B};x_0)$. Closure under inversion follows from lemma 6.1: for any $\Theta(\gamma) \in \mathcal{I}(\mathcal{B};x_0)$, the reverse cycle γ^{-1} is a closed cycle based at x_0 with holonomy $\Theta(\gamma)^{-1}$. Therefore $\mathcal{I}(\mathcal{B};x_0)$ is a subgroup of G.

The containment statement follows directly from definition 5.2: every closed cycle γ in $G(\mathcal{B})$ satisfies $\Theta(\gamma) \in \mathcal{H}^G_{\mathrm{adm}}$ by hypothesis, so every element of $\mathcal{I}(\mathcal{B};x_0)$ lies in $\mathcal{H}^G_{\mathrm{adm}}$. □

The subgroup $\mathcal{I}(\mathcal{B};x_0)$ is the holonomy image of the bundled configuration $\mathcal{B}$ at basepoint x_0. It encodes the global coherence content of the exchange connection on $\mathcal{B}$ and is the structural object that the admissibility condition constrains.

Remark 6.3 (Basepoint dependence)**.** *The holonomy image $\mathcal{I}(\mathcal{B};x_0)$ depends on the choice of basepoint x_0, and basepoints on different connected components of $G(\mathcal{B})$ yield distinct images. For basepoints x_0, x_0' connected by a coherent exchange path β in $G(\mathcal{B})$, the two images are related by conjugation,*

$$\mathcal{I}(\mathcal{B};x_0') = \Theta(\beta)\,\mathcal{I}(\mathcal{B};x_0)\,\Theta(\beta)^{-1},$$

as follows from proposition 4.7 applied to cycles based at x_0' written as $\beta * \gamma * \beta^{-1}$ with γ based at x_0. The structural content of the holonomy image is therefore a conjugacy class of subgroups of G, not a distinguished subgroup.

6.4 Bundle-level interpretation

The construction of Sec. 6.3 admits a natural interpretation at the level of the exchange graph. The closed coherent cycles in $G(\mathcal{B})$ based at x_0, with concatenation as the binary operation and reverse traversal as inversion, form a group $\Gamma(\mathcal{B};x_0)$.

Proposition 6.4 (Holonomy as a group homomorphism)**.** *The assignment*

$$\Theta : \Gamma(\mathcal{B};x_0) \longrightarrow G, \qquad \gamma \longmapsto \Theta(\gamma),$$

is a group homomorphism whose image is the holonomy image $\mathcal{I}(\mathcal{B};x_0)$ of proposition 6.2.

Proof. The composition law of proposition 4.7 states $\Theta(\gamma_2 * \gamma_1) = \Theta(\gamma_2)\,\Theta(\gamma_1)$, which is the homomorphism property of Θ with respect to concatenation. Lemma 6.1 furnishes $\Theta(\gamma^{-1}) = \Theta(\gamma)^{-1}$, consistent with the homomorphism property under inversion. The trivial cycle has holonomy $\mathbf{1}_G$, mapping the identity of $\Gamma(\mathcal{B};x_0)$ to the identity of G. The image of Θ is by definition $\mathcal{I}(\mathcal{B};x_0)$. □

Proposition 6.4 expresses the structural relationship between the bundle and the structure group: the exchange connection induces a homomorphism from the closed-cycle group of $\mathcal{B}$ into G, with image constrained to lie in the admissible class $\mathcal{H}^G_{\mathrm{adm}}$. This homomorphism encodes the global coherence content of the non-Abelian exchange sector.

Remark 6.5 (Recovery of the M5/M6 closure structure)**.** *In the Abelian case $G = \mathrm{U}(1)$, proposition 6.4 specializes to a homomorphism $\Theta : \Gamma(\mathcal{B}; x_0) \to \mathrm{U}(1)$ whose image is the cumulative orientation class of M6/M6.5 under the identification of theorem 5.11. The homomorphism property recovers the additivity of* U(1) *orientations along concatenated cycles, and the admissibility constraint reduces to the integer-charge condition. The structural framework of the present section therefore generalizes M5/M6 closure without modifying its content in the Abelian sector.*

7 Gauge Equivalence

The exchange connection of definition 4.1 is specified by its components A^a_μ on $\mathcal{M}$. Distinct component fields may, however, encode the same structural content of the exchange sector: a redefinition of the local frame in which the internal labels of Sec. 3.2 are expressed generates a corresponding redefinition of the connection. The present section formulates this redefinition as a gauge transformation and records its action on the closed-cycle holonomy and on the admissibility condition.

The conjugation structure encountered in remark 6.3, where a change of basepoint conjugated the holonomy image by a fixed group element, returns here in a manifold-wide form. The basepoint freedom of Sec. 6.4 and the gauge freedom of the present section are structurally parallel.

7.1 Local gauge transformation

Definition 7.1 (Gauge transformation of the exchange connection)**.** *A* gauge transformation *of the exchange connection is specified by a smooth map*

$$g : \mathcal{M} \longrightarrow G.$$

The transformed connection is [7, 8]

$$A \;\longmapsto\; A^g := g\, A\, g^{-1} + \mathrm{i}\, g\, dg^{-1},$$

where the products are understood in the matrix algebra of any faithful representation of G on a finite-dimensional vector space.

Remark 7.2 (Recovery of the U(1) gauge transformation)**.** *Taking $G = \mathrm{U}(1)$ and writing $g(x) = e^{\mathrm{i}\,\phi(x)}$ for a smooth real-valued function $\phi : \mathcal{M} \to \mathbb{R}$, the transformation of definition 7.1 reduces to*

$$A \;\longmapsto\; A + d\phi,$$

which is the gauge transformation of the U(1) *exchange connection in M4 and M5. The non-Abelian gauge transformation is therefore a strict generalization of the M-series Abelian gauge transformation.*

Remark 7.3 (Independence from the choice of representation)**.** *The transformation rule of definition 7.1 is formulated in a faithful representation of G for explicitness, but the structural content of the transformation depends only on the group element $g(x) \in G$ and is independent of the representation in which the matrix products are computed. Throughout the developments of the present section, we use the matrix realization as a calculational convenience without committing to a specific representation.*

7.2 Transformation of the holonomy

The closed-cycle holonomy of definition 4.5 transforms under the gauge transformation of definition 7.1 by conjugation at the basepoint of the cycle.

Proposition 7.4 (Conjugation of the holonomy)**.** *Let A be a non-Abelian exchange connection on $\mathcal{M}$, let $g : \mathcal{M} \to G$ be a smooth gauge transformation, and let γ be a piecewise-smooth closed path in $\mathcal{M}$ with basepoint x_0. Then the holonomy of the gauge-transformed connection A^g along γ satisfies*

$$\Theta_{A^g}(\gamma) = g(x_0)\,\Theta_A(\gamma)\,g(x_0)^{-1}.$$

Proof. The path-ordered holonomy $\Theta_A(\gamma)$ is the value $U(1)$ of the unique solution to the ordinary differential equation

$$\frac{dU(t)}{dt} = \mathrm{i}\,A^a_\mu(\gamma(t))\,\dot\gamma^\mu(t)\,T_a\,U(t), \qquad U(0) = \mathbf{1}_G,$$

identified in the proof of proposition 4.6. Define $V(t) := g(\gamma(t))\,U(t)\,g(\gamma(0))^{-1}$. Differentiating V and applying the chain rule yields

$$\frac{dV(t)}{dt} = \mathrm{i}\,(A^g)^a_\mu(\gamma(t))\,\dot\gamma^\mu(t)\,T_a\,V(t), \qquad V(0) = \mathbf{1}_G,$$

where the components $(A^g)^a_\mu$ are those of the transformed connection of definition 7.1. By uniqueness of solutions to this differential equation, $V(t)$ coincides with the path-ordered holonomy of A^g along γ from $\gamma(0)$ to $\gamma(t)$. For a closed path with $\gamma(1) = \gamma(0) = x_0$, evaluation at $t = 1$ yields

$$\Theta_{A^g}(\gamma) = V(1) = g(x_0)\,U(1)\,g(x_0)^{-1} = g(x_0)\,\Theta_A(\gamma)\,g(x_0)^{-1},$$

as claimed. □

Remark 7.5 (Gauge-invariance of the conjugacy class)**.** *By proposition 7.4, the gauge transformation acts on the holonomy of a closed cycle by conjugation in G. The conjugacy class of $\Theta(\gamma)$ in G is therefore invariant under gauge transformations, while the specific group element representing the holonomy is not. Gauge-invariant structural data extracted from the holonomy must depend only on its conjugacy class.*

7.3 Gauge invariance of the admissibility condition

The admissibility condition of definition 5.2 requires $\Theta(\gamma)$ to lie in the discrete subgroup $\mathcal{H}^G_{\mathrm{adm}}$ of G. Because the holonomy transforms by conjugation, gauge invariance of this condition requires the admissible class to be stable under conjugation by elements of G.

Corollary 7.6 (Gauge invariance of admissibility)**.** *Suppose the admissible class $\mathcal{H}^G_{\mathrm{adm}}$ is conjugation-stable in G, in the sense that*

$$g\,h\,g^{-1} \in \mathcal{H}^G_{\mathrm{adm}} \qquad \text{for all } g \in G \text{ and } h \in \mathcal{H}^G_{\mathrm{adm}}.$$

Then the generalized holonomic admissibility condition of definition 5.2 is invariant under gauge transformations of the exchange connection. In particular, the canonical case $\mathcal{H}^G_{\mathrm{adm}} = \{\mathbf{1}_G\}$ is gauge invariant, as is any choice of $\mathcal{H}^G_{\mathrm{adm}}$ contained in the center $Z(G)$ of G.

Proof. By proposition 7.4, a gauge transformation g maps the holonomy $\Theta(\gamma)$ to $g(x_0)\,\Theta(\gamma)\,g(x_0)^{-1}$ at the basepoint of the cycle. If $\Theta(\gamma) \in \mathcal{H}^G_{\text{adm}}$ and $\mathcal{H}^G_{\text{adm}}$ is conjugation-stable, then the transformed holonomy also lies in $\mathcal{H}^G_{\text{adm}}$, and the admissibility condition is preserved. The trivial subgroup $\{\mathbf{1}_G\}$ is conjugation-stable since $g\,\mathbf{1}_G\,g^{-1} = \mathbf{1}_G$ for all $g \in G$. Subgroups of the center $Z(G)$ are pointwise fixed under conjugation by elements of G and are therefore conjugation-stable. □

Remark 7.7 (Conjugation-stability and admissible class structure)**.** *Conjugation-stability of $\mathcal{H}^G_{\text{adm}}$ is a structural condition on the admissible class, not an additional postulate of the framework. Any normal subgroup of G, including the trivial subgroup, the center $Z(G)$, and any subgroup contained in $Z(G)$, satisfies this condition automatically. At the foundational level developed here, no specific choice of $\mathcal{H}^G_{\text{adm}}$ is fixed; corollary 7.6 records the conditions under which gauge invariance of the admissibility condition holds.*

7.4 Gauge equivalence of connections

The transformation of definition 7.1 generates an equivalence relation on the space of non-Abelian exchange connections: two connections A, A' are gauge equivalent when there exists a smooth $g : \mathcal{M} \to G$ such that $A' = A^g$. The structural content of the exchange sector is encoded in the gauge equivalence class of A rather than in A itself.

Remark 7.8 (Gauge equivalence and structural content)**.** *By proposition 7.4, gauge-equivalent connections yield holonomies that differ by conjugation at the basepoint of each cycle. The conjugacy classes of holonomies along closed coherent cycles are therefore the gauge-invariant structural data of the exchange sector. In particular, the holonomy image $\mathcal{I}(\mathcal{B}; x_0)$ of proposition 6.2 is well-defined as a conjugacy class of subgroups of G, in agreement with the basepoint-dependence remark of remark 6.3.*

The structural content established in Secs. 4–7—a gauge equivalence class of $\mathfrak{g}$-valued connections, generating a conjugacy class of holonomy homomorphisms into G, constrained to take values in a conjugation-stable discrete admissible class $\mathcal{H}^G_{\text{adm}}$—constitutes the non-Abelian generalization of the M4/M5 exchange sector. No further structural input is required for the formulation of specific sectoral reductions, which are treated in subsequent series.

8 Implications and Forward Structure

The structural framework of Secs. 4–7 specifies the non-Abelian exchange sector at the level of the foundational M-series. At this level, the gauge group G is unspecified: any Lie group consistent with the bundle taxonomy of M6 and M6.5 is admissible as a structure group of the exchange connection. The present section records three structural implications of this framework that orient subsequent sectoral developments without performing them.

8.1 Specific gauge groups as choices of input

The framework derived here treats the gauge group as a structural input to the exchange sector rather than as a derivable consequence of it. Specifying G is equivalent to specifying the labeling data (G, V, ρ) of definition 3.1, which in turn is equivalent to specifying the admissible internal structure of the open loops of the bundle.

For each choice of G, the construction proceeds uniformly:

- *the connection A takes values in the Lie algebra $\mathfrak{g}$ associated with G;*

- *the closed-cycle holonomy* $\Theta(\gamma)$ *is the path-ordered exponential of definition 4.5;*
- *the admissibility condition takes the form* $\Theta(\gamma) \in \mathcal{H}^G_{\mathrm{adm}}$ *for a conjugation-stable discrete subgroup of* G*;*
- *gauge equivalence is the action of smooth maps* $g : \mathcal{M} \to G$ *specified by definition 7.1.*

No further structural input is required at the level of the foundational framework. Specific sectoral reductions arising from particular choices of G *are treated in subsequent series. The correspondence between bundle taxonomy and structure group, and the specification of matter content as choices of representation* $\rho : G \to \mathrm{GL}(V)$*, are sectoral commitments that fall outside the scope of the present paper.*

8.2 Structural features absent in the Abelian sector

The non-Abelian generalization admits structural features that have no Abelian counterpart. Three are recorded here for orientation; none is developed in the present paper.

Non-commutative cycle composition. *By proposition 4.7, the holonomy of a concatenated cycle depends on the order of concatenation when* G *is non-Abelian. Closed cycles whose order of traversal cannot be permuted without modifying the holonomy are therefore admissible only under admissibility classes consistent with the order-dependent product. This feature is absent in the Abelian sector, where order of concatenation is structurally immaterial.*

Conjugation-orbit structure of the admissible class. *By corollary 7.6 and the basepoint-dependence of remark 6.3, the gauge-invariant content of the admissibility condition is its conjugation-orbit structure in* G*. For non-Abelian* G*, this structure is generally nontrivial: distinct conjugacy classes of* G *provide distinct admissibility constraints, none of which is reducible to a scalar invariant. In the Abelian case, conjugation acts trivially and the admissibility condition reduces to a scalar (integer-valued) constraint, recovering the M5/M6 closure condition.*

Group-singlet composite configurations. *Bundles whose individual open loops carry nontrivial internal labels in a non-Abelian representation may admit composite configurations whose cumulative holonomy lies in* $\mathcal{H}^G_{\mathrm{adm}}$ *even when the holonomies of the individual constituents do not. Such configurations are admissible as composites but not as isolated constituents. This is a structural feature with no Abelian analog: in the Abelian case, additivity of orientation labels makes admissibility of constituents and admissibility of composites equivalent. The structural origin of confinement-type behavior in specific non-Abelian sectoral reductions lies in this feature; its sectoral development is treated in subsequent series.*

8.3 Preservation of the M-series Abelian sector

The non-Abelian generalization developed in the present paper preserves the M-series Abelian sector without modification. This preservation is structural and follows from the explicit recovery results of Secs. 4 and 5:

- *proposition 4.11 recovers the M5* $\mathrm{U}(1)$ *holonomy as the Abelian special case of the path-ordered holonomy;*

- *theorem 5.11 identifies the M6/M6.5 bundle closure condition as the Abelian projection of the generalized admissibility condition;*
- *remark 7.2 recovers the M-series Abelian gauge transformation as the Abelian special case of definition 7.1;*
- *remark 6.5 recovers the M5/M6 closure structure as the Abelian specialization of the bundle-level homomorphism of proposition 6.4.*

No previous result of M4, M5, M6, or M6.5 concerning the $\mathrm{U}(1)$ *exchange sector is modified, qualified, or replaced by the present extension. Bundled configurations whose open loops carry* $\mathrm{U}(1)$ *orientation labels remain admissible exactly as specified in M6 and M6.5, and the integer-charge condition associated with the* $\mathrm{U}(1)$ *exchange sector is preserved as the Abelian special case of the generalized admissibility condition.*

The non-Abelian extension is therefore an enlargement of the M-series exchange sector: structures admissible under the Abelian framework remain admissible under the non-Abelian framework, while bundles whose internal labeling exceeds the $\mathrm{U}(1)$ *structure become admissible under the group-valued admissibility condition for the appropriate choice of* G *and* $\mathcal{H}^G_{\mathrm{adm}}$.

9 Interpretation and Conclusion

The M-series exchange sector developed in M4 and M5 specifies the exchange connection as a smooth real-valued 1-form on the scalar–conformal manifold, with closed-cycle holonomy taking values in $\mathrm{U}(1)$ *and admissibility expressed as a discrete return-class condition. This specification is structurally complete for the Abelian sector of the exchange dynamics and is incorporated into the bundle taxonomy of M6 and M6.5 through the integer-charge closure condition.*

The present paper has generalized this specification to the non-Abelian level. The exchange connection

$$A = A^a_\mu(x)\, T_a\, dx^\mu$$

takes values in the Lie algebra $\mathfrak{g}$ *of a structure group* G*. The closed-cycle holonomy*

$$\Theta(\gamma) = \mathcal{P}\exp\Big(\mathrm{i}\oint_\gamma A^a\, T_a\Big) \in G$$

is defined by the path-ordered exponential, composes by group multiplication along concatenated cycles, and transforms by conjugation under gauge transformations of the connection. The holonomic admissibility condition takes the form $\Theta(\gamma) \in \mathcal{H}^G_{\mathrm{adm}}$*, with* $\mathcal{H}^G_{\mathrm{adm}}$ *a conjugation-stable discrete subgroup of* G *determined by the coherence requirements of the bundle.*

The framework satisfies the structural requirements set out in Sec. 1.2. It is expressed entirely in terms of a smooth $\mathfrak{g}$*-valued 1-form on* $\mathcal{M}$*; it admits a well-defined closed-cycle holonomy whose composition law reflects the group structure; it supports an admissibility condition compatible with the bundle taxonomy of M6 and M6.5; it reduces exactly to the M4/M5 construction in the Abelian case; and it introduces no postulates beyond those of the foundational M-series framework. Specific sectoral reductions corresponding to particular choices of* G *are not addressed in the present paper and are deferred to subsequent series.*

The structural specification of the exchange sector at the foundational level is therefore complete. In the Abelian case, it specializes exactly to the M-series electromagnetic sector developed in M4, M5, and M6. In the non-Abelian case, it provides the structural framework within which sectoral reductions to specific gauge groups are formulated. The bundle taxonomy of M6 and M6.5, the *holonomic coherence framework of M5, and the canonical exchange sector of M4 are preserved without modification; the non-Abelian generalization extends this taxonomy to bundles whose open loops carry internal labels in non-Abelian representations of a structure group.*

This concludes the structural development of the exchange sector at the foundational M-series level. The framework derived here serves as the structural input to subsequent non-Abelian sectoral developments of the NUVO program.

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M5 *R. W. Austin,* Holonomic Coherence and Geometric Quantization on a Scalar–Conformal Manifold. *St Claire Scientific Research, Development, and Publishing. Closed-cycle holonomy of the exchange connection; holonomic admissibility condition* $\Theta(\gamma) \in \mathcal{H}_{\mathrm{adm}}$*; integer-charge condition in the* $\mathrm{U}(1)$ *case.*

M6 *R. W. Austin,* Bundled Loop Structures and Persistent Matter on Scalar–Conformal NUVO Space. *St Claire Scientific Research, Development, and Publishing. Bundle taxonomy* $\mathcal{B} = (\mathbf{C}, \mathbf{O}, \mathcal{R}, \sigma)$*; exchange graph* $G(\mathcal{B})$*; cumulative orientation closure condition.*

M6.5 *R. W. Austin,* Anchors, Capacity Delivery, and Flux Imbalance in Scalar–Conformal NUVO Space. *St Claire Scientific Research, Development, and Publishing. Boundary-flux representation of bundled structures; refinement of the bundle taxonomy through anchor and delivery formulations.*

M10 *R. W. Austin,* Composition of Scalar Modulation in Scalar–Conformal NUVO Space. *St Claire Scientific Research, Development, and Publishing. Multiplicative composition law* $\lambda_{\mathrm{eff}} = \lambda_{\mathrm{amb}} \cdot \lambda_{\mathrm{loc}}$*; immediate predecessor in the M-series.*

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