QB5 – Consistency of Weight Assignments and Emergence of Quadratic Structure in Scalar–Conformal NUVO Systems

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Rickey W. Austin
St Claire Scientific Research, Development, and Publishing

Abstract

We determine the form of admissible weight assignments on the projector-based event structure derived in QB4. Weight assignments are defined as maps on the projector algebra of the hydrogenic representational space and are constrained by normalization, additivity over orthogonal decompositions, and context independence.

We show that these conditions imply the existence of a positive frame function on the unit sphere of the representational space. Applying a Gleason-type theorem, we obtain a unique trace representation of the form

$$\mu(P) = \mathrm{Tr}(\rho P),$$

for a positive operator ρ with unit trace.

In the pure-state case, this reduces to the quadratic form

$$\mu(P_\Phi) = |\langle \Phi, \Psi \rangle|^2,$$

which is formally identical to the Born rule. Within the NUVO framework, this expression is interpreted as a squared coherence overlap derived from the invariant inner product established in QB3.

No probabilistic postulates are assumed. The quadratic weighting rule emerges as a consequence of structural consistency applied to a closure-based event algebra, providing a geometric origin for Born-type structure within scalar–conformal NUVO systems.

1 Introduction

In the preceding papers, we developed a structural framework for the hydrogenic sector of scalar–conformal NUVO systems based on closure and holonomic coherence principles. In Paper 1 [1], we showed that the stationary closure hierarchy admits a natural complex pre-Hilbert structure derived from an invariant coherence functional. In Paper 2 [2], we demonstrated that interaction-induced constraints partition the closure-class structure and induce a corresponding projector algebra on the representational space.

The present work addresses the next fundamental question: given an event structure represented by projectors, what form can a consistent assignment of weights to these events take?

[*] Bibliography is provisional. Cross-references to companion NUVO-series papers (M-, SR-, Q-, QB-, QM-series) will be updated with Zenodo DOIs in subsequent versions.

In standard formulations of quantum theory, such weights are interpreted as probabilities and are given by the Born rule [3]. In this work, however, we do not assume any probabilistic interpretation a priori. Instead, we consider weight assignments purely as maps on the projector algebra satisfying structural consistency conditions derived from the event framework established in Paper 2.

The central objective of this paper is to show that these consistency conditions strongly constrain the form of admissible weight assignments. In particular, in sufficiently rich finite-dimensional sectors, we will show that any assignment satisfying normalization, additivity over orthogonal decompositions, and context independence must take a trace form with respect to a positive operator.

This result follows from a Gleason-type theorem [4] applied to the projector algebra of the representational space. Rather than re-deriving Gleason's theorem, we will identify the conditions under which it applies in the present framework and interpret its consequences in terms of the structures developed in the previous papers.

The key point is that the emergence of the trace form is not introduced as a postulate, but arises as a consequence of the internal consistency of the event structure. In this sense, the weighting rule is inherited from the geometry of the representational space and its associated projector algebra.

Only after establishing this structural result will we relate the trace form back to the underlying NUVO quantities. In particular, we will show that, when expressed in terms of the coherence functional introduced in Paper 1, the trace expression reduces to a quadratic form involving inner products of represented states. This provides the bridge to the familiar Born-type expression.

It is important to emphasize the scope of this work. We do not assume that weight assignments correspond to empirical probabilities, nor do we introduce measurement postulates. The analysis is purely structural: it determines the form of admissible weight functions on the event algebra derived from closure and coherence principles.

The organization of the paper is as follows. In Section 2, we recall the event structure and projector algebra constructed in Paper 2. In Section 3, we define weight assignments on this algebra. Section 4 introduces the structural consistency conditions required of such assignments. In Section 5, we analyze the consequences of these conditions. In Section 6, we apply a Gleason-type theorem to obtain the trace representation. Section 7 specializes this result to rank-one projectors, yielding a quadratic form. In Section 8, we interpret this result in terms of holonomic coherence. Sections 9 and 10 discuss dimensional requirements and interpretation, and Section 11 outlines further directions.

2 Event Structure from Closure and Coherence

We summarize the structural framework established in the preceding papers that will serve as the foundation for the present analysis. The emphasis here is on the representational space and its associated projector algebra, which together define the domain of admissible weight assignments.

2.1 Representational Space

In Paper 1, it was shown that the hydrogenic stationary sector admits a complex pre-Hilbert structure. The corresponding representational space is given by

$$\mathcal{V}_H^{\text{fin}} := \operatorname{span}_{\mathbb{C}}\{\Psi_n\},$$

where $\{\Psi_n\}$ is the family of stationary closure modes.

This space is equipped with an inner product

$$\langle \Phi, \Xi \rangle_H = \int_{\Gamma_H} \Phi^* \Xi \, d\mu_H,$$

derived from an invariant coherence functional. The stationary modes form an orthonormal basis with respect to this inner product.

Origin in scalar-modulated closure structure. The discrete family of stationary modes $\{\Psi_n\}$ arises from the scalar-modulated return condition established in Q2 [5],

$$k \oint_\gamma \lambda_{\text{eff}}(x,u)\, ds = L_\gamma.$$

This condition determines the admissible closure-compatible configurations of the hydrogenic sector and thereby fixes the index set $\mathcal{I}_H$ underlying the representational construction.

2.2 Projector Algebra

In Paper 2, we showed that interaction-induced closure constraints give rise to a family of orthogonal projectors on $\mathcal{V}_H^{\text{fin}}$. These projectors were initially constructed from closure-class partitions and subsequently extended to the full projector algebra associated with the inner product structure.

We denote by

$$\mathsf{P}(\mathcal{V}_H^{\text{fin}})$$

the set of all orthogonal projectors on $\mathcal{V}_H^{\text{fin}}$.

Each projector $P \in \mathsf{P}(\mathcal{V}_H^{\text{fin}})$ is a linear operator satisfying

$$P^2 = P, \qquad P^\dagger = P.$$

2.3 Event Interpretation

The elements of $\mathsf{P}(\mathcal{V}_H^{\text{fin}})$ are interpreted as events. That is, each projector corresponds to a subspace of the representational space and represents a structurally defined outcome class.

Orthogonality of projectors encodes exclusivity:

$$PQ = 0 \quad \Longleftrightarrow \quad \text{events } P \text{ and } Q \text{ are exclusive.}$$

2.4 Orthogonal Decompositions

A collection of projectors $\{P_\alpha\}$ is said to form an orthogonal decomposition of the identity if

$$\sum_\alpha P_\alpha = I, \qquad P_\alpha P_\beta = 0 \quad (\alpha \neq \beta).$$

Such decompositions represent complete resolutions of the system into mutually exclusive outcome classes.

2.5 Context Independence

A crucial structural property established in Paper 2 is that the projectors are context independent. That is, if two interaction configurations induce the same projector, then the corresponding event is the same, independent of the details of the interaction.

Formally, this implies that any assignment defined on $\mathsf{P}(\mathcal{V}_H^{\text{fin}})$ must depend only on the projector itself and not on the particular decomposition in which it appears.

2.6 Finite-Dimensional Structure

The representational space $\mathcal{V}_H^{\text{fin}}$ is finite dimensional in the present analysis. This ensures that:

- every subspace admits an orthonormal basis,
- every projector can be expressed as a finite sum of rank-one projectors,
- and the full projector algebra is well-defined.

This finite-dimensional structure will be essential for the application of a Gleason-type theorem in later sections.

2.7 Summary

The hydrogenic sector thus provides:

- a complex inner product space $\mathcal{V}_H^{\text{fin}}$,
- a complete projector algebra $\mathsf{P}(\mathcal{V}_H^{\text{fin}})$,
- an interpretation of projectors as events,
- and a context-independent event structure.

These ingredients define the domain on which weight assignments will be introduced and constrained in the following sections.

3 Weight Assignments

Having established the event structure associated with the projector algebra $\mathsf{P}(\mathcal{V}_H^{\text{fin}})$, we now introduce the notion of weight assignments on this structure. These assignments will be subject to structural consistency conditions in the following section.

3.1 Definition of Weight Assignment

Definition 3.1 (Weight assignment)**.** *A weight assignment on the hydrogenic event structure is a map*

$$\mu : \mathsf{P}(\mathcal{V}_H^{\text{fin}}) \to [0, 1].$$

Thus, to each projector P, the assignment μ associates a real number between 0 and 1.

Domain of admissible weight assignments. While the inner product structure of $\mathcal{V}_H^{\text{fin}}$ supports the full projector algebra, not all projectors necessarily correspond to physically realizable events arising from closure partitions.

Accordingly, weight assignments are understood to be defined on the class of projectors that are admissible under the closure-based event structure and its consistent extensions. The extension of μ to the full projector algebra is justified by the representational completeness of the inner product structure, but the physical interpretation of such extensions remains constrained by closure compatibility.

3.2 Interpretational Scope

At this stage, the map μ is not assumed to represent a probability measure. It is introduced as a general assignment of weights to events, without specifying any interpretation in terms of measurement or statistical frequency.

The purpose of introducing μ is to determine what forms such assignments can take when subject to the structural constraints of the event algebra.

3.3 Dependence on Projectors

The domain of μ is the set of projectors itself, rather than any particular decomposition of the identity. In particular, the value $\mu(P)$ is associated directly with the event represented by P.

This reflects the context-independence property established in Section 2: the assignment must depend only on the projector and not on the manner in which it arises within a given decomposition.

3.4 Elementary and Composite Events

The definition of μ applies uniformly to all projectors:

- For a rank-one projector $P_\Phi = |\Phi\rangle \langle\Phi|$, the value $\mu(P_\Phi)$ assigns a weight to the elementary event associated with the one-dimensional subspace spanned by Φ.
- For higher-rank projectors P, the value $\mu(P)$ assigns a weight to the composite event represented by the corresponding subspace.

No additional structure is assumed at this stage relating the values of μ on different projectors.

3.5 Normalization Condition

We impose a minimal normalization requirement:

Assumption 3.2 (Normalization)**.**

$$\mu(I) = 1,$$

where I is the identity operator on $\mathcal{V}_H^{\text{fin}}$.

This condition reflects the interpretation of I as the total event corresponding to the full representational space.

3.6 Remarks

- The definition of μ is deliberately minimal. No additivity or further constraints are imposed at this stage.
- The goal is to derive the form of admissible weight assignments from structural consistency conditions rather than to assume it.
- In the next section, we will introduce the additional conditions required for consistency with the event structure, including additivity over orthogonal decompositions.

4 Consistency Conditions

We now impose structural consistency conditions on weight assignments defined on the projector algebra $\mathsf{P}(\mathcal{V}_H^{\mathrm{fin}})$. These conditions are motivated by the event structure established in Section 2 and are required for internal coherence of the assignment.

4.1 Additivity over Orthogonal Decompositions

The event structure admits decompositions of the identity into mutually exclusive events:

$$\sum_\alpha P_\alpha = I, \qquad P_\alpha P_\beta = 0 \quad (\alpha \neq \beta).$$

Such decompositions represent complete resolutions of the system into distinct outcome classes.

Assumption 4.1 (Additivity)**.** *Let $\{P_\alpha\}$ be a finite family of mutually orthogonal projectors. Then*

$$\mu\left(\sum_\alpha P_\alpha\right) = \sum_\alpha \mu(P_\alpha).$$

In particular, for a decomposition of the identity,

$$\sum_\alpha \mu(P_\alpha) = \mu(I) = 1.$$

This condition expresses the requirement that the weight assigned to a composite event equals the sum of the weights assigned to its mutually exclusive components.

4.2 Noncontextuality

As established in Paper 2, projectors represent intrinsic structural events that are independent of the particular interaction context used to realize them.

Assumption 4.2 (Noncontextuality)**.** *The value $\mu(P)$ depends only on the projector P and is independent of any decomposition in which P appears.*

Thus, if a projector P appears in multiple decompositions of the identity,

$$P + \sum_i Q_i = I \quad \text{and} \quad P + \sum_j R_j = I,$$

then the value $\mu(P)$ is the same in both contexts.

This condition is essential: without noncontextuality, the assignment $f(\Phi)$ could depend on the choice of orthonormal frame in which Φ is embedded, and the frame function would not be well-defined. Noncontextuality therefore ensures that the weight assigned to a one-dimensional subspace is intrinsic to that subspace and not an artifact of a particular decomposition.

This condition follows from the context-independence of projectors established in QB4. Since projectors are determined uniquely by the closure-class partitions they represent, their identification is independent of the specific interaction configuration used to realize them. Accordingly, any assignment defined on projectors must inherit this context independence.

4.3 Finite Additivity and Frame Consistency

The additivity condition implies that for any orthonormal basis $\{\Phi_i\}$ of $\mathcal{V}_H^{\text{fin}}$, the associated rank-one projectors

$$P_i = |\Phi_i\rangle \langle\Phi_i|$$

satisfy

$$\sum_i \mu(P_i) = 1.$$

Such sets of rank-one projectors will be referred to as *frames*.

Definition 4.3 (Frame function). *A function f defined on unit vectors by*

$$f(\Phi) := \mu(|\Phi\rangle \langle\Phi|)$$

is called a frame function if for every orthonormal basis $\{\Phi_i\}$,

$$\sum_i f(\Phi_i) = 1.$$

Thus, the weight assignment μ induces a frame function on the unit sphere of $\mathcal{V}_H^{\text{fin}}$.

4.4 Positivity

Since $\mu(P) \in [0, 1]$ by definition, the induced frame function satisfies

$$f(\Phi) \geq 0 \quad \text{for all unit vectors } \Phi.$$

4.5 Summary of Conditions

We have established that any admissible weight assignment μ must satisfy:

- normalization: $\mu(I) = 1$,
- additivity over orthogonal projectors,
- noncontextuality,
- and positivity.

These conditions imply that μ defines a positive frame function on the unit sphere of $\mathcal{V}_H^{\text{fin}}$.

4.6 Role in Subsequent Analysis

The conditions above place strong constraints on the possible forms of μ. In particular, in finite-dimensional spaces of dimension at least three, it is known that such frame functions admit a specific representation in terms of the inner product structure.

In the next section, we analyze the structural consequences of these conditions and prepare the ground for the application of a Gleason-type theorem.

5 Structural Consequences

The consistency conditions introduced in the previous section imply strong structural restrictions on admissible weight assignments. In this section, we record the consequences that are needed for the Gleason-type representation theorem.

5.1 Reduction to Rank-One Projectors

Because $\mathcal{V}_H^{\text{fin}}$ is finite dimensional, every projector can be decomposed into a finite sum of mutually orthogonal rank-one projectors.

Lemma 5.1 (Reduction to rank-one projectors). *Let $P \in \mathsf{P}(\mathcal{V}_H^{\text{fin}})$ be a projector onto a subspace $\mathcal{H} \subseteq \mathcal{V}_H^{\text{fin}}$. Let $\{\Phi_i\}_{i=1}^r$ be an orthonormal basis of $\mathcal{H}$. Then*

$$P = \sum_{i=1}^{r} |\Phi_i\rangle \langle\Phi_i| \,,$$

and

$$\mu(P) = \sum_{i=1}^{r} \mu(|\Phi_i\rangle \langle\Phi_i|).$$

Proof. Since $\{\Phi_i\}_{i=1}^r$ is an orthonormal basis of $\mathcal{H}$, the orthogonal projector onto $\mathcal{H}$ is

$$P = \sum_{i=1}^{r} |\Phi_i\rangle \langle\Phi_i| \,.$$

The rank-one projectors $|\Phi_i\rangle \langle\Phi_i|$ are mutually orthogonal, so additivity yields

$$\mu(P) = \mu\left(\sum_{i=1}^{r} |\Phi_i\rangle \langle\Phi_i|\right) = \sum_{i=1}^{r} \mu(|\Phi_i\rangle \langle\Phi_i|).$$

□

This shows that the full weight assignment is determined by its values on rank-one projectors.

5.2 Induced Frame Function

Let Φ be a unit vector in $\mathcal{V}_H^{\text{fin}}$, and define

$$f(\Phi) := \mu(|\Phi\rangle \langle\Phi|).$$

By construction, f is nonnegative and depends only on the one-dimensional subspace spanned by Φ.

Lemma 5.2 (Frame property). *Let $\{\Phi_i\}_{i=1}^N$ be an orthonormal basis of $\mathcal{V}_H^{\text{fin}}$. Then*

$$\sum_{i=1}^{N} f(\Phi_i) = 1.$$

Proof. For an orthonormal basis $\{\Phi_i\}_{i=1}^N$, the associated rank-one projectors satisfy

$$\sum_{i=1}^{N} |\Phi_i\rangle \langle\Phi_i| = I.$$

Applying additivity and normalization,

$$\sum_{i=1}^{N} f(\Phi_i) = \sum_{i=1}^{N} \mu(|\Phi_i\rangle \langle\Phi_i|) = \mu(I) = 1.$$

□

Thus, f is a positive frame function of weight one on the unit sphere of $\mathcal{V}_H^{\text{fin}}$.

5.3 Basis Independence

The preceding lemma holds for every orthonormal basis. Therefore, the sum of the values of f over a complete orthonormal frame is independent of the basis chosen.

Corollary 5.3 (Basis independence)**.** *If $\{\Phi_i\}_{i=1}^N$ and $\{\Xi_j\}_{j=1}^N$ are orthonormal bases of $\mathcal{V}_H^{\text{fin}}$, then*

$$\sum_{i=1}^{N} f(\Phi_i) = \sum_{j=1}^{N} f(\Xi_j) = 1.$$

This basis independence is the essential frame consistency condition required for a Gleason-type representation.

5.4 Determination of Composite Events

The previous results imply that once the values of μ are known on rank-one projectors, the values on all higher-rank projectors are fixed.

Corollary 5.4 (Determination by rank-one data)**.** *The weight assignment μ is uniquely determined by the induced frame function f.*

Proof. Every projector P is a finite sum of mutually orthogonal rank-one projectors, and by the first lemma its weight is the sum of the corresponding frame-function values. Hence knowledge of f determines μ on all projectors. □

5.5 Dimensional Requirement

The representation theorem to be used in the next section requires that the underlying inner product space have dimension at least three. We therefore record this requirement explicitly.

Assumption 5.5 (Dimensional richness)**.** *Throughout the remainder of this paper, we assume*

$$\dim \mathcal{V}_H^{\text{fin}} \geq 3.$$

This condition is the minimal dimensional hypothesis under which the relevant Gleason-type result applies.

5.6 Summary

The consistency conditions of Section 4 imply that:

- the weight assignment μ is determined entirely by its values on rank-one projectors,
- these values define a positive frame function of weight one,
- and the resulting frame function is basis independent.

These consequences place the weight assignment precisely in the class governed by Gleason-type theorems. In the next section, we use this fact to obtain the trace representation of μ.

5.7 Reduction to Gleason Conditions

The preceding results establish that the induced function

$$f(\Phi) := \mu(|\Phi\rangle\langle\Phi|)$$

satisfies the defining properties of a positive frame function of weight one on the unit sphere of $\mathcal{V}_H^{\text{fin}}$.

In particular:

- $f(\Phi) \geq 0$ for all unit vectors Φ,
- for every orthonormal basis $\{\Phi_i\}$,
$$\sum_i f(\Phi_i) = 1,$$
- and f is defined on all one-dimensional subspaces via noncontextuality.

Together with the dimensional assumption

$$\dim \mathcal{V}_H^{\text{fin}} \geq 3,$$

these conditions place f precisely within the class of functions governed by Gleason-type representation theorems.

6 Gleason-Type Representation

In the previous section, we showed that the consistency conditions imposed on the weight assignment μ imply the existence of a positive frame function

$$f(\Phi) := \mu(|\Phi\rangle\langle\Phi|)$$

of weight one on the unit sphere of $\mathcal{V}_H^{\text{fin}}$.

In this section, we apply a Gleason-type theorem to obtain the explicit form of μ.

The structural conditions established in the preceding sections place the weight assignment precisely within the class of functions governed by Gleason-type representation theorems. The role of the theorem in this context is therefore not to introduce new structure, but to identify the unique form compatible with the already established consistency conditions.

6.1 Statement of the Representation Theorem

We require the following standard result.

Theorem 6.1 (Gleason-type representation [4]). *Let $\mathcal{H}$ be a real or complex inner product space with*

$$\dim \mathcal{H} \geq 3.$$

Let f be a nonnegative function on the unit sphere such that for every orthonormal basis $\{\Phi_i\}$ of $\mathcal{H}$,

$$\sum_i f(\Phi_i) = 1.$$

Then there exists a unique positive semidefinite, trace-class operator ρ on $\mathcal{H}$ such that

$$f(\Phi) = \langle \Phi | \, \rho \, | \Phi \rangle \quad \textit{for all unit } \Phi,$$

and

$$\mathrm{Tr}(\rho) = 1.$$

This theorem applies directly to the frame function constructed in Section 5.

6.2 Application to $\mathcal{V}_H^{\text{fin}}$

Since $\mathcal{V}_H^{\text{fin}}$ is finite dimensional and satisfies

$$\dim \mathcal{V}_H^{\text{fin}} \geq 3,$$

the Gleason-type theorem yields the existence of a positive semidefinite operator

$$\rho : \mathcal{V}_H^{\text{fin}} \to \mathcal{V}_H^{\text{fin}}$$

with

$$\mathrm{Tr}(\rho) = 1$$

such that for all unit vectors Φ,

$$\mu(|\Phi\rangle \langle \Phi|) = \langle \Phi | \, \rho \, | \Phi \rangle \, .$$

6.3 Extension to General Projectors

We now extend this result from rank-one projectors to arbitrary projectors.

Theorem 6.2 (Trace representation of μ). *For every projector $P \in \mathsf{P}(\mathcal{V}_H^{\text{fin}})$,*

$$\mu(P) = \mathrm{Tr}(\rho P).$$

Proof. Let P be a projector onto a subspace $\mathcal{H} \subseteq \mathcal{V}_H^{\text{fin}}$, and let $\{\Phi_i\}_{i=1}^r$ be an orthonormal basis of $\mathcal{H}$. Then

$$P = \sum_{i=1}^r |\Phi_i\rangle \langle \Phi_i| \, .$$

By additivity,

$$\mu(P) = \sum_{i=1}^r \mu(|\Phi_i\rangle \langle \Phi_i|).$$

Applying the Gleason representation,

$$\mu(P) = \sum_{i=1}^{r} \langle\Phi_i| \rho |\Phi_i\rangle .$$

Using the definition of the trace,

$$\sum_{i=1}^{r} \langle\Phi_i| \rho |\Phi_i\rangle = \mathrm{Tr}(\rho P).$$

This establishes the result. □

6.4 Uniqueness

Theorem 6.3 (Uniqueness of the representation)**.** *The operator ρ is uniquely determined by the weight assignment μ.*

Proof. If ρ_1 and ρ_2 both satisfy

$$\mu(|\Phi\rangle\langle\Phi|) = \langle\Phi| \rho_i |\Phi\rangle$$

for all unit Φ, then

$$\langle\Phi| (\rho_1 - \rho_2) |\Phi\rangle = 0 \quad \text{for all } \Phi.$$

This implies $\rho_1 - \rho_2 = 0$, and hence $\rho_1 = \rho_2$. □

6.5 Uniqueness and Structural Necessity

The preceding results show that the trace representation is not merely one admissible form of the weight assignment, but the only form compatible with the structural conditions imposed in Section 4.

In particular, any map

$$\mu : \mathsf{P}(\mathcal{V}_H^{\mathrm{fin}}) \to [0, 1]$$

satisfying normalization, additivity over orthogonal decompositions, and noncontextuality must take the form

$$\mu(P) = \mathrm{Tr}(\rho P),$$

for a unique positive operator ρ with unit trace.

No alternative functional form is compatible with these conditions within the class of assignments satisfying positivity, additivity, and noncontextuality in dimensions greater than or equal to three.

6.6 Interpretive Boundary

At this stage, the operator ρ arises purely as a mathematical representation of the consistency conditions imposed on μ.

No probabilistic interpretation is assumed. In particular:

- μ is a structural weight assignment on projectors,
- ρ is the unique operator encoding this assignment,
- and the trace formula is a representation theorem, not a postulate.

6.7 Summary

We have shown that any admissible weight assignment μ satisfying the structural conditions of Section 4 must take the form

$$\mu(P) = \mathrm{Tr}(\rho P),$$

for a unique positive semidefinite operator ρ with unit trace.

This result completes the structural characterization of μ. In the next section, we interpret this representation within the transport closure framework and establish the emergence of the Born rule [3].

7 Pure-State Form

In the previous section, we showed that any admissible weight assignment μ admits a trace representation

$$\mu(P) = \mathrm{Tr}(\rho P),$$

where ρ is a positive semidefinite operator with unit trace.

We now consider the special case in which ρ is a rank-one projector.

7.1 Pure-State Case

Definition 7.1 (Pure state)**.** *A weight assignment is said to be in a pure-state form if the operator ρ takes the form*

$$\rho = |\Psi\rangle \langle\Psi| \,,$$

for some normalized vector $\Psi \in \mathcal{V}_H^{\mathrm{fin}}$.

Corollary 7.2 (Pure-state representation)**.** *Let $\rho = |\Psi\rangle \langle\Psi|$ with $\|\Psi\| = 1$. Then for any rank-one projector*

$$P_\Phi = |\Phi\rangle \langle\Phi| \,,$$

we have

$$\mu(P_\Phi) = |\langle\Phi, \Psi\rangle|^2.$$

Proof. Using the trace representation,

$$\mu(P_\Phi) = \mathrm{Tr}(\rho P_\Phi) = \mathrm{Tr}(|\Psi\rangle \langle\Psi|\Phi\rangle \langle\Phi|).$$

Using cyclicity of the trace,

$$\mathrm{Tr}(|\Psi\rangle \langle\Psi|\Phi\rangle \langle\Phi|) = \mathrm{Tr}(|\Phi\rangle \langle\Phi|\Psi\rangle \langle\Psi|).$$

Evaluating,

$$|\Phi\rangle \langle\Phi|\Psi\rangle = \langle\Phi, \Psi\rangle \,|\Phi\rangle \,,$$

so

$$\mathrm{Tr}(|\Phi\rangle \langle\Phi|\Psi\rangle \langle\Psi|) = |\langle\Phi, \Psi\rangle|^2.$$

□

This result is therefore not an additional assumption, but the unique quadratic form compatible with the event structure and consistency conditions established in the preceding sections.

7.2 Extension to Higher-Rank Projectors

For a general projector P with orthonormal basis $\{\Phi_i\}$,

$$P = \sum_i |\Phi_i\rangle \langle\Phi_i| \,,$$

we obtain

$$\mu(P) = \sum_i |\langle\Phi_i, \Psi\rangle|^2.$$

Thus, the weight assigned to a composite event is the sum of squared inner products over an orthonormal basis of the corresponding subspace.

7.3 Interpretive Boundary

The expression

$$\mu(P_\Phi) = |\langle\Phi, \Psi\rangle|^2$$

is formally identical to the Born rule. However, in the present framework:

- it arises as a consequence of the structural consistency conditions imposed on μ,
- it is derived from the inner product induced by holonomic coherence,
- and it is not introduced as a probabilistic postulate.

7.4 Summary

In the pure-state case, the general trace representation reduces to a quadratic form in the inner product:

$$\mu(P_\Phi) = |\langle\Phi, \Psi\rangle|^2.$$

This provides the precise mathematical form of the weight assignment associated with a single represented state and prepares the ground for interpretation in terms of coherence structure.

8 Interpretation in Terms of Holonomic Coherence

In the previous sections, we established that any admissible weight assignment on the hydrogenic event structure must take the trace form

$$\mu(P) = \mathrm{Tr}(\rho P),$$

and that, in the pure-state case,

$$\mu(P_\Phi) = |\langle\Phi, \Psi\rangle|^2.$$

We now interpret this result within the NUVO framework in terms of closure density and holonomic coherence.

8.1 Inner Product as Coherence Functional

In Paper 1, the inner product on $\mathcal{V}_H^{\mathrm{fin}}$ was derived from an invariant coherence functional of the form

$$\langle \Phi, \Xi \rangle_H = \int_{\Gamma_H} \Phi^* \Xi \, d\mu_H,$$

where Γ_H denotes the hydrogenic cycle family and $d\mu_H$ is invariant under the return map.

This inner product measures the compatibility of closure density and transport phase between two represented configurations. In particular, the magnitude

$$|\langle \Phi, \Psi \rangle_H|$$

quantifies the degree of holonomic coherence between the configurations represented by Φ and Ψ.

8.2 Quadratic Form and Coherence Weighting

In the pure-state case, the weight assignment takes the form

$$\mu(P_\Phi) = |\langle \Phi, \Psi \rangle_H|^2.$$

This expression has a direct interpretation:

- $\langle \Phi, \Psi \rangle_H$ measures the coherence overlap between the two configurations,
- the squared magnitude $|\langle \Phi, \Psi \rangle_H|^2$ represents a quadratic coherence weighting,
- and the normalization condition ensures that the total weight over a complete orthonormal frame is unity.

Thus, the weight assigned to an event is determined by the squared coherence overlap between the represented state and the subspace defining the event.

8.3 Closure Density and Represented States

The representation

$$\Psi = \sqrt{\rho}\, e^{i\phi/\Phi_0}$$

encodes closure density ρ and transport phase ϕ.

The quadratic form

$$|\langle \Phi, \Psi \rangle_H|^2$$

therefore depends on both:

- the overlap of closure densities,
- and the relative transport phase structure.

In this sense, the weighting rule reflects a combined measure of structural compatibility in both density and phase.

8.4 Relation to Closure Partitions

In Paper 2, events were identified with projectors corresponding to subspaces of $\mathcal{V}_H^{\text{fin}}$ induced by closure partitions.

For a general projector P, the weight

$$\mu(P) = \mathrm{Tr}(\rho P)$$

can be expressed as a sum over an orthonormal basis $\{\Phi_i\}$ of the corresponding subspace:

$$\mu(P) = \sum_i |\langle \Phi_i, \Psi \rangle_H|^2.$$

This shows that the weight assigned to a composite event is the total coherence weight distributed across the corresponding closure-compatible configurations.

8.5 Interpretive Boundary

It is important to maintain the distinction between the present structural result and a full probabilistic interpretation.

- The quantity $\mu(P)$ has been derived as a consistency-constrained weight assignment.
- The quadratic form arises from the inner product induced by holonomic coherence.
- No assumption has been made that $\mu(P)$ represents an empirical probability.

The present result establishes that any admissible weighting consistent with the event structure must take a coherence-squared form. The interpretation of this quantity as probability, if made, must be justified separately.

8.6 Summary

The trace representation derived in the previous sections acquires a natural interpretation within the NUVO framework:

- the inner product encodes holonomic coherence,
- the quadratic form $|\langle \Phi, \Psi \rangle_H|^2$ represents coherence weighting,
- and the resulting structure provides a geometric origin for the Born-type expression.

Thus, the familiar quadratic weighting rule emerges as a direct consequence of closure density, transport phase, and the coherence structure they induce on the representational space.

9 Hydrogenic Sector and Dimensionality

The representation theorem applied in Section 6 requires that the underlying inner product space have dimension at least three. We now clarify how this requirement is satisfied within the hydrogenic sector.

9.1 Dimensional Structure of $\mathcal{V}_H^{\mathrm{fin}}$

The representational space

$$\mathcal{V}_H^{\mathrm{fin}} = \mathrm{span}_{\mathbb{C}}\{\Psi_n\}$$

is generated by the family of stationary closure modes indexed by $\mathcal{I}_H$.

Even when restricted to a finite subset of modes, the space admits dimension

$$\dim \mathcal{V}_H^{\mathrm{fin}} \geq 3,$$

provided at least three distinct closure classes are included.

9.2 Hydrogenic Spectral Richness

The hydrogenic closure hierarchy naturally provides an infinite sequence of distinct stationary modes. Any physically relevant finite truncation containing at least three such modes satisfies the dimensional requirement.

Thus, the Gleason-type theorem applies generically within the hydrogenic sector.

9.3 Minimal Sector for Representation

It is sufficient to consider any three-dimensional subspace

$$\mathcal{W} \subset \mathcal{V}_H^{\mathrm{fin}}$$

spanned by three orthonormal stationary modes. The restriction of μ to projectors on $\mathcal{W}$ already determines the trace representation within that subspace.

This local application is sufficient to establish the global structure.

9.4 Interpretation

The dimensional requirement reflects the need for a sufficiently rich event structure to constrain weight assignments uniquely. In lower-dimensional spaces, additional degrees of freedom in the assignment remain unconstrained.

The hydrogenic sector satisfies this requirement without introducing additional assumptions.

10 Interpretation

We now summarize the structural result obtained in this paper and clarify its conceptual meaning.

10.1 From Structure to Quadratic Form

Starting from the closure and coherence framework of NUVO:

- Paper 1 established a pre-Hilbert structure derived from holonomic coherence,
- Paper 2 constructed a projector-based event structure from closure partitions,
- the present work imposed consistency conditions on weight assignments over this structure.

These steps lead uniquely to a quadratic form

$$\mu(P_\Phi) = |\langle \Phi, \Psi \rangle_H|^2$$

in the pure-state case.

10.2 Absence of Probabilistic Postulates

At no stage in this derivation have probabilistic axioms been assumed. In particular:

- no probability measure was postulated,
- no Born rule was introduced,
- and no measurement axioms were invoked.

The quadratic form arises entirely from the internal consistency of the event structure and the underlying coherence geometry.

10.3 Geometric Origin of the Quadratic Rule

Within the NUVO framework:

- the inner product encodes holonomic coherence,
- orthogonality encodes closure incompatibility,
- and the projector algebra encodes outcome structure.

The quadratic dependence on the inner product is therefore a direct consequence of this geometric structure.

10.4 Relation to Standard Quantum Formalism

The resulting expression is formally identical to the Born rule of quantum theory. However, its origin here is fundamentally different:

- in standard formulations, the Born rule is postulated,
- in the present framework, it is derived from structural consistency.

This establishes a correspondence between the NUVO coherence framework and the mathematical structure of quantum theory, while maintaining a distinct conceptual foundation.

10.5 Interpretive Boundary

The present result determines the form of admissible weight assignments but does not identify these weights with empirical probabilities.

The interpretation of μ as probability, if adopted, requires additional physical assumptions beyond the scope of this work.

11 Outlook

The results of this paper complete a structural program connecting closure, coherence, and quadratic weighting within the hydrogenic sector of scalar–conformal NUVO systems. We now outline directions for further development.

11.1 Extension Beyond the Hydrogenic Sector

The present analysis has been restricted to the hydrogenic stationary sector. Extending the construction to:

- dynamical configurations,
- multi-particle systems,
- and non-stationary closure structures

will be necessary to establish the generality of the framework [6].

11.2 Connection to Measurement Theory

While the present work determines the form of admissible weight assignments, it does not specify how these weights are realized in physical measurement processes.

A complete theory will require:

- a dynamical model of interaction,
- a mechanism for outcome selection,
- and a connection between structural weights and empirical frequencies.

11.3 Relation to Exchange and Transport Structure

The coherence functional underlying the inner product arises from transport and closure properties of the system. A deeper connection between:

- exchange dynamics,
- transport coherence,
- and weight assignment

may provide additional insight into the physical origin of the quadratic rule.

11.4 Toward a Unified NUVO Framework

Together with Papers 1 and 2, the present work establishes:

- a geometric origin for inner-product structure,
- a closure-based event algebra,
- and a uniquely constrained quadratic weighting rule.

These results suggest that key elements of quantum structure may emerge from deeper geometric and coherence principles within the NUVO framework.

11.5 Final Remarks

The derivation presented here shows that the quadratic form commonly associated with the Born rule is not an independent postulate, but a consequence of consistency conditions applied to a closure-based event structure.

This provides a conceptually unified route from geometric coherence to quantum-like structure, forming a foundation for further development within the NUVO program.

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