

# QB3 – Holonomic Coherence and the Emergence of Pre-Hilbert Structure in Scalar–Conformal NUVO Systems

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## Abstract

We develop the representational structure of the hydrogenic sector in scalar–conformal NUVO systems from first principles of closure and holonomic coherence. Starting from the invariant return structure of exchange-cycle dynamics, we construct a family of stationary closure modes and show that their coherence relations induce a natural complex inner product.

This inner product is not postulated but arises from an invariant coherence functional defined over the hydrogenic cycle family. Distinct closure classes are shown to correspond to orthogonal modes under this functional, yielding a pre-Hilbert structure on the finite representational span of stationary configurations.

The resulting structure provides a comparison space in which closure density and transport phase are unified into a complex representation. No probabilistic or measurement-theoretic assumptions are introduced. The construction establishes the geometric and coherence-based origin of inner-product structure required for subsequent development of event and weighting frameworks.

## 1 Introduction

The emergence of probabilistic structure in quantum theory is typically formulated through the Born rule [1], which assigns weights to measurement outcomes based on the squared modulus of a complex state. In conventional formulations, this rule is either postulated or motivated through statistical or operational arguments. In the present work, we pursue a different route: rather than introducing probability at the outset, we seek to identify the structural conditions under which a Hilbert-type framework arises intrinsically from the underlying geometry of the system.

The setting is scalar–conformal NUVO space, in which physical states are described by closure density and transport-derived phase, and admissibility is governed by global closure constraints. In this framework, the hydrogenic proton–electron system occupies a distinguished role, as it provides the simplest nontrivial realization of a discrete closure hierarchy generated by repeated exchange-cycle structure. The existence of this hierarchy suggests that the hydrogenic sector may serve as a natural starting point for investigating the emergence of orthogonality, spectral decomposition, and ultimately measurement structure.

The central objective of this paper is to demonstrate that the hydrogenic stationary sector admits a natural complex comparison structure with the properties of a pre-Hilbert space, derived entirely from its holonomic coherence structure. Importantly, this construction does not assume

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\*Bibliography is provisional. Cross-references to companion NUVO-series papers (M-, SR-, Q-, QB-, QM-series) will be updated with Zenodo DOIs in subsequent versions.

any probabilistic interpretation, measurement postulate, or Hilbert-space framework. Instead, it proceeds from three structural ingredients:

- the existence of a canonical hydrogenic return structure generated by a fixed traversal defect,
- the identification of stationary closure modes as eigenmodes of the induced return map,
- and the invariance of the associated coherence functional under this return symmetry.

The key result is that distinct stationary closure modes correspond to distinct characters of the return map, and orthogonality follows from the invariance of the coherence functional under this symmetry. This yields a normalized orthogonal family and an induced inner product on the finite representational span of the hydrogenic modes. The resulting structure is a complex pre-Hilbert space constructed directly from closure density and transport phase.

It is important to emphasize the scope of the present work. We do not introduce probability, measurement, or statistical ensembles. The construction here is purely structural: it establishes the comparison geometry of the hydrogenic sector. In particular, we do not assume that arbitrary superpositions correspond to physically realized states; rather, the representational span serves as a comparison space in which coherence relations can be evaluated.

This paper forms the first step in a program aimed at deriving probabilistic structure as a consequence of underlying geometric and coherence properties. In subsequent work [2], we will show how selector-induced partitions of the closure-class structure give rise to a projector algebra, and how context-independent weight assignments on that algebra lead, via Gleason-type arguments [3], to the Born rule as the unique admissible weighting scheme.

The organization of the paper is as follows. In Section 2, we recall the hydrogenic closure structure and define the associated discrete hierarchy of closure classes. Section 3 introduces the represented transport states used to encode closure density and phase. In Sections 4 and 5, we define the hydrogenic return map and characterize stationary closure modes as its eigenmodes. Section 6 introduces the invariant coherence functional. In Section 7, we prove the orthogonality of distinct stationary modes using return-map symmetry. Section 8 establishes the induced inner product, and Section 9 constructs the associated pre-Hilbert space. Finally, Sections 10 and 11 discuss interpretation and outline the connection to subsequent developments.

## 2 Hydrogenic Closure Structure

The hydrogenic proton–electron system provides the simplest nontrivial realization of closure-constrained dynamics in scalar–conformal NUVO space. In this setting, admissible configurations are not determined by orbital trajectories or force balance, but by global closure compatibility of the exchange cycle.

### 2.1 Exchange Cycle and Closure Compatibility

The hydrogenic system is modeled as a coupled exchange process between a proton source and an electron sink, mediated by the scalar–conformal transport structure. The fundamental dynamical object is the exchange cycle, consisting of a complete transport loop from source to sink and back.

Admissibility of a configuration is governed by a global closure condition: the accumulated transport phase along a full cycle must match a discrete compatibility condition determined by the

underlying scalar–conformal geometry. This condition takes the general form [4]

$$\oint_{\gamma} d\phi = m \Phi_c,$$

where  $\gamma$  is an admissible exchange cycle,  $\phi$  is the transport-derived phase,  $\Phi_c$  is the characteristic closure scale, and  $m \in \mathbb{Z}$  labels the closure class.

Thus, admissible hydrogenic configurations are indexed by discrete closure classes determined by the integer  $m$ .

**Relation to the scalar-modulated closure functional.** The closure condition expressed here in terms of phase accumulation is a representation of the scalar-modulated return condition established in Q2 [5],

$$k \oint_{\gamma} \lambda_{\text{eff}}(x, u) ds = L_{\gamma}.$$

In the hydrogenic stationary regime, this condition reduces to a cycle-level compatibility expressed through the accumulated phase increment. The phase-based formulation therefore encodes the same closure constraint in transport variables rather than geometric arc-length variables.

## 2.2 Discrete Closure Hierarchy

The closure condition induces a discrete hierarchy of admissible hydrogenic configurations [6]. Each integer  $m$  corresponds to a distinct closure-compatible exchange structure, and hence to a distinct stationary configuration of the system.

We denote the set of all admissible hydrogenic closure classes by

$$\mathcal{I}_H := \{m \in \mathbb{Z}_{>0} \mid \text{closure-compatible hydrogenic configurations}\}.$$

Each element  $m \in \mathcal{I}_H$  represents a distinct global closure configuration of the proton–electron exchange cycle. These configurations are not distinguished by local dynamical variables alone, but by their compatibility with the global return structure of the system.

## 2.3 Stationary Closure Modes

A hydrogenic closure class is said to be stationary if it is compatible with repeated traversal of the exchange cycle under the same closure condition. That is, after one full cycle, the system returns to a configuration of the same closure class, up to the intrinsic phase increment.

Stationary closure modes therefore correspond to configurations that are invariant under repeated application of the exchange cycle, modulo the discrete phase structure imposed by the closure condition.

This motivates the identification of stationary closure modes as the fundamental building blocks of the hydrogenic sector. In subsequent sections, these modes will be represented by complex-valued functions encoding closure density and transport phase, and will be shown to furnish a natural orthogonal family under an induced comparison structure.

## 2.4 Remarks on Physical Interpretation

It is important to emphasize that the closure-class label  $m$  does not correspond to a classical orbital parameter. Rather, it encodes a global compatibility condition on the exchange cycle. The

resulting hierarchy reflects the discrete structure of admissible configurations in scalar–conformal NUVO space, not a discretization of continuous classical motion.

In particular, the hydrogenic closure hierarchy arises purely from geometric and coherence constraints, without reference to forces, potentials, or probabilistic interpretations. This perspective will be essential in the derivation of orthogonality and comparison structure in later sections.

### 3 Represented Transport States

In scalar–conformal NUVO systems, the fundamental state variables are closure density and transport-derived phase. These quantities arise from the underlying transport and closure structure and are sufficient to describe the admissible configuration of the system in the integrable regime.

In this section, we introduce a complex representation of these variables. This representation will be used as a mathematical encoding of the underlying structure, but it is important to emphasize that it does not introduce any probabilistic interpretation.

#### 3.1 Closure Density and Transport Phase

Let  $\rho$  denote the closure density associated with a given configuration, and let  $\phi$  denote the transport-derived phase accumulated along admissible paths in the scalar–conformal geometry.

The pair  $(\rho, \phi)$  fully characterizes the state of the system in the transport closure formulation. In particular:

- $\rho$  measures the local closure content or capacity participation of the configuration,
- $\phi$  encodes the accumulated transport phase consistent with admissible exchange cycles.

These quantities are governed by the transport and closure laws of the NUVO framework and are not introduced as auxiliary constructs.

**Origin in closure compatibility under transport.** The transport phase  $\phi$  appearing in this formulation is not an independent degree of freedom. As established in Q8–Q10 [7, 8, 4], it arises as a representation of closure compatibility under transport, encoding the accumulation of exchange-modulated transport required to restore admissible closure conditions along a cycle.

#### 3.2 Complex Representation

We define the represented transport state by [9]

$$\Psi := \sqrt{\rho} e^{i\phi/\Phi_0},$$

where  $\Phi_0$  is a fixed representation scale.

This representation encodes the full information of  $(\rho, \phi)$ :

$$|\Psi|^2 = \rho, \quad \arg(\Psi) = \frac{\phi}{\Phi_0}.$$

Thus, the mapping

$$(\rho, \phi) \longleftrightarrow \Psi$$

is invertible, and no information is lost in passing to the complex representation.

### 3.3 Interpretational Scope

The complex quantity  $\Psi$  should be regarded strictly as a representational device. It is not assumed to be a wavefunction in the quantum-mechanical sense, nor is  $|\Psi|^2$  interpreted as a probability density.

Instead,  $\Psi$  provides a convenient encoding of closure density and transport phase that allows coherence relations between different configurations to be expressed in a compact form. In particular, products of the form

$$\Psi_m^* \Psi_n = \sqrt{\rho_m \rho_n} e^{i(\phi_n - \phi_m)/\Phi_0}$$

naturally combine closure-density overlap with relative phase information.

These quantities will play a central role in the construction of the coherence functional and the induced comparison structure in subsequent sections.

### 3.4 Stationary Modes in Represented Form

For each hydrogenic closure class  $n \in \mathcal{I}_H$ , we associate a represented stationary mode

$$\Psi_n = \sqrt{\rho_n} e^{i\phi_n/\Phi_0},$$

where  $(\rho_n, \phi_n)$  correspond to the closure density and transport phase of the stationary configuration in that class.

The collection  $\{\Psi_n\}$  will serve as the fundamental family of represented modes in the hydrogenic sector. In later sections, these modes will be shown to exhibit orthogonality properties derived from the underlying return structure, and will form the basis of the induced comparison space.

### 3.5 Remark on Representational Linearity

The complex representation naturally allows the formation of linear combinations of the form

$$\Phi = \sum_n c_n \Psi_n, \quad c_n \in \mathbb{C}.$$

At this stage, such combinations are introduced purely as elements of a representational span used for comparison and analysis. We do not assume that arbitrary linear combinations correspond to physically realized configurations of the system.

This linear structure reflects the comparison geometry induced by the coherence functional and does not imply superposability of physical configurations.

This distinction between representational structure and physical admissibility will be maintained throughout the paper and will be essential in the interpretation of the resulting comparison space.

## 4 Hydrogenic Return Structure

The discrete closure hierarchy described in the previous section arises from a repeated-return structure inherent to the hydrogenic exchange cycle. In this section, we formalize this structure through the introduction of a canonical return map. This provides the symmetry framework from which orthogonality will later be derived.

## 4.1 Traversal Defect and Return Structure

In scalar–conformal NUVO systems, transport along an exchange cycle is not strictly periodic in the naive geometric sense. Instead, each traversal carries a fixed structural defect, or advance, determined by the underlying scalar–conformal geometry of the system.

In the hydrogenic regime, this defect is constant across successive traversals [10, 11]. That is, each complete traversal of the exchange cycle contributes the same net increment to the global transport structure. Repeated traversals therefore generate a cumulative return structure governed by a fixed increment at each step.

This repeated-defect structure is the origin of the discrete closure hierarchy: admissible configurations correspond to those for which the accumulated defect after a finite number of traversals is compatible with the global closure condition.

## 4.2 Definition of the Return Map

We formalize this structure by introducing a return map on the hydrogenic exchange-cycle family.

**Definition 4.1** (Hydrogenic return map). *Let  $\Gamma_H$  denote the reference hydrogenic exchange-cycle family. The hydrogenic return map*

$$T : \Gamma_H \rightarrow \Gamma_H$$

*is defined as the transformation induced by one full traversal of the exchange cycle, including the fixed traversal defect.*

Thus, the action of  $T$  represents the advancement of the system by one complete exchange cycle.

## 4.3 Iterated Return and Closure Compatibility

Repeated application of the return map generates the sequence

$$\gamma, \quad T(\gamma), \quad T^2(\gamma), \quad \dots$$

corresponding to successive traversals of the exchange cycle.

The global closure condition may then be expressed in terms of this iterated structure: a configuration is closure-compatible if, after a finite number of iterations, the accumulated transport phase satisfies the discrete closure condition introduced in Section 2.

In this formulation, the closure class label is determined by the compatibility of the configuration with the iterated return structure generated by  $T$ .

## 4.4 Return Symmetry and Stationarity

A hydrogenic closure mode is stationary if it is compatible with repeated application of the return map in a consistent manner. That is, after each traversal, the configuration returns to the same closure class, up to the intrinsic phase increment determined by the closure condition.

This motivates the following structural principle:

- The hydrogenic stationary sector is invariant under the action of the return map  $T$ .

This invariance will be made precise in the next section, where stationary modes are characterized as eigenmodes of the induced action of  $T$  on represented transport states.

## 4.5 Invariant Structure

The existence of a fixed traversal defect implies that the return map acts uniformly across the hydrogenic cycle family. In particular, there is no distinguished traversal step: each application of  $T$  contributes the same structural increment.

This uniformity is the key property that allows the construction of an invariant coherence functional in later sections. Specifically, it motivates the introduction of a measure on  $\Gamma_H$  that is invariant under the action of  $T$ .

## 4.6 Remark on Abstraction

It is important to emphasize that the return map  $T$  is an abstract representation of the repeated-return structure induced by the exchange cycle. It should not be interpreted as a dynamical time evolution operator in the conventional sense.

Rather,  $T$  encodes the geometric and closure properties of the system under repeated traversal of the exchange cycle. Its role in this paper is to provide a symmetry structure from which coherence relations and orthogonality can be derived.

# 5 Stationary Modes as Return-Map Eigenmodes

In the previous section, we introduced the hydrogenic return map  $T$  as the structural transformation associated with one full traversal of the exchange cycle. We now characterize the stationary closure modes of the hydrogenic sector in terms of this return map.

## 5.1 Induced Action on Represented States

The return map  $T$  acts on functions defined on the hydrogenic cycle family  $\Gamma_H$  via pullback. For a represented transport state  $\Psi$ , we define

$$T^*\Psi := \Psi \circ T.$$

This defines a linear action of  $T$  on the space of represented transport states.

## 5.2 Stationary Modes as Eigenmodes

We now formalize the notion of stationarity in terms of this induced action.

**Definition 5.1** (Return-map eigenmode). *A represented hydrogenic stationary closure mode  $\Psi_n$  is said to be a return-map eigenmode if*

$$T^*\Psi_n = \chi_n \Psi_n,$$

where  $\chi_n \in U(1)$  is a unit-modulus complex number.

The quantity  $\chi_n$  represents the phase accumulated by the mode under one application of the return map.

### 5.3 Relation to Closure Classes

The eigencharacter  $\chi_n$  is determined by the closure class of the mode. Indeed, from the closure condition

$$\oint_{\gamma} d\phi_n = m_n \Phi_c,$$

it follows that one traversal of the exchange cycle produces a fixed phase increment. In the represented formulation, this implies

$$\chi_n = \exp\left(\frac{i}{\Phi_0} \oint_{\gamma} d\phi_n\right) = \exp\left(\frac{im_n \Phi_c}{\Phi_0}\right).$$

Thus, the eigencharacter is directly determined by the closure-class label  $m_n$ .

### 5.4 Distinct Closure Characters

The discrete closure hierarchy induces a corresponding distinction among the eigencharacters.

**Assumption 5.2** (Distinct closure characters). *Distinct closure classes correspond to distinct eigencharacters, as a consequence of the discrete closure compatibility condition and the non-degeneracy of the associated phase increments under the return structure.*

This assumption reflects the fact that different closure-compatible exchange configurations correspond to different accumulated phase increments under the return map.

### 5.5 Interpretation

The return-map eigenmode condition expresses stationarity in a precise algebraic form: a stationary closure mode is invariant under the return structure up to a phase factor.

In this sense, the hydrogenic stationary sector decomposes into a family of modes characterized by their response to the return map. The eigencharacters  $\chi_n$  encode the holonomic phase structure associated with each closure class.

This characterization will play a central role in the next section, where we introduce a coherence functional invariant under the return map and use the distinct eigencharacters to establish orthogonality of the stationary modes.

## 6 Invariant Coherence Functional

In the preceding sections, we introduced the hydrogenic return map  $T$  and characterized stationary closure modes as its eigenmodes. We now construct a coherence functional that is compatible with this return symmetry. This functional will provide the foundation for the induced comparison structure.

### 6.1 Invariant Measure

The return map  $T$  encodes the repeated traversal structure of the hydrogenic exchange cycle. In order to compare different represented states in a manner consistent with this structure, we require a measure that is invariant under the action of  $T$ .

**Definition 6.1** (Invariant measure). *A measure  $d\mu_H$  on the hydrogenic cycle family  $\Gamma_H$  is said to be invariant under the return map  $T$  if, for all integrable functions  $f$ ,*

$$\int_{\Gamma_H} f d\mu_H = \int_{\Gamma_H} T^* f d\mu_H.$$

Such a measure reflects the absence of any preferred location on the return structure: each traversal step contributes identically, and integration over the cycle family respects this symmetry.

This invariance follows from the uniformity of the return structure: each traversal contributes an identical structural increment, and no distinguished point exists along the cycle family. The invariant measure therefore reflects the homogeneity of the return structure rather than an independently imposed integration rule.

## 6.2 Mixed Coherence Functional

Using the invariant measure, we define the central object of this section.

**Definition 6.2** (Mixed coherence functional). *For represented stationary modes  $\Psi_m$  and  $\Psi_n$ , define*

$$\mathcal{C}_{mn} := \int_{\Gamma_H} \Psi_m^* \Psi_n d\mu_H.$$

This functional combines closure-density overlap with relative phase information. Using the representation

$$\Psi_k = \sqrt{\rho_k} e^{i\phi_k/\Phi_0},$$

we have

$$\Psi_m^* \Psi_n = \sqrt{\rho_m \rho_n} \exp\left(\frac{i}{\Phi_0}(\phi_n - \phi_m)\right).$$

Thus,  $\mathcal{C}_{mn}$  encodes both the overlap of closure densities and the compatibility of transport phases between two configurations.

## 6.3 Invariance Under the Return Map

The coherence functional is constructed to be compatible with the return symmetry.

**Lemma 6.3** (Return invariance of coherence functional). *Let  $\Psi_m, \Psi_n$  be represented states. Then*

$$\mathcal{C}_{mn} = \int_{\Gamma_H} T^*(\Psi_m^* \Psi_n) d\mu_H.$$

*Proof.* This follows directly from the invariance of the measure:

$$\int_{\Gamma_H} \Psi_m^* \Psi_n d\mu_H = \int_{\Gamma_H} T^*(\Psi_m^* \Psi_n) d\mu_H.$$

□

## 6.4 Interpretation

The mixed coherence functional provides a natural pairing between represented states that respects the intrinsic symmetry of the hydrogenic return structure.

It is important to emphasize that  $\mathcal{C}_{mn}$  is not introduced as an inner product by assumption. Rather, it is defined directly from closure density and transport phase, combined with the invariant measure induced by the return map.

In the next section, we will show that the invariance of this functional, together with the eigencharacter structure of stationary modes, implies orthogonality of distinct closure classes. This will allow us to promote  $\mathcal{C}_{mn}$  to an inner product on the stationary sector.

## 7 Orthogonality from Return-Map Symmetry

In this section, we establish the central structural result of the paper: distinct hydrogenic stationary closure modes are orthogonal with respect to the invariant coherence functional. This result follows directly from the interplay between return-map symmetry and the eigencharacter structure of the stationary modes.

### 7.1 Transformation of Mixed Coherence

Let  $\Psi_m$  and  $\Psi_n$  be represented stationary modes satisfying

$$T^*\Psi_k = \chi_k\Psi_k, \quad k = m, n,$$

with  $\chi_k \in U(1)$ .

We consider the transformation of the mixed product under the return map:

$$T^*(\Psi_m^*\Psi_n) = (T^*\Psi_m)^*(T^*\Psi_n).$$

Using the eigenmode property, we obtain

$$T^*(\Psi_m^*\Psi_n) = (\chi_m\Psi_m)^*(\chi_n\Psi_n) = \overline{\chi_m}\chi_n\Psi_m^*\Psi_n.$$

### 7.2 Orthogonality Theorem

**Theorem 7.1** (Orthogonality of distinct stationary modes). *Let  $\Psi_m$  and  $\Psi_n$  be hydrogenic stationary modes with return-map eigencharacters  $\chi_m$  and  $\chi_n$ . If*

$$\chi_m \neq \chi_n,$$

*then*

$$\mathcal{C}_{mn} = 0.$$

*Proof.* By definition,

$$\mathcal{C}_{mn} = \int_{\Gamma_H} \Psi_m^*\Psi_n d\mu_H.$$

Using the invariance of the measure (Section 6), we also have

$$\mathcal{C}_{mn} = \int_{\Gamma_H} T^*(\Psi_m^*\Psi_n) d\mu_H.$$

Substituting the transformation law,

$$\mathcal{C}_{mn} = \int_{\Gamma_H} \overline{\chi_m \chi_n} \Psi_m^* \Psi_n d\mu_H = \overline{\chi_m \chi_n} \mathcal{C}_{mn}.$$

Rearranging,

$$(1 - \overline{\chi_m \chi_n}) \mathcal{C}_{mn} = 0.$$

If  $\chi_m \neq \chi_n$ , then  $\overline{\chi_m \chi_n} \neq 1$ , and therefore

$$\mathcal{C}_{mn} = 0.$$

□

### 7.3 Discussion

This result shows that orthogonality of stationary modes is a direct consequence of return-map symmetry. No assumptions about uniform density, oscillatory cancellation, or external Hilbert structure are required.

The essential ingredients are:

- invariance of the coherence functional under the return map,
- eigencharacter structure of stationary modes,
- and the distinction of closure classes via their return-phase characters.

This establishes orthogonality as a structural property of the hydrogenic closure hierarchy.

### 7.4 Interpretational Remark

The vanishing of  $\mathcal{C}_{mn}$  for distinct modes expresses the incompatibility of different closure classes at the level of holonomic coherence. Distinct closure-compatible configurations carry distinct return-phase characters, and their mixed coherence cancels under the invariant return structure.

This interpretation is purely geometric and coherence-based. No probabilistic or measurement-theoretic assumptions are involved.

### 7.5 Normalization

For  $m = n$ , we obtain

$$\mathcal{C}_{nn} = \int_{\Gamma_H} |\Psi_n|^2 d\mu_H = \int_{\Gamma_H} \rho_n d\mu_H.$$

Since  $\rho_n > 0$  for admissible configurations, it follows that

$$\mathcal{C}_{nn} > 0.$$

Thus, each stationary mode has strictly positive self-coherence, and may be normalized by rescaling:

$$\Psi_n \mapsto \frac{\Psi_n}{\sqrt{\mathcal{C}_{nn}}}.$$

After normalization, we obtain

$$\mathcal{C}_{mn} = \delta_{mn}.$$

## 8 Induced Comparison Structure

In the previous section, we established that the mixed coherence functional vanishes between distinct stationary modes and is strictly positive on each mode individually. We now show that this functional induces a natural inner product on the stationary sector.

### 8.1 Definition of the Inner Product

**Definition 8.1** (Induced inner product). *For represented stationary modes  $\Psi_m$  and  $\Psi_n$ , define*

$$\langle \Psi_m, \Psi_n \rangle_H := C_{mn} = \int_{\Gamma_H} \Psi_m^* \Psi_n d\mu_H.$$

This definition extends sesquilinearly to the finite representational span.

### 8.2 Hermitian Structure

**Theorem 8.2** (Hermitian properties). *The induced form  $\langle \cdot, \cdot \rangle_H$  satisfies:*

1. *Conjugate symmetry:*

$$\langle \Psi_m, \Psi_n \rangle_H = \overline{\langle \Psi_n, \Psi_m \rangle_H}.$$

2. *Linearity in the second argument and antilinearity in the first.*

3. *Positivity:*

$$\langle \Psi_n, \Psi_n \rangle_H > 0.$$

*Proof.* Conjugate symmetry follows directly from complex conjugation:

$$\langle \Psi_m, \Psi_n \rangle_H = \int \Psi_m^* \Psi_n d\mu_H = \overline{\int \Psi_n^* \Psi_m d\mu_H} = \overline{\langle \Psi_n, \Psi_m \rangle_H}.$$

Linearity follows from linearity of the integral.

Positivity follows from

$$\langle \Psi_n, \Psi_n \rangle_H = \int |\Psi_n|^2 d\mu_H = \int \rho_n d\mu_H > 0,$$

since  $\rho_n$  is strictly positive for admissible configurations. □

### 8.3 Orthonormal Family

From Theorem 7.5, we have

$$\langle \Psi_m, \Psi_n \rangle_H = 0 \quad \text{for } m \neq n.$$

Together with positivity, this implies that the stationary modes form an orthogonal family. By normalization, we obtain:

**Corollary 8.3** (Orthonormal basis). *There exists a normalization of the stationary modes such that*

$$\langle \Psi_m, \Psi_n \rangle_H = \delta_{mn}.$$

## 8.4 Extension to Representational Span

Let

$$\mathcal{V}_H^{\text{fin}} := \text{span}_{\mathbb{C}}\{\Psi_n\}.$$

For any two elements

$$\Phi = \sum_n a_n \Psi_n, \quad \Xi = \sum_n b_n \Psi_n,$$

we define

$$\langle \Phi, \Xi \rangle_H := \sum_{m,n} \overline{a_m} b_n \langle \Psi_m, \Psi_n \rangle_H.$$

Using orthonormality, this reduces to

$$\langle \Phi, \Xi \rangle_H = \sum_n \overline{a_n} b_n.$$

## 8.5 Positive Definiteness

**Theorem 8.4.** *The induced form  $\langle \cdot, \cdot \rangle_H$  is positive definite on  $\mathcal{V}_H^{\text{fin}}$ , i.e.,*

$$\langle \Phi, \Phi \rangle_H > 0 \quad \text{for all } \Phi \neq 0.$$

*Proof.* Let  $\Phi = \sum_n a_n \Psi_n$ . Then

$$\langle \Phi, \Phi \rangle_H = \sum_n |a_n|^2.$$

This is strictly positive unless all  $a_n = 0$ , i.e., unless  $\Phi = 0$ . □

## 8.6 Interpretation

The coherence functional  $\mathcal{C}_{mn}$ , originally defined in terms of closure density and transport phase, induces a complete comparison structure on the stationary sector.

This structure is not postulated but derived from:

- return-map symmetry,
- closure compatibility,
- and invariant integration over the hydrogenic cycle family.

The resulting inner product equips the representational span with a geometry in which orthogonality reflects incompatibility of closure classes.

It is important to emphasize that this structure arises prior to any notion of probability or measurement. It provides a purely geometric and coherence-based framework that will serve as the foundation for subsequent developments.

## 9 Pre-Hilbert Realization

In the previous section, we established that the invariant coherence functional induces a positive-definite Hermitian form on the representational span of the hydrogenic stationary modes. We now formalize this structure.

## 9.1 Representational Space

**Definition 9.1** (Hydrogenic representational space). *Let*

$$\mathcal{V}_H^{\text{fin}} := \text{span}_{\mathbb{C}}\{\Psi_n\},$$

where  $\{\Psi_n\}$  is the family of normalized hydrogenic stationary modes.

This space consists of finite complex linear combinations of stationary closure modes and serves as a comparison space for evaluating coherence relations.

## 9.2 Pre-Hilbert Structure

**Theorem 9.2** (Pre-Hilbert realization). *The space  $\mathcal{V}_H^{\text{fin}}$ , equipped with the inner product  $\langle \cdot, \cdot \rangle_H$ , is a complex pre-Hilbert space.*

*Proof.* From Section 8, the form  $\langle \cdot, \cdot \rangle_H$  is:

- conjugate symmetric,
- linear in the second argument and antilinear in the first,
- positive definite.

Thus,  $\langle \cdot, \cdot \rangle_H$  defines an inner product on  $\mathcal{V}_H^{\text{fin}}$ . Since  $\mathcal{V}_H^{\text{fin}}$  is not assumed to be complete with respect to the induced norm, it follows that it is a pre-Hilbert space.  $\square$

## 9.3 Induced Norm

The inner product induces a norm on  $\mathcal{V}_H^{\text{fin}}$  given by

$$\|\Phi\|_H := \sqrt{\langle \Phi, \Phi \rangle_H}.$$

For  $\Phi = \sum_n a_n \Psi_n$ , this reduces to

$$\|\Phi\|_H^2 = \sum_n |a_n|^2.$$

## 9.4 Orthonormal Basis Structure

The normalized stationary modes  $\{\Psi_n\}$  form an orthonormal basis for  $\mathcal{V}_H^{\text{fin}}$ :

$$\langle \Psi_m, \Psi_n \rangle_H = \delta_{mn}.$$

Thus,  $\mathcal{V}_H^{\text{fin}}$  is isomorphic, as an inner product space, to a finite-dimensional complex Euclidean space.

## 9.5 Remark on Completion

The construction above does not assume completeness of  $\mathcal{V}_H^{\text{fin}}$ . One may consider its completion with respect to the induced norm to obtain a Hilbert space. However, for the purposes of the present work, the finite representational span is sufficient.

In particular, all subsequent structural results will be formulated on finite-dimensional subspaces of  $\mathcal{V}_H^{\text{fin}}$ , and no assumptions about infinite-dimensional completeness are required at this stage.

## 9.6 Interpretation

The pre-Hilbert structure obtained here is not postulated but arises directly from the holonomic coherence properties of the hydrogenic closure hierarchy. The inner product encodes compatibility of closure density and transport phase under the invariant return structure.

This result provides a purely geometric and coherence-based origin for a complex comparison space. It establishes the structural foundation required for the development of measurement and probability in subsequent work, without introducing probabilistic or statistical assumptions.

## 9.7 Transition to Subsequent Work

In the next stage of the program, we will consider how interaction-induced partitions of the closure-class structure give rise to a family of subspaces of  $\mathcal{V}_H^{\text{fin}}$ . These subspaces will be shown to define a projector algebra associated with admissible outcome structures.

This will allow the introduction of consistent weight assignments on the resulting event structure and will provide the framework for deriving probabilistic rules from purely structural considerations.

# 10 Interpretation

The results of this paper establish that the hydrogenic stationary sector of scalar-conformal NUVO systems admits a natural complex comparison structure with the properties of a pre-Hilbert space. It is important to clarify the precise meaning and scope of this result.

## 10.1 Separation from Quantum-Mechanical Postulates

Although the resulting structure resembles the mathematical framework of quantum theory, it has not been introduced through quantum-mechanical postulates. In particular:

- The complex representation  $\Psi = \sqrt{\rho} e^{i\phi/\Phi_0}$  is an encoding of closure density and transport phase, not a wavefunction postulate.
- The inner product  $\langle \cdot, \cdot \rangle_H$  is derived from an invariant coherence functional, not assumed a priori.
- Orthogonality arises from return-map symmetry and distinct eigencharacters, not from an imposed Hilbert-space structure.

Thus, the appearance of a complex inner-product space is a consequence of the underlying geometric and coherence properties of the hydrogenic closure hierarchy.

## 10.2 Absence of Probabilistic Interpretation

No probabilistic interpretation has been introduced in this work. In particular:

- The quantity  $|\Psi|^2 = \rho$  represents closure density, not probability density.
- The inner product does not encode measurement likelihoods.
- No statistical ensembles or measurement postulates have been assumed.

The structure obtained here is therefore prior to any probabilistic framework. It provides a geometry of comparison between closure-compatible configurations, not a theory of measurement outcomes.

### 10.3 Geometric Meaning of Orthogonality

Orthogonality in the induced inner product has a direct geometric interpretation within the NUVO framework.

Two stationary modes  $\Psi_m$  and  $\Psi_n$  are orthogonal if and only if their mixed coherence vanishes:

$$\langle \Psi_m, \Psi_n \rangle_H = 0.$$

This vanishing arises from incompatibility of their return-map eigencharacters. Distinct closure classes correspond to distinct holonomic phase structures, and their mixed coherence cancels under the invariant return symmetry.

In this sense, orthogonality expresses a fundamental geometric incompatibility between different closure configurations, rather than a probabilistic exclusivity.

### 10.4 Representational vs. Physical Structure

The space  $\mathcal{V}_H^{\text{fin}}$  is a representational construction. While it admits linear combinations of stationary modes, we do not assume that arbitrary elements of this space correspond to physically realized configurations.

Instead,  $\mathcal{V}_H^{\text{fin}}$  should be understood as a comparison space in which coherence relations can be evaluated and structural properties can be expressed. The distinction between representational structure and physical admissibility will remain important in subsequent developments.

### 10.5 Summary

The central conclusion of this paper is that a complex pre-Hilbert structure emerges naturally from the holonomic coherence properties of the hydrogenic closure hierarchy. This structure is:

- derived from return-map symmetry,
- independent of probabilistic assumptions,
- and grounded in closure density and transport phase.

It provides the geometric foundation upon which further structural and interpretational developments can be built.

## 11 Outlook

The construction of a pre-Hilbert structure on the hydrogenic stationary sector provides a foundational step toward a broader program: the derivation of measurement and probabilistic structure from underlying geometric and coherence principles.

### 11.1 From Closure Partitions to Projectors

The hydrogenic closure hierarchy is naturally organized into discrete classes. In the next stage of development, we will consider how interaction-induced constraints partition this closure-class structure into mutually exclusive outcome sets.

These partitions will be shown to induce subspaces of  $\mathcal{V}_H^{\text{fin}}$ , and hence a family of orthogonal projectors. The resulting projector structure will provide a representation of admissible outcome alternatives within the comparison space.

## 11.2 Event Structure and Context Independence

A key requirement for the development of a consistent measurement framework is that the identification of outcomes be independent of the specific interaction context. In subsequent work, we will show that the projector structure induced by closure partitions satisfies this context-independence property within the hydrogenic spectral sector.

This will allow the formulation of an intrinsic event algebra associated with the system.

## 11.3 Toward Probabilistic Structure

Once a context-independent projector algebra has been established, one may consider assigning weights to these projectors in a manner consistent with the structural properties of the system.

We will show that any such assignment satisfying natural consistency conditions is strongly constrained. In particular, in sufficiently rich finite-dimensional sectors, these constraints lead to a unique form of weight assignment characterized by an inner-product expression.

## 11.4 Connection to the Born Rule

The development outlined above provides a route by which Born-type weighting emerges not as a postulate, but as a consequence of the structural properties of the comparison space and its associated event algebra.

In this program:

- the pre-Hilbert structure arises from holonomic coherence,
- the projector algebra arises from closure partitions,
- and the weighting rule is determined by consistency requirements on this algebra.

This establishes a conceptual framework in which probabilistic structure is inherited from geometric and coherence principles rather than introduced independently.

## 11.5 Scope and Limitations

The present work is restricted to the hydrogenic stationary sector and to finite-dimensional representational subspaces. Extension to more general systems, including dynamical regimes and multi-particle configurations, will require additional analysis.

Nevertheless, the hydrogenic case provides a minimal setting in which the essential structural features can be identified and developed in a controlled manner.

## 11.6 Final Remarks

The results of this paper suggest that key elements of the mathematical structure commonly associated with quantum theory may arise from deeper geometric and coherence properties of scalar-conformal systems.

The subsequent development of projector structure and weight assignments will determine to what extent this correspondence can be made precise and complete.

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