QM1 — Normalization and the Complete Hilbert Space

in Scalar–Conformal NUVO Systems *Preprint, Version 1.0*[*]

Rickey W. Austin
St Claire Scientific Research, Development, and Publishing

Notation and Conventions

- $\mathcal{M}$ denotes the spacetime manifold.
- η denotes the reference Lorentzian metric (typically Minkowski in a global chart).
- g denotes the physical metric.
- The scalar field $\Lambda : \mathcal{M} \to \mathbb{R}_{>0}$ is the NUVO modulation field.
- The physical metric is scalar–conformal:

$$g_{\mu\nu} = \Lambda^2 \, \eta_{\mu\nu}.$$

- $\Lambda_0 > 0$ denotes the baseline scalar availability level supported by the intrinsic delivery structure of the underlying field. In the absence of localized structural occupation the scalar field satisfies $\Lambda(x) = \Lambda_0$.
- The dimensionless scalar diagnostic is

$$\lambda(x) \coloneqq \frac{\Lambda(x)}{\Lambda_0}.$$

- The scalar field represents the *locally available structural capacity* of the underlying delivery field. Localized structures may reduce this availability through occupation or transport, but the intrinsic delivery baseline Λ_0 remains fixed.
- Greek indices $\mu, \nu, \ldots$ range over spacetime coordinates $0, 1, 2, 3$.
- We use the Einstein summation convention unless explicitly stated otherwise.

Remark 0.1. *Unless otherwise stated, the background signature is* $(-, +, +, +)$.

[*]Bibliography is provisional. Cross-references to companion NUVO-series papers (M-, SR-, Q-, QB-, QM-series) will be updated with Zenodo DOIs in subsequent versions.

Program scope.

Abstract

The QB-series established a pre-Hilbert space $\mathcal{H}^{\mathrm{fin}}$ of stationary closure modes, equipped with a holonomic inner product, and demonstrated that observable outcomes and Born-rule frequencies emerge from coherence-gated transport dynamics in this finite-dimensional setting. The present paper completes the transition to a full, infinite-dimensional Hilbert space appropriate to the general scalar–conformal transport closure system.

We show that the continuity relation governing closure density implies a conservation law for the total integrated closure, and that this conservation law elevates normalization from a conventional choice to a structural constraint on admissible states. We then extend the pre-Hilbert framework of the QB-series to the separable L^2 Hilbert space $\mathcal{H}$, establishing completeness, the inner product, and the continuous linear structure required for the remainder of the QM-series.

The transport generators established in QB2 are promoted to self-adjoint operators on $\mathcal{H}$ with explicitly identified domains. Their spectra are shown to decompose into discrete and continuous parts, with generalized eigenstates introduced via a rigged Hilbert space triple to accommodate the continuous spectrum sector. Completeness relations and the resolution of the identity are derived as consequences of the spectral theorem for self-adjoint transport generators.

No probabilistic postulates, collapse assumptions, or new ontological commitments are introduced. The normalization condition, the inner product, and the spectral structure all arise as consequences of the transport closure geometry established in the prior series.

1 Introduction

1.1 Position Within the QM-Series

The scalar–conformal NUVO program has developed through a sequence of internally consistent sector papers, each building strictly on the results of its predecessors. The M-series fixed the foundational geometry: a spacetime manifold $\mathcal{M}$ equipped with a scalar capacity field Λ that modulates the reference Lorentzian metric through the conformal relation $g_{\mu\nu} = \Lambda^2 \eta_{\mu\nu}$, and established the delivery-substrate ontology, the loop taxonomy, and the program-wide variational structure. The Q-series then developed the exchange sector, deriving closure conditions, coherence constraints, quantization from holonomic return, the hydrogenic correspondence, and the emergence of a Schrödinger-type representation directly from the transport closure system. The QB-series completed the first stage of the quantum-mechanical development: QB1 established the complex state encoding of transport closure, QB2 derived the momentum and energy operators from transport generators together with the canonical commutation relation, QB3 constructed a holonomic inner product and showed that distinct closure classes are orthogonal, and QB4 through QB7 established the projector-valued observable structure, the Born frequency law, and the identification of measurement with coherence-gated interaction events—all within a finite-dimensional pre-Hilbert space of stationary closure modes.

The QB-series result is structurally complete within its stated scope. However, the finite-dimensional pre-Hilbert space $\mathcal{H}^{\mathrm{fin}}$ of QB3 is insufficient to support the full scope of quantum-mechanical structure that the QM-series must establish. Specifically, $\mathcal{H}^{\mathrm{fin}}$ does not accommodate: (a) non-stationary transport configurations, which are required for the time-dependent Schrödinger dynamics of QM4; (b) continuous-spectrum states, which arise in scattering and tunneling problems and which are treated in QM10; (c) tensor-product multi-particle state spaces, required for the configuration-space treatment of identical particles in QM7 and for the construction of entangled states in QM9; and (d) the full spectral theory of unbounded self-adjoint operators, which underlies the uncertainty relations of QM3 and the angular momentum structure of QM5. The QM-series begins by closing this gap.

The present paper, QM1, occupies the foundational position within the QM-series. Its principal results—the derivation of normalization as a structural constraint, the construction of the separable complex Hilbert space $\mathcal{H}$ of closure states, the promotion of the QB2 transport generators to essentially self-adjoint operators on $\mathcal{H}$, and the spectral theorem together with the resolution of the identity—are prerequisites for every subsequent paper in the series. Nothing in QM2 through QM11 is independent of the framework established here.

The immediate successor, QM2, exploits the complete Hilbert space and the associated superposition structure to derive the interference behavior of transport closure under two-path configurations, providing the scalar–conformal account of the double-slit experiment. That derivation requires the continuous-spectrum expansion established in Sec. 6 and Sec. 7 of the present paper, and could not be undertaken within the finite setting of the QB-series.

1.2 Objective of the Present Work

The central objective of the present paper is to establish the complete Hilbert-space framework for the scalar–conformal NUVO transport closure system. Specifically, the paper aims to establish five related claims.

1. The continuity relation governing closure density, recalled from the Q-series, implies that the total integrated closure is a conserved quantity under admissible transport evolution. This conservation law elevates the normalization condition $\int_{\mathbb{R}^3} |\Psi|^2 \, \mathrm{d}^3x = 1$ from a conventional choice to a structural constraint on admissible states.

2. The transport closure system admits a state space that is a separable complex Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C})$, equipped with an inner product inherited from the closure density integral. This space extends the finite-dimensional pre-Hilbert structure of QB3 in a manner that is consistent with—and strictly contains—the holonomic coherence functional constructed there.

3. The momentum and energy transport generators of QB2, $\hat{p}_j = -\mathrm{i}\Phi_0 \, \partial_j$ and $\hat{E} = \mathrm{i}\Phi_0 \, \partial_t$, extend to essentially self-adjoint operators on $\mathcal{H}$ with explicitly identified domains. Their essential self-adjointness ensures that each generator possesses a unique self-adjoint closure, so that no additional boundary conditions or extension choices are required.

4. The spectrum of each self-adjoint transport generator decomposes into a discrete part, corresponding to the normalizable holonomic closure modes of the prior series, and a continuous part. Generalized eigenstates for the continuous spectrum are introduced within a rigged Hilbert space triple and shown to satisfy distributional orthogonality and completeness relations.

5. The resolution of the identity holds for each transport generator: every closure state in $\mathcal{H}$ admits a spectral expansion over the discrete and continuous eigenstates, and the squared coefficients sum or integrate to unity. This Parseval identity is interpreted within the NUVO framework as total closure conservation expressed in the spectral coefficient representation.

These five results are not independent; they form a logically ordered sequence in which each depends on those that precede it. The derivations proceed entirely from the transport closure structure of the Q-series and the operator and inner-product structure of the QB-series, without introducing new postulates, new physical fields, or new ontological commitments.

1.3 What Is Not Assumed

The present work maintains the interpretive discipline established in the Q-series and continued throughout the QB-series. The following exclusions apply without exception.

No probabilistic postulate is introduced. The normalization condition $\int |\Psi|^2 \, \mathrm{d}^3x = 1$ is derived from the closure density continuity relation and the choice of closure units. The identification of $|\Psi|^2$ with a position-probability density follows from the Born frequency law established in QB6 and is not assumed here. The two results are numerically consistent but logically independent within the NUVO framework.

No collapse mechanism is introduced or implied. The extension of the state space from $\mathcal{H}^{\mathrm{fin}}$ to the full Hilbert space $\mathcal{H}$ is a representational extension. It does not alter the deterministic character of the transport evolution and does not introduce any discontinuous state-change rule.

No new ontology is introduced. The Hilbert space $\mathcal{H}$ is a mathematical representation of the transport closure states; it is not a physical arena, a medium, or a substrate. Its introduction is entirely analogous to the introduction of the complex encoding Ψ in QB1: both are representational objects that encode transport structure compactly without altering the underlying geometry.

The rigged Hilbert space triple, introduced in Sec. 6 to accommodate generalized eigenstates of the continuous spectrum, is a mathematical device that extends the representational framework to handle distributional completeness relations. It carries no additional physical content, and its elements outside $\mathcal{H}$ do not represent admissible physical states.

1.4 Structure of the Paper

Sec. 2 recalls the transport closure structure, the complex state encoding, the holonomic inner product, and the operator results from the Q- and QB-series that are needed for the present development, and identifies precisely the structural gap that the present paper closes. Sec. 3 derives the total-closure conservation law and establishes normalization as the structural constraint it implies for the complex state encoding. Sec. 4 constructs the L^2 Hilbert space of closure states, verifies completeness and separability, and establishes consistency with the QB3 holonomic inner product. Sec. 5 promotes the QB2 transport generators to essentially self-adjoint operators on $\mathcal{H}$ and promotes the canonical commutation relation to the complete setting. Sec. 6 introduces the spectral decomposition of self-adjoint operators, the rigged Hilbert space triple, and the generalized eigenstates of the continuous-spectrum sector. Sec. 7 derives the resolution of the identity and the Parseval identity, and records their NUVO interpretation. Sec. 8 collects interpretive clarifications and records the scope of the present construction. Sec. 9 summarizes the results, records their programmatic significance, and prepares the transition to QM2.

2 Recalled Structure from the Q- and QB-Series

2.1 Transport Closure and the Continuity Relation

The Q-series established that the exchange sector of the scalar–conformal NUVO framework admits a local description in terms of two scalar quantities: a closure density $\rho : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ and a transport-derived phase $\phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$. The closure density measures the local distribution of admissible closure configurations and is defined as a geometric quantity representing closure content. The phase arises from the cumulative exchange interaction experienced along admissible transport paths and encodes transport consistency. Neither quantity is introduced as a probabilistic or wave-theoretic object.

The evolution of these quantities is governed by a deterministic coupled system. The closure density satisfies the continuity relation

$$\partial_t \rho + \nabla \cdot (\rho\, v) = 0, \tag{1}$$

where $v : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ is the transport velocity field. The phase satisfies the transport-consistent evolution

$$\partial_t \phi + v \cdot \nabla \phi = \mathcal{E}(x,t), \tag{2}$$

where $\mathcal{E}(x,t)$ is a scalar function determined by the exchange interaction along the transport trajectory. The velocity field v is not an independent degree of freedom but is determined through the transport structure by the local phase gradient and the scalar geometry; accordingly the system closes as a deterministic coupled evolution for (ρ, ϕ).

These relations are recalled from the unified transport law established in the Q-series and are not re-derived here.

Remark 2.1. *The key property of Eq.* (1) *used throughout the present paper is that ρ satisfies a divergence-form conservation law with no source term. This structure is independent of the specific form of v and holds for all admissible transport configurations. It is this property, and not any probabilistic interpretation of ρ, that gives rise to the normalization constraint derived in Sec. 3.*

2.2 The Complex State Encoding

QB1 established that the pair (ρ, ϕ) constitutes the minimal local state description of the exchange-sector transport system in the integrable regime, and that this pair admits a lossless complex encoding. Specifically, the complex-valued function

$$\Psi(x,t) \coloneqq \sqrt{\rho(x,t)}\, \mathrm{e}^{\,\mathrm{i}\phi(x,t)/\Phi_0} \tag{3}$$

encodes both ρ and ϕ without loss of information: the closure density is recovered as $\rho = |\Psi|^2$ and the phase as $\phi = \Phi_0 \arg \Psi$.

The constant Φ_0 carries the dimensions of action and was identified with $\hbar$ through the hydrogenic correspondence established in the Q-series. It enters the encoding as a unit-fixing parameter that translates between the geometric phase accumulation of the transport system and the conventional phase normalization of the quantum-mechanical formalism.

The function Ψ is a representational object. It encodes transport closure structure and carries no independent ontological status as a physical wave or oscillatory medium. The complex form of the encoding is a consequence of the two-component structure of the minimal local state (ρ, ϕ); it is not introduced as a primitive. These interpretive constraints are inherited from QB1 and remain in force throughout the present paper.

2.3 The Finite-Dimensional Pre-Hilbert Space

QB3 constructed a holonomic coherence functional over the finite family of stationary closure modes of the exchange sector. Starting from the invariant return structure of exchange-cycle dynamics, QB3 identified a family of stationary closure modes indexed by holonomy class and showed that their coherence relations induce a natural complex inner product on the finite representational span $\mathcal{H}^{\mathrm{fin}}$.

Specifically, for any two elements $\Psi_1, \Psi_2 \in \mathcal{H}^{\mathrm{fin}}$, the holonomic coherence functional takes the form

$$\langle \Psi_1, \Psi_2 \rangle_{\mathcal{H}^{\mathrm{fin}}} = \int_{\mathcal{M}} \overline{\Psi_1(x)}\, \Psi_2(x)\, \mathrm{d}^3x. \tag{4}$$

Distinct closure classes, corresponding to distinct holonomy eigenvalues, are orthogonal under this functional. QB3 verified that Eq. (4) satisfies the axioms of a complex inner product on $\mathcal{H}^{\text{fin}}$ and thereby established a pre-Hilbert structure on the finite representational span of stationary configurations.

The pre-Hilbert space $\mathcal{H}^{\text{fin}}$ is finite-dimensional by construction: it spans only the discrete family of holonomically closed stationary modes. This is entirely appropriate to the setting of QB3, which was concerned with stationary closure classes and their observable projector structure. However, as recorded in Sec. 2.5 below, this finite-dimensional setting is not adequate for the full scope of the QM-series.

2.4 Operators and the Born Frequency Law

QB2 established that infinitesimal spacetime transport induces a natural class of differential generators acting on the complex state Ψ, and that these generators reproduce the operator structure associated with momentum and energy. The spatial transport generator, identified as the momentum operator, is

$$\hat{p}_j = -\mathrm{i}\Phi_0\,\partial_j, \tag{5}$$

and the temporal transport generator, identified as the energy operator, is

$$\hat{E} = \mathrm{i}\Phi_0\,\partial_t. \tag{6}$$

These operators were not postulated but emerged as representations of transport generators encoded through phase evolution. Their algebraic relation with the position operator $\hat{x}^j$ was derived as a representation identity, yielding the canonical commutation relation

$$[\hat{x}^j, \hat{p}_k]\,\Psi = \mathrm{i}\Phi_0\,\delta^j{}_k\,\Psi \tag{7}$$

for $\Psi \in \mathcal{H}^{\text{fin}}$.

Subsequently, QB6 established the Born frequency law: for any projector $\hat{P}$ in the projector algebra of $\mathcal{H}^{\text{fin}}$, the asymptotic frequency of the associated coherence-gated interaction event satisfies

$$\lim_{T\to\infty} \frac{N_P(T)}{N_{\text{tot}}(T)} = \Big\langle \Psi, \hat{P}\Psi, \Big\rangle \mathcal{H}^{\text{fin}}. \tag{8}$$

This was shown to reproduce the Born rule as an asymptotic event-frequency law rather than as an assumed probability measure. QB7 completed the correspondence by identifying the quantum-mechanical formalism—states, observables, measurement structure, and statistical interpretation—with the coherence-gated dynamics of the NUVO transport framework.

All results recorded in this subsection hold within the finite-dimensional pre-Hilbert space $\mathcal{H}^{\text{fin}}$. The present paper extends them to the complete Hilbert space $\mathcal{H}$.

2.5 Structural Gap: Limitations of the Finite Setting

The pre-Hilbert space $\mathcal{H}^{\text{fin}}$ of QB3 is defined as the finite representational span of stationary holonomic closure modes. Four structural limitations of this setting must be resolved before the QM-series can proceed.

First, $\mathcal{H}^{\text{fin}}$ contains only stationary configurations: those whose transport phase satisfies the holonomy closure condition with integer winding number. Non-stationary transport configurations, whose phase evolves continuously in time, do not belong to $\mathcal{H}^{\text{fin}}$. The time-dependent Schrödinger dynamics of QM4 require a state space that accommodates arbitrary time-evolving closure states, not only stationary modes.

Second, the spectrum of a Hamiltonian acting on $\mathcal{H}^{\text{fin}}$ is discrete by construction. Scattering states—transport configurations in which the closure density does not remain spatially localized—are associated with a continuous spectrum and cannot be represented in a finite-dimensional space. The treatment of scattering and tunneling in QM10 requires a state space that accommodates continuous-spectrum transport.

Third, the construction of multi-particle state spaces in QM7 requires the tensor product of single-particle Hilbert spaces. The tensor product of pre-Hilbert spaces is a pre-Hilbert space, but its completion to a genuine Hilbert space—essential for the analysis of exchange symmetry and the Pauli exclusion principle—requires each factor to be complete. This completeness is not available in $\mathcal{H}^{\text{fin}}$.

Fourth, the spectral theorem for unbounded self-adjoint operators, which underlies the uncertainty relations of QM3, the angular momentum structure of QM5, and the harmonic oscillator ladder structure of QM6, holds for operators on Hilbert spaces. The momentum and energy operators of QB2 are unbounded differential operators. The finite-dimensional pre-Hilbert setting does not provide the domain theory needed to treat such operators rigorously as self-adjoint objects with well-defined spectra.

The present paper addresses all four limitations by constructing the complete separable Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C})$, establishing the self-adjointness and spectral theory of the transport generators on this space, and providing the generalized eigenstate framework needed for the continuous spectrum. With this foundation in place, each of the four limitations is resolved, and the QM-series may proceed.

3 Closure Density Conservation and Normalization

3.1 The Continuity Relation as a Conservation Law

The continuity relation Eq. (1), recalled from the Q-series in Sec. 2.1, states that the closure density ρ satisfies a divergence-form balance equation with no source term. This structure has an immediate integral consequence: the total closure integrated over all of space is a conserved quantity under admissible transport evolution. The following lemma makes this precise.

Lemma 3.1 (Total-closure conservation). *Let $\rho : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ satisfy the transport continuity relation Eq. (1), with $\rho(\,\cdot\,, t)$ integrable over $\mathbb{R}^3$ for each t and with $\rho(x, t) \to 0$ sufficiently rapidly as $|x| \to \infty$ that the flux $\rho\, v$ vanishes on every bounding surface as the surface recedes to infinity. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \rho(x, t)\, \mathrm{d}^3x = 0.$$

In particular, the total closure

$$C_{\text{tot}} := \int_{\mathbb{R}^3} \rho(x, t)\, \mathrm{d}^3x$$

is independent of t.

Proof. Integrate the continuity relation Eq. (1) over a bounded spatial region $\Omega \subset \mathbb{R}^3$:

$$\int_\Omega \partial_t \rho\, \mathrm{d}^3x + \int_\Omega \nabla \cdot (\rho\, v)\, \mathrm{d}^3x = 0.$$

Since ρ is integrable and the time derivative may be exchanged with the integral under the stated integrability conditions, the first term equals $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho\,\mathrm{d}^3x$. Applying the divergence theorem to the second term yields a surface integral of the flux $\rho\,v$ over $\partial\Omega$. Taking $\Omega \uparrow \mathbb{R}^3$, the surface integral vanishes by the assumed decay of $\rho\,v$ at spatial infinity, giving

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \rho(x,t)\,\mathrm{d}^3x = 0.$$

The constancy of C_{tot} follows immediately. □

Remark 3.2. *The decay condition on $\rho\,v$ at spatial infinity is satisfied by all spatially localized closure configurations, in particular by the stationary holonomic modes of the QB-series and by any superposition of finitely many such modes. For scattering configurations treated in QM10, the appropriate condition is replaced by a flux-balance condition at infinity; the conclusion of Lemma 3.1 continues to hold in that setting with a suitable redefinition of the boundary term. The present paper is concerned with the localized case throughout.*

3.2 Normalization as a Structural Constraint

Lemma 3.1 establishes that C_{tot} is a time-invariant positive real number for any non-trivial admissible closure configuration. Its value depends on the choice of units in which the closure density ρ is measured. Since the transport closure system is linear in ρ—the continuity relation Eq. (1) is homogeneous—there is no dynamical mechanism that fixes the absolute scale of ρ. This scale is therefore a representational degree of freedom, entirely analogous to the choice of the baseline scalar level Λ_0 in the M-series, which fixes the unit of structural capacity without affecting the dynamics.

We fix this representational degree of freedom by working in units in which $C_{\mathrm{tot}} = 1$. This choice defines the normalized closure density.

Definition 3.3 (Normalized closure density)**.** *Given a transport closure state with total closure $C_{\mathrm{tot}} = \int_{\mathbb{R}^3} \rho(x,t)\,\mathrm{d}^3x > 0$, the* normalized closure density *is*

$$\tilde\rho(x,t) := \frac{\rho(x,t)}{C_{\mathrm{tot}}}.$$

By Lemma 3.1 and the linearity of integration, $\tilde\rho$ satisfies

$$\int_{\mathbb{R}^3} \tilde\rho(x,t)\,\mathrm{d}^3x = 1$$

for all t, and $\tilde\rho$ satisfies the same continuity relation Eq. (1) *as ρ.*

The unit-normalization of Definition 3.3 is not a probabilistic postulate. It is the selection of a canonical representative within the equivalence class of closure densities that differ only by a positive overall scale factor. No statistical interpretation of $\tilde\rho$ is assumed at this stage; the identification of $\tilde\rho$ with a probability density follows from the Born frequency law of QB6, which is a separate and logically independent result. The present construction establishes only that the total-closure integral is conserved and that its value may be fixed to unity without loss of generality.

3.3 Normalization of the Complex State

The normalization condition on $\tilde{\rho}$ translates directly into a normalization condition on the complex state encoding Ψ introduced in QB1.

Proposition 3.4 (State normalization from closure conservation)**.** *Let* $\Psi = \sqrt{\tilde{\rho}}\, e^{i\phi/\Phi_0}$ *be the complex encoding of the normalized closure state, as in Eq.* (3) *with* ρ *replaced by* $\tilde{\rho}$*. Then*

$$\int_{\mathbb{R}^3} |\Psi(x,t)|^2 \, d^3x = 1$$

for all t*.*

Proof. By construction of the complex encoding, $|\Psi(x,t)|^2 = \tilde{\rho}(x,t)$ pointwise. The result follows immediately from Definition 3.3. □

Remark 3.5. *Proposition 3.4 establishes the relation* $\int |\Psi|^2 \, d^3x = 1$ *as a structural consequence of the continuity relation and the choice of closure units. In the standard quantum-mechanical formalism this relation is the normalization condition on the wave function and is conventionally introduced as a requirement of the probabilistic interpretation. In the present framework the relation arises prior to and independently of any probabilistic interpretation. The two accounts are numerically consistent: the Born frequency law of QB6, now extended to* $\mathcal{H}$ *in Sec. 5, assigns frequency* $\int_B |\Psi|^2 \, d^3x$ *to a spatial region* B*, in agreement with the normalized closure content. But the derivation of the normalization condition does not depend on the Born law, and the Born law does not depend on the present derivation.*

3.4 Invariance of Normalization Under Transport Evolution

The conservation law of Lemma 3.1 implies not only that C_{tot} is constant but that the normalization of the complex state is preserved for all time under admissible transport. This result is recorded as a corollary because it will be used in QM4 as the geometric underpinning of the unitarity of Schrödinger evolution: the full time-dependent dynamics preserves state normalization not by postulate but as a consequence of the divergence-free transport structure.

Corollary 3.6 (Norm preservation under transport)**.** *Let* $\Psi(\,\cdot\,,t)$ *be the complex encoding of an admissible transport closure evolution with* $\|\Psi(\,\cdot\,,0)\|_{\mathcal{H}} = 1$*. Then*

$$\|\Psi(\,\cdot\,,t)\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^3} |\Psi(x,t)|^2 \, d^3x = 1$$

for all $t \in \mathbb{R}$*.*

Proof. Since $|\Psi(\,\cdot\,,t)|^2 = \tilde{\rho}(\,\cdot\,,t)$ and $\tilde{\rho}$ satisfies the continuity relation Eq. (1), Lemma 3.1 gives $\int_{\mathbb{R}^3} \tilde{\rho}(x,t) \, d^3x = \int_{\mathbb{R}^3} \tilde{\rho}(x,0) \, d^3x = 1$ for all t. □

Remark 3.7. *Corollary 3.6 establishes norm preservation at the level of the transport closure system, prior to any operator or dynamical formulation. In QM4 this result will be identified as the geometric origin of the unitarity of the time-evolution operator: the fact that Schrödinger evolution preserves the* $\mathcal{H}$*-norm is a direct consequence of the divergence-free structure of the underlying transport, not an additional requirement imposed on the dynamics.*

4 The L^2 Hilbert Space of Closure States

4.1 Function Space Setting

The complex state encoding Ψ established in QB1 and recalled in Sec. 2.2 is, for each fixed time t, a complex-valued function on $\mathbb{R}^3$. Proposition 3.4 establishes that the normalized closure state satisfies $\int_{\mathbb{R}^3} |\Psi(x,t)|^2 \, \mathrm{d}^3x = 1$, which in particular requires that $|\Psi(\,\cdot\,,t)|^2$ is integrable over $\mathbb{R}^3$. The natural ambient space for such functions is the space of square-integrable complex-valued functions on $\mathbb{R}^3$.

Definition 4.1 (L^2 space of closure states)**.** *The* space of square-integrable closure states *is*

$$L^2(\mathbb{R}^3,\mathbb{C}) \coloneqq \left\{ f : \mathbb{R}^3 \to \mathbb{C} \;\middle|\; \int_{\mathbb{R}^3} |f(x)|^2 \, \mathrm{d}^3x < \infty \right\},$$

where two functions are identified if they agree almost everywhere with respect to Lebesgue measure on $\mathbb{R}^3$.

Remark 4.2. *The identification of functions that agree almost everywhere is a standard measure-theoretic convention. In the NUVO transport setting it is physically natural: two closure density fields that differ only on a set of measure zero carry identical total closure and are indistinguishable by any spatial integral of ρ. The identification therefore introduces no physical ambiguity.*

The space $L^2(\mathbb{R}^3,\mathbb{C})$ is equipped with a canonical inner product that is consistent with the closure density integral.

Definition 4.3 (L^2 inner product)**.** *The* closure inner product *on $L^2(\mathbb{R}^3,\mathbb{C})$ is*

$$\langle \Psi_1, \Psi_2 \rangle_{\mathcal{H}} \coloneqq \int_{\mathbb{R}^3} \overline{\Psi_1(x)} \, \Psi_2(x) \, \mathrm{d}^3x.$$

We verify that Definition 4.3 defines a genuine complex inner product by checking the four required properties.

Lemma 4.4 (Inner product axioms)**.** *The pairing $\langle \cdot\,, \cdot \rangle_{\mathcal{H}}$ of Definition 4.3 satisfies:*

(i) Conjugate symmetry: $\langle \Psi_1, \Psi_2 \rangle_{\mathcal{H}} = \overline{\langle \Psi_2, \Psi_1 \rangle_{\mathcal{H}}}$ *for all* $\Psi_1, \Psi_2 \in L^2(\mathbb{R}^3,\mathbb{C})$.

(ii) Linearity in the second argument: $\langle \Psi_1, \alpha\Psi_2 + \beta\Psi_3 \rangle_{\mathcal{H}} = \alpha \langle \Psi_1, \Psi_2 \rangle_{\mathcal{H}} + \beta \langle \Psi_1, \Psi_3 \rangle_{\mathcal{H}}$ *for all* $\alpha, \beta \in \mathbb{C}$.

(iii) Positive semi-definiteness: $\langle \Psi, \Psi \rangle_{\mathcal{H}} \geq 0$, *with equality if and only if* $\Psi = 0$ *almost everywhere.*

(iv) Well-definedness: $|\langle \Psi_1, \Psi_2 \rangle_{\mathcal{H}}| < \infty$ *for all* $\Psi_1, \Psi_2 \in L^2(\mathbb{R}^3,\mathbb{C})$.

Proof. Properties (i) through (iii) follow directly from the linearity of the Lebesgue integral and the pointwise properties of complex conjugation and modulus. For (i), interchange Ψ_1 and Ψ_2 in the integrand and take the complex conjugate; the result is immediate. For (ii), linearity of the integral in the integrand gives linearity of $\langle \cdot\,, \cdot \rangle_{\mathcal{H}}$ in the second argument; conjugate linearity in the first argument follows from (i) and (ii) together. For (iii), $|\Psi(x)|^2 \geq 0$ pointwise, so the integral is non-negative; it vanishes if and only if $|\Psi|^2 = 0$ almost everywhere, which is equivalent to $\Psi = 0$ almost everywhere. For (iv), the Cauchy–Schwarz inequality for integrals gives

$$\left| \int_{\mathbb{R}^3} \overline{\Psi_1} \, \Psi_2 \, \mathrm{d}^3x \right| \leq \left(\int_{\mathbb{R}^3} |\Psi_1|^2 \, \mathrm{d}^3x \right)^{1/2} \left(\int_{\mathbb{R}^3} |\Psi_2|^2 \, \mathrm{d}^3x \right)^{1/2} < \infty,$$

since both factors are finite by assumption that $\Psi_1, \Psi_2 \in L^2(\mathbb{R}^3,\mathbb{C})$. □

Remark 4.5. *The convention adopted in Definition 4.3 places the complex conjugate in the first argument, so that* $\langle \cdot , \cdot \rangle_{\mathcal{H}}$ *is conjugate-linear in the first argument and linear in the second. This is the physics convention and is consistent with the holonomic coherence functional of QB3, recalled in Eq.* (4)*.*

The induced norm is

$$\|\Psi\|_{\mathcal{H}} := \langle \Psi, \Psi \rangle_{\mathcal{H}}^{1/2} = \left(\int_{\mathbb{R}^3} |\Psi(x)|^2 \, \mathrm{d}^3 x \right)^{1/2},$$

and the normalization condition of Proposition 3.4 is precisely the statement $\|\Psi\|_{\mathcal{H}} = 1$.

4.2 Extension of the QB3 Inner Product

The inner product of Definition 4.3 is not introduced as a new structure. It extends the holonomic coherence functional of QB3 in a manner that is strictly consistent with the finite-dimensional setting already established.

Proposition 4.6 (Consistency with QB3)**.** *The closure inner product of Definition 4.3, when restricted to the finite representational span* $\mathcal{H}^{\mathrm{fin}}$ *of the QB3 closure mode family, coincides with the holonomic coherence functional established in QB3.*

Proof. The holonomic coherence functional of QB3 is given by the integral Eq. (4): for $\Psi_1, \Psi_2 \in \mathcal{H}^{\mathrm{fin}}$,

$$\langle \Psi_1, \Psi_2 \rangle_{\mathcal{H}^{\mathrm{fin}}} = \int_{\mathcal{M}} \overline{\Psi_1(x)} \, \Psi_2(x) \, \mathrm{d}^3 x.$$

Since the stationary holonomic closure modes of $\mathcal{H}^{\mathrm{fin}}$ are spatially localized, they belong to $L^2(\mathbb{R}^3, \mathbb{C})$, and the integral over $\mathcal{M}$ coincides with the integral over $\mathbb{R}^3$ in the spatial sector. The right-hand side therefore equals $\langle \Psi_1, \Psi_2 \rangle_{\mathcal{H}}$ of Definition 4.3, and the two functionals agree on $\mathcal{H}^{\mathrm{fin}}$. □

Proposition 4.6 confirms that no discontinuity in inner product structure occurs at the transition from the QB-series to the QM-series. The holonomic orthogonality relations of QB3—distinct closure classes are mutually orthogonal under the inner product—are preserved in $\mathcal{H}$, and the finite-dimensional spectral structure of the prior series is faithfully embedded in the present infinite-dimensional framework.

4.3 Completeness and Separability

The space $L^2(\mathbb{R}^3, \mathbb{C})$ equipped with the inner product of Definition 4.3 is not merely a pre-Hilbert space: it is complete. Completeness is the property that distinguishes a Hilbert space from a general inner product space and is the property that makes the spectral theory of self-adjoint operators, developed in Secs. 5–7, available.

Theorem 4.7 (Hilbert space of closure states)**.** *The space* $\mathcal{H} := L^2(\mathbb{R}^3, \mathbb{C})$ *equipped with the closure inner product of Definition 4.3 is a separable complex Hilbert space. In particular:*

(i) $\mathcal{H}$ *is complete: every Cauchy sequence in the norm* $\|\cdot\|_{\mathcal{H}}$ *converges to an element of* $\mathcal{H}$*.*

(ii) $\mathcal{H}$ *is separable: it admits a countable orthonormal basis.*

Proof. Completeness is the content of the Riesz–Fischer theorem, a classical result of measure theory, which establishes that L^2 over any sigma-finite measure space is complete [2]. Separability follows from the denseness of the space of compactly supported smooth functions $C_c^\infty(\mathbb{R}^3, \mathbb{C})$ in $L^2(\mathbb{R}^3, \mathbb{C})$, combined with the separability of $\mathbb{R}^3$ as a metric space; a countable orthonormal basis may be constructed by applying the Gram–Schmidt procedure to a countable dense subset [2]. These are cited as classical results; their derivation lies outside the scope of the present paper. □

Remark 4.8. *The role of Theorem 4.7 in the NUVO program is confirmatory rather than constructive. The Hilbert space $\mathcal{H}$ is not built from scratch here; it is the standard L^2 space of mathematical analysis. What the theorem confirms is that the transport closure states, encoded as normalized complex functions via Eq.* (3)*, inhabit a complete inner-product space. This completeness is the property that guarantees the spectral theorem applies to the self-adjoint transport generators of QB2, and that Cauchy-convergent sequences of closure states—arising, for example, as limits of approximating sequences of stationary modes—converge to well-defined elements of the state space.*

4.4 Admissible Closure States Within $\mathcal{H}$

Theorem 4.7 establishes $\mathcal{H}$ as the ambient mathematical space for closure state representations. It is appropriate to record the relationship between this ambient space and the physically admissible closure states of the transport system.

Not every element of $\mathcal{H}$ arises from an admissible transport closure configuration. Admissible states are those consistent with the full transport closure system of the Q-series: they must arise as solutions of the coupled (ρ, ϕ) system with non-negative closure density and transport-consistent phase. An arbitrary element of $L^2(\mathbb{R}^3, \mathbb{C})$ need satisfy neither of these geometric constraints.

The use of the full $\mathcal{H}$ as the state space is justified on three grounds.

First, norm-preserving limits of admissible transport sequences remain in $\mathcal{H}$. If a sequence of admissible closure states $\{\Psi_n\}$ is Cauchy in the $\mathcal{H}$-norm, Theorem 4.7 guarantees that it converges to some $\Psi \in \mathcal{H}$. Working in the complete space ensures that such limits are always available as representational objects, even when a direct transport-closure interpretation of the limit state is not immediately apparent.

Second, superpositions of admissible states are admissible. The linearity of the transport closure equation, established in the Q-series, implies that any finite linear combination of closure states satisfying the transport law also satisfies it. This superposition principle, developed fully in QM2, ensures that the physically relevant states are closed under the algebraic operations appropriate to the Hilbert space structure.

Third, the spectral and functional-analytic theory required for the remainder of the QM-series is fully developed for operators on $\mathcal{H}$. Restricting to a proper subset of $\mathcal{H}$ would forfeit this structure without compensating physical benefit.

Remark 4.9. *A precise characterization of the admissible closure states as a subset of $\mathcal{H}$—including the identification of regularity conditions, domain restrictions, and the sense in which the transport system selects physical states within the ambient Hilbert space—is a structural question that goes beyond the scope of the present paper. It is carried forward as an open item within the QM-series and will be addressed in the context of specific physical sectors as they arise.*

5 Self-Adjoint Transport Generators on $\mathcal{H}$

The operator structure established in QB2 was derived within the finite-dimensional pre-Hilbert space $\mathcal{H}^{\text{fin}}$. The momentum and energy generators were identified as differential operators acting on the complex state encoding, and their canonical commutation relation was derived as a representation identity. The present section promotes these results to the complete Hilbert space $\mathcal{H}$ constructed in Sec. 4. The central technical requirement is that the generators, which are unbounded differential operators, extend from their initial domain to genuinely self-adjoint operators on $\mathcal{H}$. Self-adjointness—as opposed to mere symmetry—is the property that guarantees a real spectrum, a spectral theorem, and a well-defined functional calculus, all of which are required in subsequent sections and throughout the QM-series.

5.1 Recall of the Momentum and Energy Operators

The transport generators of QB2 are recalled here in the form in which they will be promoted to operators on $\mathcal{H}$. For each spatial index $j \in \{1, 2, 3\}$, the momentum transport generator acts on a differentiable state Ψ by

$$\hat{p}_j \, \Psi(x, t) := -\mathrm{i}\Phi_0 \, \partial_j \Psi(x, t), \tag{9}$$

and the energy transport generator acts by

$$\hat{E} \, \Psi(x, t) := \mathrm{i}\Phi_0 \, \partial_t \Psi(x, t). \tag{10}$$

In QB2 these were defined on the finite representational span $\mathcal{H}^{\mathrm{fin}}$ and shown to arise from infinitesimal spacetime transport acting on the phase structure of the complex encoding. Their derivation is not repeated here; they are recalled as the objects to be promoted.

Both operators are differential operators of first order. On any finite-dimensional space of smooth functions, first-order differential operators are automatically well-defined; no domain theory is needed. On the infinite-dimensional space $\mathcal{H}$, however, differential operators are in general unbounded—their operator norm is infinite—and the question of self-adjointness requires careful attention to domain. The analysis proceeds in two steps: symmetry on a dense domain, and then essential self-adjointness.

5.2 Formal Self-Adjointness and Domain

The natural initial domain for the transport generators is the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ of rapidly decreasing smooth functions. This space is dense in $\mathcal{H}$ in the $\|\cdot\|_{\mathcal{H}}$-norm, a classical fact of functional analysis, so operators defined on $\mathcal{S}(\mathbb{R}^3)$ can in principle be extended to all of $\mathcal{H}$ by continuity or closure. Moreover, $\mathcal{S}(\mathbb{R}^3)$ is stable under differentiation and under multiplication by polynomials, making it the natural domain for first-order differential operators with polynomial-phase structure.

Lemma 5.1 (Symmetry of the momentum generator)**.** *The momentum transport generator $\hat{p}_j = -\mathrm{i}\Phi_0 \, \partial_j$, defined on the dense domain $\mathcal{D}(\hat{p}_j) = \mathcal{S}(\mathbb{R}^3) \subset \mathcal{H}$, is symmetric; that is,*

$$\langle f, \hat{p}_j \, g \rangle_{\mathcal{H}} = \langle \hat{p}_j \, f, g \rangle_{\mathcal{H}}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^3)$.

Proof. Expand the left-hand side using Definition 4.3 and Eq. (9):

$$\langle f, \hat{p}_j \, g \rangle_{\mathcal{H}} = \int_{\mathbb{R}^3} \overline{f(x)} \left(-\mathrm{i}\Phi_0 \, \partial_j g(x) \right) \mathrm{d}^3 x.$$

Integrate by parts in the x^j coordinate. Since $f, g \in \mathcal{S}(\mathbb{R}^3)$, the product $\overline{f} \, g$ and all its derivatives decay faster than any polynomial as $|x| \to \infty$, so the boundary term at spatial infinity vanishes.

The integration by parts transfers the derivative from g to $\overline{f}$, giving

$$\langle f, \hat{p}_j\, g\rangle_{\mathcal{H}} = \int_{\mathbb{R}^3} \big(\mathrm{i}\Phi_0\, \partial_j \overline{f(x)}\big)\, g(x)\, \mathrm{d}^3x = \int_{\mathbb{R}^3} \overline{\big(-\mathrm{i}\Phi_0\, \partial_j f(x)\big)}\, g(x)\, \mathrm{d}^3x = \langle \hat{p}_j\, f, g\rangle_{\mathcal{H}}\,,$$

where in the last step the complex conjugate of i contributes a sign that reproduces the factor $-\mathrm{i}\Phi_0$ in the definition of $\hat{p}_j$. □

Remark 5.2. *Lemma 5.1 establishes that $\hat{p}_j$ is* symmetric *on $\mathcal{S}(\mathbb{R}^3)$, meaning $\langle f, \hat{p}_j\, g\rangle_{\mathcal{H}} = \langle \hat{p}_j\, f, g\rangle_{\mathcal{H}}$ for all f, g in the domain. Symmetry is necessary but not sufficient for self-adjointness. An operator is self-adjoint only if, in addition, its domain equals the domain of its adjoint. For bounded operators these conditions coincide; for the unbounded differential operators considered here, symmetry on a dense domain implies only that a self-adjoint extension may exist. The question of whether the extension is unique—essential self-adjointness—is addressed in Sec. 5.3.*

An analogous symmetry argument applies to the energy generator $\hat{E} = \mathrm{i}\Phi_0\, \partial_t$, with the integration-by-parts argument carried out in the time variable on any finite time interval with periodic or vanishing boundary conditions. Since the energy generator acts on the time evolution of states rather than on their spatial structure, its treatment on $\mathcal{H}$ is parallel to that of $\hat{p}_j$ and will not be repeated in detail.

5.3 Essential Self-Adjointness

An operator is essentially self-adjoint on a given domain if it is symmetric there and its closure—the smallest closed extension—is self-adjoint. Essential self-adjointness on the Schwartz domain is the key property for the transport generators: it implies that each generator has a unique self-adjoint extension to a larger domain, and that this extension is unambiguous. In the NUVO setting, essential self-adjointness means that the transport generators define unique, physically unambiguous observables on $\mathcal{H}$ without requiring any additional choice of boundary conditions or extension data.

Theorem 5.3 (Essential self-adjointness of transport generators)**.** *The momentum transport generators $\hat{p}_j = -\mathrm{i}\Phi_0\, \partial_j$, $j = 1, 2, 3$, defined on the dense domain $\mathcal{S}(\mathbb{R}^3) \subset \mathcal{H}$, are essentially self-adjoint. Their unique self-adjoint closures $\overline{\hat{p}_j}$ have domains equal to the first-order Sobolev space*

$$\mathcal{D}(\overline{\hat{p}_j}) = H^1(\mathbb{R}^3) := \big\{ f \in L^2(\mathbb{R}^3, \mathbb{C}) \;\big|\; \partial_j f \in L^2(\mathbb{R}^3, \mathbb{C}) \big\},$$

where $\partial_j f$ is understood in the distributional sense.

Proof. This is a classical result of spectral theory for constant-coefficient differential operators on $\mathbb{R}^3$. The operator $-\mathrm{i}\Phi_0\, \partial_j$ is unitarily equivalent, via the Fourier transform at scale Φ_0, to the multiplication operator by the real-valued function p_j on $L^2(\mathbb{R}^3, \mathbb{C})$. A real-valued multiplication operator on L^2 is self-adjoint on its natural domain $\{f \in L^2 : p_j f \in L^2\}$, and the Fourier transform carries the Schwartz domain to itself densely. It follows that $-\mathrm{i}\Phi_0\, \partial_j$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^3)$, with closure having domain $H^1(\mathbb{R}^3)$ as stated. The result is cited from [3]; the details of the Fourier analysis are standard and are not reproduced here. □

Remark 5.4. *The program role of Theorem 5.3 is to confirm that the transport generators derived in QB2 from transport closure structure extend unambiguously to self-adjoint operators on the complete Hilbert space $\mathcal{H}$. No new derivation is required beyond the classical result cited; the content of the theorem in the NUVO program is the identification of the physical operators with the* *abstract self-adjoint operators to which the standard functional analysis applies. In particular, the Sobolev domain $H^1(\mathbb{R}^3)$ contains all functions whose first spatial derivatives are square-integrable, which is precisely the regularity required for the phase gradient structure of the transport closure encoding.*

Remark 5.5. *The uniqueness of the self-adjoint closure has a direct physical interpretation within the NUVO framework. A symmetric operator that is not essentially self-adjoint admits multiple distinct self-adjoint extensions, each corresponding to a different choice of boundary condition and hence a different observable. Essential self-adjointness of $\hat{p}_j$ on $\mathcal{S}(\mathbb{R}^3)$ implies that the momentum transport generator defines a single, unambiguous observable on $\mathcal{H}$. This is consistent with the derivation of $\hat{p}_j$ in QB2, where no choice of boundary conditions was required: the operator emerged uniquely from the transport phase structure.*

5.4 The Canonical Commutation Relation on $\mathcal{H}$

With the transport generators established as essentially self-adjoint on the Schwartz domain, the canonical commutation relation of QB2 can be promoted to the complete Hilbert space setting.

Proposition 5.6 (Canonical commutation relation on $\mathcal{H}$)**.** *On the dense domain $\mathcal{S}(\mathbb{R}^3) \subset \mathcal{H}$, the position and momentum transport generators satisfy*

$$[\hat{x}^j, \hat{p}_k]\,\Psi = \mathrm{i}\Phi_0\,\delta^j{}_k\,\Psi \tag{11}$$

for all $\Psi \in \mathcal{S}(\mathbb{R}^3)$.

Proof. The commutator is computed directly on $\mathcal{S}(\mathbb{R}^3)$. For $\Psi \in \mathcal{S}(\mathbb{R}^3)$ and fixed indices j, k:

$$[\hat{x}^j, \hat{p}_k]\,\Psi = \hat{x}^j(\hat{p}_k\,\Psi) - \hat{p}_k(\hat{x}^j\,\Psi).$$

Substituting Eq. (9) and the action of $\hat{x}^j$ as multiplication by x^j:

$$\hat{x}^j(\hat{p}_k\,\Psi) = x^j\big(-\mathrm{i}\Phi_0\,\partial_k\Psi\big) = -\mathrm{i}\Phi_0\,x^j\,\partial_k\Psi,$$

$$\hat{p}_k(\hat{x}^j\,\Psi) = -\mathrm{i}\Phi_0\,\partial_k(x^j\,\Psi) = -\mathrm{i}\Phi_0\left(\delta^j{}_k\,\Psi + x^j\,\partial_k\Psi\right).$$

Subtracting:

$$[\hat{x}^j, \hat{p}_k]\,\Psi = -\mathrm{i}\Phi_0\,x^j\,\partial_k\Psi - \big(-\mathrm{i}\Phi_0\,\delta^j{}_k\,\Psi - \mathrm{i}\Phi_0\,x^j\,\partial_k\Psi\big) = \mathrm{i}\Phi_0\,\delta^j{}_k\,\Psi.$$

□

Remark 5.7. *Equation (11) extends the relation derived in QB2 from the finite representational span $\mathcal{H}^{\mathrm{fin}}$ to the complete Hilbert space $\mathcal{H}$. The derivation in QB2 obtained this relation as a consequence of the representation of transport generators through phase gradients; the present proof confirms that the same algebraic identity holds on the full Schwartz domain without modification. The canonical commutation relation Eq. (11) is the foundation for the uncertainty relations to be established in QM3: the Robertson bound $\Delta\hat{x}^j \cdot \Delta\hat{p}_k \geq \frac{1}{2}\Phi_0\,\delta^j{}_k$ will follow from Eq. (11) by an application of the Cauchy–Schwarz inequality on $\mathcal{H}$.*

Remark 5.8. *It is a classical result—the Stone–von Neumann theorem—that, up to unitary equivalence, the Schrödinger representation on $L^2(\mathbb{R}^3, \mathbb{C})$ is the unique irreducible representation of the canonical commutation relations Eq. (11) for a finite number of degrees of freedom [2]. In the NUVO framework this uniqueness theorem confirms that the operator structure emerging from transport closure is not one possible representation among many, but the essentially unique representation consistent with the commutation structure derived from transport generators. This result is cited for orientation; its proof lies outside the scope of the present paper.*

6 Continuous Spectrum and Generalized Eigenstates

The self-adjoint transport generators established in Sec. 5 act on the complete Hilbert space $\mathcal{H}$ and, by Theorem 5.3, possess well-defined spectra contained in $\mathbb{R}$. For the stationary holonomic closure modes of the QB-series, the relevant operators—in particular the Hamiltonian of the hydrogenic sector—possess isolated eigenvalues with normalizable eigenfunctions in $\mathcal{H}$. These constitute the discrete part of the spectrum and are directly identified with the closure modes of $\mathcal{H}^{\mathrm{fin}}$. However, the full spectrum of the transport generators is not exhausted by such eigenvalues. The momentum operator $\hat{p}_j$, for instance, has no normalizable eigenfunctions in $\mathcal{H}$ at all: no square-integrable function satisfies $\hat{p}_j\,\Psi = p_j\,\Psi$ for any fixed $p_j \in \mathbb{R}$. The present section develops the framework required to handle this continuous-spectrum sector systematically.

6.1 Discrete and Continuous Spectrum

For a self-adjoint operator A on $\mathcal{H}$, the spectrum $\sigma(A)$ is the set of all $a \in \mathbb{R}$ for which the resolvent $(A - a\,\hat{\mathbf{1}})^{-1}$ fails to be a bounded operator defined on all of $\mathcal{H}$. The spectrum decomposes into qualitatively distinct parts according to the nature of the spectral values and the existence of associated eigenfunctions.

Definition 6.1 (Discrete and continuous spectrum)**.** *Let A be a self-adjoint operator on $\mathcal{H}$. The* discrete spectrum $\sigma_{\mathrm{disc}}(A)$ *consists of all values $a \in \sigma(A)$ for which there exists a normalizable eigenfunction $\Psi_a \in \mathcal{H}$ satisfying $A\,\Psi_a = a\,\Psi_a$ and $\|\Psi_a\|_{\mathcal{H}} = 1$. The* continuous spectrum $\sigma_{\mathrm{cont}}(A)$ *consists of all values $a \in \sigma(A)$ for which no such L^2 eigenfunction exists. The full spectrum decomposes as*

$$\sigma(A) = \sigma_{\mathrm{disc}}(A) \cup \sigma_{\mathrm{cont}}(A),$$

where the two parts are disjoint.

Remark 6.2. *Within the NUVO transport framework, the two parts of the spectrum carry distinct physical interpretations. Elements of $\sigma_{\mathrm{disc}}(A)$ correspond to holonomically closed, spatially localized closure configurations: the normalizable eigenfunction Ψ_a represents a bound closure mode, and the eigenvalue a is the associated observable value. These are precisely the configurations represented in $\mathcal{H}^{\mathrm{fin}}$, so the discrete spectrum of the Hamiltonian acting on $\mathcal{H}$ recovers the finite-dimensional spectral structure of the QB-series. Elements of $\sigma_{\mathrm{cont}}(A)$, by contrast, correspond to non-localizable transport configurations whose closure density does not decay sufficiently at spatial infinity to be square-integrable. These include scattering configurations, treated in QM10, in which the closure transport propagates across unbounded regions. No L^2 eigenfunction exists for such configurations, but they are nonetheless physically relevant and must be represented in the formalism.*

Two instructive examples of this decomposition are provided by the transport generators established in Sec. 5.

For the momentum operator $\hat{p}_j = -\mathrm{i}\Phi_0\,\partial_j$ on $L^2(\mathbb{R}^3, \mathbb{C})$, the spectrum is purely continuous: $\sigma_{\mathrm{disc}}(\hat{p}_j) = \emptyset$ and $\sigma_{\mathrm{cont}}(\hat{p}_j) = \mathbb{R}$. A formal eigenequation $\hat{p}_j\,\Psi = p_j\,\Psi$ requires $\Psi(x) \propto \mathrm{e}^{\,\mathrm{i}p_j x^j/\Phi_0}$, which is not square-integrable over $\mathbb{R}^3$.

For the hydrogenic Hamiltonian $\hat{H}_{\mathrm{H}}$ of the Q-series, the spectrum decomposes as $\sigma_{\mathrm{disc}}(\hat{H}_{\mathrm{H}}) = \{E_n\}_{n\geq 1}$, a discrete sequence of negative eigenvalues corresponding to the bound holonomic closure modes, and $\sigma_{\mathrm{cont}}(\hat{H}_{\mathrm{H}}) = [0, \infty)$, the continuum of positive-energy scattering configurations. Both parts of the spectrum are needed for a complete representation of arbitrary closure states.

6.2 The Rigged Hilbert Space Framework

The continuous spectrum presents a representational difficulty: its associated "eigenstates" are not elements of $\mathcal{H}$ and therefore cannot be treated as closure states in the sense of Sec. 4. Yet completeness of the spectral representation requires that they participate in the expansion of physical states. The rigged Hilbert space framework resolves this difficulty by embedding $\mathcal{H}$ in a larger distributional space within which generalized eigenstates are well-defined objects, while retaining $\mathcal{H}$ as the space of physical states.

Definition 6.3 (Rigged Hilbert space)**.** *The* rigged Hilbert space *associated to $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C})$ is the Gelfand triple*

$$\mathcal{S}(\mathbb{R}^3) \subset \mathcal{H} \subset \mathcal{S}'(\mathbb{R}^3),$$

where $\mathcal{S}(\mathbb{R}^3)$ is the Schwartz space of rapidly decreasing smooth functions, equipped with its natural locally convex topology, and $\mathcal{S}'(\mathbb{R}^3)$ is its topological dual, the space of tempered distributions. The inclusions are continuous and dense: $\mathcal{S}(\mathbb{R}^3)$ is dense in $\mathcal{H}$ in the norm $\|\cdot\|_{\mathcal{H}}$, and $\mathcal{H}$ is continuously embedded in $\mathcal{S}'(\mathbb{R}^3)$ via the identification of each $f \in \mathcal{H}$ with the functional $\varphi \mapsto \langle f, \varphi \rangle_{\mathcal{H}}$.

The three spaces in Definition 6.3 play distinct and complementary roles. The Schwartz space $\mathcal{S}(\mathbb{R}^3)$ is the domain of the transport generators established in Sec. 5: it is the space of test functions on which the operators are defined and on which the canonical commutation relation Eq. (11) holds. The Hilbert space $\mathcal{H}$ is the physical state space: all admissible normalized closure states reside in $\mathcal{H}$, and all expectation values and frequency laws are computed within $\mathcal{H}$. The distribution space $\mathcal{S}'(\mathbb{R}^3)$ is the extended representational space within which generalized eigenstates of continuous-spectrum operators are defined.

Remark 6.4. *The rigged Hilbert space triple is introduced solely to provide a consistent mathematical setting for generalized eigenstates and completeness relations. It does not alter the physical state space $\mathcal{H}$, does not introduce new ontological content, and does not affect any result established in the prior series. Physical, normalizable closure states reside in $\mathcal{H}$; elements of $\mathcal{S}'(\mathbb{R}^3) \setminus \mathcal{H}$ are formal representational tools that appear in expansions of physical states but do not themselves represent admissible transport configurations. The framework is analogous in spirit to the introduction of the complex state encoding in QB1: it is a representational extension that encodes existing structure more compactly, without altering the underlying geometry or dynamics.*

The nuclear spectral theorem, due to Gelfand and Vilenkin, guarantees that every self-adjoint operator that maps $\mathcal{S}(\mathbb{R}^3)$ continuously into itself admits a complete family of generalized eigenstates in $\mathcal{S}'(\mathbb{R}^3)$ [1]. Both the momentum operators $\hat{p}_j$ and the Hamiltonian $\hat{H}$, which act by differentiation and multiplication on Schwartz-class functions, satisfy this condition. The completeness of the generalized eigenstate expansion is therefore guaranteed by the nuclear spectral theorem rather than assumed.

6.3 Plane Waves as Generalized Momentum Eigenstates

The generalized eigenstates of the momentum operator are constructed explicitly. For each $p \in \mathbb{R}^3$, a formal eigenequation for $\hat{p}_j$ with eigenvalue p_j requires a function ψ_p satisfying $-\mathrm{i}\Phi_0 \, \partial_j \psi_p(x) = p_j \, \psi_p(x)$ for each $j = 1, 2, 3$ simultaneously. This system is solved by exponential functions of the form $\mathrm{e}^{\mathrm{i}p\cdot x/\Phi_0}$, modulo an overall normalization constant to be fixed by the completeness relation.

Definition 6.5 (Generalized momentum eigenstate)**.** *For* $p \in \mathbb{R}^3$*, the* generalized momentum eigenstate *is the tempered distribution*

$$\psi_p(x) \coloneqq \frac{1}{(2\pi\Phi_0)^{3/2}}\, \mathrm{e}^{\,\mathrm{i}p\cdot x/\Phi_0}, \tag{12}$$

understood as an element of $\mathcal{S}'(\mathbb{R}^3)$*.*

Proposition 6.6 (Eigenvalue equation in the distributional sense)**.** *For each* $p \in \mathbb{R}^3$ *and each* $j \in \{1, 2, 3\}$*, the generalized momentum eigenstate* ψ_p *of Definition 6.5 satisfies*

$$\hat{p}_j\, \psi_p = p_j\, \psi_p \tag{13}$$

in the distributional sense; that is, $\langle \hat{p}_j\, \psi_p, \varphi\rangle = p_j\, \langle \psi_p, \varphi\rangle$ *for all* $\varphi \in \mathcal{S}(\mathbb{R}^3)$*, where* $\langle\,\cdot\,,\,\cdot\,\rangle$ *denotes the duality pairing between* $\mathcal{S}'(\mathbb{R}^3)$ *and* $\mathcal{S}(\mathbb{R}^3)$*.*

Proof. For any $\varphi \in \mathcal{S}(\mathbb{R}^3)$:

$$\langle \hat{p}_j\, \psi_p, \varphi\rangle = \langle \psi_p, \hat{p}_j^\dagger\, \varphi\rangle = \langle \psi_p, \hat{p}_j\, \varphi\rangle,$$

where the second equality uses the symmetry of $\hat{p}_j$ established in Lemma 5.1. Computing $\hat{p}_j\, \varphi = -\mathrm{i}\Phi_0\, \partial_j \varphi$ and evaluating the duality pairing against ψ_p:

$$\langle \psi_p, \hat{p}_j\, \varphi\rangle = \int_{\mathbb{R}^3} \overline{\psi_p(x)}\, (-\mathrm{i}\Phi_0\, \partial_j \varphi(x))\, \mathrm{d}^3x.$$

Integrating by parts and using the fact that $\varphi \in \mathcal{S}(\mathbb{R}^3)$ decays at infinity:

$$= \int_{\mathbb{R}^3} \mathrm{i}\Phi_0\, (\partial_j \overline{\psi_p(x)})\, \varphi(x)\, \mathrm{d}^3x = \int_{\mathbb{R}^3} \overline{(-\mathrm{i}\Phi_0\, \partial_j \psi_p(x))}\, \varphi(x)\, \mathrm{d}^3x.$$

Since $-\mathrm{i}\Phi_0\, \partial_j \psi_p(x) = p_j\, \psi_p(x)$ pointwise by direct differentiation of Eq. (12), and $p_j \in \mathbb{R}$, the last expression equals $p_j\, \langle \psi_p, \varphi\rangle$, which is the required result. □

The generalized momentum eigenstates satisfy a distributional orthogonality relation that replaces the discrete orthonormality of the bound modes.

Proposition 6.7 (Generalized orthogonality)**.** *For* $p, p' \in \mathbb{R}^3$*, the generalized momentum eigenstates of Definition 6.5 satisfy*

$$\int_{\mathbb{R}^3} \overline{\psi_p(x)}\, \psi_{p'}(x)\, \mathrm{d}^3x = \delta^{(3)}(p - p'), \tag{14}$$

where the integral is understood in the distributional sense and $\delta^{(3)}$ *is the three-dimensional Dirac delta distribution.*

Proof. Substituting Definition 6.5:

$$\int_{\mathbb{R}^3} \overline{\psi_p(x)}\, \psi_{p'}(x)\, \mathrm{d}^3x = \frac{1}{(2\pi\Phi_0)^3} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}p\cdot x/\Phi_0}\, \mathrm{e}^{\,\mathrm{i}p'\cdot x/\Phi_0}\, \mathrm{d}^3x = \frac{1}{(2\pi\Phi_0)^3} \int_{\mathbb{R}^3} \mathrm{e}^{\,\mathrm{i}(p'-p)\cdot x/\Phi_0}\, \mathrm{d}^3x.$$

The integral on the right is the distributional Fourier transform of the constant function 1 evaluated at $(p' - p)/\Phi_0$, which yields $(2\pi\Phi_0)^3\, \delta^{(3)}(p' - p) = (2\pi\Phi_0)^3\, \delta^{(3)}(p - p')$. Dividing by $(2\pi\Phi_0)^3$ gives the result Eq. (14). □

Remark 6.8. *Within the NUVO framework, the generalized momentum eigenstates ψ_p represent idealized transport configurations of unbounded spatial extent: closure transport modes in which the phase gradient $\nabla\phi = p/\Phi_0$ is spatially uniform and the closure density is spread uniformly across all of $\mathbb{R}^3$. Such configurations are not physically realizable as normalized closure states, since they are not square-integrable; their integrated closure diverges. They enter the formalism as a complete basis for expanding physical closure states in $\mathcal{H}$, as established in Sec. 7, but they do not themselves represent admissible transport configurations. This is consistent with the role of the extended space $\mathcal{S}'(\mathbb{R}^3)$ in the rigged Hilbert space triple: it provides a representational extension of $\mathcal{H}$ without enlarging the physical state space.*

7 Completeness and the Resolution of the Identity

The self-adjoint transport generators established in Sec. 5 and the generalized eigenstate framework developed in Sec. 6 together provide the ingredients for a complete spectral representation of arbitrary closure states in $\mathcal{H}$. The present section assembles these ingredients into three results. The spectral theorem provides a canonical decomposition of each self-adjoint generator as an integral over its spectrum with respect to a projection-valued measure. The resolution of the identity expresses this decomposition concretely in terms of the generalized momentum eigenstates of Sec. 6.3. The Parseval identity then translates the normalization condition of Sec. 3 into a conservation law for the spectral coefficients of any closure state, closing the logical arc of the paper.

7.1 The Spectral Theorem for Self-Adjoint Generators

The spectral theorem is the central structural result of the theory of self-adjoint operators on Hilbert spaces. It generalizes the diagonalization of Hermitian matrices to the infinite-dimensional setting and is the mathematical foundation for the decomposition of observables into their eigenvalue contributions. The precise formulation requires the notion of a projection-valued measure.

Definition 7.1 (Projection-valued measure)**.** *A* projection-valued measure *(PVM) on* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *with values in $\mathcal{H}$ is a map $E : \mathcal{B}(\mathbb{R}) \to \mathcal{B}(\mathcal{H})$ from the Borel sigma-algebra of $\mathbb{R}$ to the set of bounded self-adjoint projections on $\mathcal{H}$, satisfying:*

(i) $E(\emptyset) = 0$ *and* $E(\mathbb{R}) = \hat{\mathbf{1}}$.

(ii) *For disjoint $B_1, B_2 \in \mathcal{B}(\mathbb{R})$: $E(B_1 \cup B_2) = E(B_1) + E(B_2)$.*

(iii) *For any $B_1, B_2 \in \mathcal{B}(\mathbb{R})$: $E(B_1)\, E(B_2) = E(B_1 \cap B_2)$.*

(iv) *For any $\Psi \in \mathcal{H}$, the scalar measure $B \mapsto \langle \Psi, E(B)\, \Psi \rangle_{\mathcal{H}}$ is a regular Borel measure on $\mathbb{R}$.*

Remark 7.2. *The projectors $\{E(B)\}$ appearing in Definition 7.1 are the infinite-dimensional counterparts of the projector-valued observable structure established in the QB-series. In QB4 through QB7, projectors $\hat{P}_\alpha$ in the finite-dimensional algebra $\mathsf{P}(\mathcal{H}^{\text{fin}})$ were associated with measurement outcome channels. The projection-valued measure E_A of the spectral theorem is the continuum generalization of this structure: the projector $E_A(B)$ projects onto the subspace of $\mathcal{H}$ spanned by eigenstates of A with eigenvalues in the Borel set $B \subset \mathbb{R}$. The observable structure of the QB-series is thereby embedded in the spectral theory of the present section.*

Theorem 7.3 (Spectral theorem for transport generators)**.** *Let A be a self-adjoint operator on $\mathcal{H}$ with domain $\mathcal{D}(A) \subset \mathcal{H}$. Then there exists a unique projection-valued measure E_A on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that*

$$A = \int_{\mathbb{R}} a \, \mathrm{d}E_A(a), \tag{15}$$

in the sense that $\langle \Psi, A\,\Phi\rangle_{\mathcal{H}} = \int_{\mathbb{R}} a \, \mathrm{d}\, \langle \Psi, E_A(a)\,\Phi\rangle_{\mathcal{H}}$ for all $\Psi \in \mathcal{H}$ and $\Phi \in \mathcal{D}(A)$. Moreover, for any $\Psi \in \mathcal{H}$, the spectral decomposition

$$\Psi = \int_{\sigma(A)} \mathrm{d}E_A(a)\,\Psi \tag{16}$$

holds in the $\mathcal{H}$-norm, and the map E_A decomposes over the discrete and continuous parts of $\sigma(A)$ as

$$\int_{\sigma(A)} \mathrm{d}E_A(a)\,\Psi = \sum_{a\,\in\,\sigma_{\mathrm{disc}}(A)} E_A(\{a\})\,\Psi + \int_{\sigma_{\mathrm{cont}}(A)} \mathrm{d}E_A(a)\,\Psi, \tag{17}$$

where the sum over isolated eigenvalues converges in $\mathcal{H}$ and the integral over the continuous spectrum is a Hilbert-space-valued Lebesgue–Stieltjes integral.

Proof. This is the spectral theorem for self-adjoint operators on Hilbert spaces, a classical result of functional analysis [2]. Its proof is outside the scope of the present paper. The theorem applies to each of the self-adjoint transport generators $\hat{p}_j$ and $\hat{H}$ by virtue of Theorem 5.3, which established their essential self-adjointness and the existence of unique self-adjoint closures on the Sobolev domains $H^1(\mathbb{R}^3)$. □

Remark 7.4. *The content of Theorem 7.3 within the NUVO program is the identification of the physical transport generators with the abstract self-adjoint operators to which the spectral theorem applies. The theorem itself is cited as a classical result; the derivation of its hypotheses from transport closure structure is the work of the present paper. Specifically: the self-adjointness required by the theorem was established in Sec. 5, and the Hilbert space structure on which the theorem acts was constructed in Sec. 4. The spectral theorem therefore represents the convergence of those two threads of the present paper into a single, complete representational result.*

7.2 The Resolution of the Identity

For the momentum operator, whose spectrum is purely continuous, the abstract spectral decomposition Eq. (16) takes a concrete form in terms of the generalized eigenstates of Sec. 6.3. This concrete form is the resolution of the identity: an expression of the identity operator $\hat{\mathbf{1}}$ on $\mathcal{H}$ as an integral over the rank-one operators $|\psi_p\rangle\langle\psi_p|$ associated with the generalized momentum eigenstates.

Proposition 7.5 (Resolution of the identity)**.** *The generalized momentum eigenstates $\{\psi_p\}_{p\in\mathbb{R}^3}$ of Definition 6.5 satisfy the completeness relation*

$$\int_{\mathbb{R}^3} \psi_p(x)\,\overline{\psi_p(y)}\,\mathrm{d}^3p = \delta^{(3)}(x-y) \tag{18}$$

in the distributional sense. Consequently, every $\Psi \in \mathcal{H}$ admits the expansion

$$\Psi(x) = \int_{\mathbb{R}^3} \tilde{\Psi}(p)\,\psi_p(x)\,\mathrm{d}^3p, \tag{19}$$

where the momentum-space coefficient function *is*

$$\tilde{\Psi}(p) := \int_{\mathbb{R}^3} \overline{\psi_p(x)}\, \Psi(x)\, \mathrm{d}^3x, \tag{20}$$

and the expansion Eq. (19) *holds in the* $\mathcal{H}$*-norm.*

Proof. Substituting Definition 6.5 into the left-hand side of Eq. (18):

$$\int_{\mathbb{R}^3} \psi_p(x)\, \overline{\psi_p(y)}\, \mathrm{d}^3p = \frac{1}{(2\pi\Phi_0)^3} \int_{\mathbb{R}^3} \mathrm{e}^{\,\mathrm{i}p\cdot x/\Phi_0}\, \mathrm{e}^{-\mathrm{i}p\cdot y/\Phi_0}\, \mathrm{d}^3p = \frac{1}{(2\pi\Phi_0)^3} \int_{\mathbb{R}^3} \mathrm{e}^{\,\mathrm{i}p\cdot(x-y)/\Phi_0}\, \mathrm{d}^3p.$$

The substitution $q = p/\Phi_0$ transforms this to

$$\frac{\Phi_0^3}{(2\pi\Phi_0)^3} \int_{\mathbb{R}^3} \mathrm{e}^{\,\mathrm{i}q\cdot(x-y)}\, \mathrm{d}^3q = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \mathrm{e}^{\,\mathrm{i}q\cdot(x-y)}\, \mathrm{d}^3q = \delta^{(3)}(x-y),$$

where the last equality is the standard distributional Fourier representation of the Dirac delta [2]. This establishes Eq. (18).

To obtain Eq. (19), multiply both sides of Eq. (18) by $\Psi(y)$ and integrate over $y \in \mathbb{R}^3$:

$$\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \psi_p(x)\, \overline{\psi_p(y)}\, \mathrm{d}^3p \right) \Psi(y)\, \mathrm{d}^3y = \int_{\mathbb{R}^3} \delta^{(3)}(x-y)\, \Psi(y)\, \mathrm{d}^3y = \Psi(x).$$

Interchanging the order of integration on the left-hand side, which is justified for $\Psi \in \mathcal{H}$ by the Fubini–Tonelli theorem applied to the distributional kernel, yields

$$\Psi(x) = \int_{\mathbb{R}^3} \psi_p(x) \left(\int_{\mathbb{R}^3} \overline{\psi_p(y)}\, \Psi(y)\, \mathrm{d}^3y \right) \mathrm{d}^3p = \int_{\mathbb{R}^3} \tilde{\Psi}(p)\, \psi_p(x)\, \mathrm{d}^3p,$$

with $\tilde{\Psi}(p)$ as defined in Eq. (20). That the expansion holds in the $\mathcal{H}$-norm follows from the classical Plancherel theorem for the Fourier transform at scale Φ_0, which identifies the map $\Psi \mapsto \tilde{\Psi}$ as a unitary isomorphism of $\mathcal{H}$ onto itself [2]. □

Remark 7.6. *The map* $\Psi \mapsto \tilde{\Psi}$ *defined by Eq.* (20) *is the Fourier transform at scale* Φ_0. *In the NUVO framework,* $\tilde{\Psi}(p)$ *is the momentum-space representation of the closure state: it encodes the amplitude with which the transport configuration of* Ψ *participates in the generalized momentum mode* ψ_p. *The resolution of the identity Eq.* (19) *is therefore a decomposition of any physical closure state into idealized momentum transport modes, weighted by these amplitudes. The unitarity of the Fourier transform at scale* Φ_0*—the Plancherel theorem—is identified in the present framework with the preservation of total closure under the change of representation from position space to momentum space.*

7.3 Discrete–Continuous Decomposition of a General State

The resolution of the identity in Sec. 7.2 treats the purely continuous-spectrum case of the momentum operator. For a Hamiltonian $\hat{H}$ with both discrete and continuous spectrum—the generic situation for physically relevant closure systems, including the hydrogenic sector of the Q-series—the spectral decomposition Eq. (17) takes a mixed form involving both a sum over discrete eigenstates and an integral over generalized continuous-spectrum states.

Let $\{\psi_n\}_{n\geq 1} \subset \mathcal{H}$ denote the orthonormal family of normalizable eigenstates of $\hat{H}$ corresponding to the discrete spectrum $\sigma_{\mathrm{disc}}(\hat{H}) = \{E_n\}_{n\geq 1}$, so that $\hat{H}\,\psi_n = E_n\,\psi_n$ and $\langle \psi_m, \psi_n \rangle_{\mathcal{H}} = \delta_{mn}$.

Let $\{\psi_k\}_{k\in\sigma_{\text{cont}}(\hat{H})} \subset \mathcal{S}'(\mathbb{R}^3)$ denote the corresponding family of generalized eigenstates for the continuous spectrum, satisfying the eigenvalue equation in the distributional sense and the generalized orthogonality $\langle \psi_k, \psi_{k'} \rangle_{\text{gen}} = \delta(k - k')$. The spectral theorem then yields the following decomposition.

Proposition 7.7 (Discrete–continuous spectral decomposition)**.** *For any normalized $\Psi \in \mathcal{H}$, the spectral decomposition of Ψ with respect to $\hat{H}$ takes the form*

$$\Psi = \sum_{n\geq 1} c_n\, \psi_n + \int_{\sigma_{\text{cont}}(\hat{H})} c(k)\, \psi_k\, \mathrm{d}k, \tag{21}$$

where the discrete coefficients are $c_n = \langle \psi_n, \Psi \rangle_{\mathcal{H}}$, the continuous coefficients are $c(k) = \langle \psi_k, \Psi \rangle_{\text{gen}}$, and the expansion holds in the $\mathcal{H}$-norm.

Proof. The decomposition Eq. (21) is the concrete expression of the abstract spectral theorem, Theorem 7.3, applied to $\hat{H}$ and combined with the nuclear spectral theorem of Sec. 6.2, which guarantees that the continuous spectrum contributes generalized eigenstates in $\mathcal{S}'(\mathbb{R}^3)$. The coefficients c_n and $c(k)$ are determined by the spectral measure $E_{\hat{H}}$ via $c_n = \langle \psi_n, \Psi \rangle_{\mathcal{H}}$ and $c(k) = \langle \psi_k, \Psi \rangle_{\text{gen}}$, respectively. Convergence of the sum and the integral in the $\mathcal{H}$-norm follows from the completeness of the spectral measure [2]. □

The decomposition Eq. (21) has a direct interpretation in the NUVO framework. The discrete terms $c_n\, \psi_n$ represent the projection of the closure state Ψ onto the bound holonomic closure modes identified in the Q-series and the QB-series. The integral term $\int c(k)\, \psi_k\, \mathrm{d}k$ represents the projection onto the unbound, continuous-transport sector, which will be treated fully in QM10. The normalization of Ψ in $\mathcal{H}$ then yields the following conservation identity.

Proposition 7.8 (Parseval identity as closure conservation)**.** *Let $\Psi \in \mathcal{H}$ be a normalized closure state with spectral decomposition Eq. (21). Then*

$$\|\Psi\|^2_{\mathcal{H}} = \sum_{n\geq 1} |c_n|^2 + \int_{\sigma_{\text{cont}}(\hat{H})} |c(k)|^2\, \mathrm{d}k = 1. \tag{22}$$

Proof. Compute $\|\Psi\|^2_{\mathcal{H}} = \langle \Psi, \Psi \rangle_{\mathcal{H}}$ by substituting the expansion Eq. (21) into both arguments of the inner product. The discrete eigenstates ψ_n are orthonormal in $\mathcal{H}$ and orthogonal to the continuous generalized eigenstates in the sense of the spectral measure. The cross terms between the discrete sum and the continuous integral vanish by the orthogonality of the spectral subspaces. The result is

$$\langle \Psi, \Psi \rangle_{\mathcal{H}} = \sum_{m,n\geq 1} \overline{c_m}\, c_n\ \langle \psi_m, \psi_n \rangle_{\mathcal{H}} + \iint \overline{c(k)}\, c(k')\ \langle \psi_k, \psi_{k'} \rangle_{\text{gen}}\, \mathrm{d}k\, \mathrm{d}k' = \sum_{n\geq 1} |c_n|^2 + \int |c(k)|^2\, \mathrm{d}k,$$

where the first equality uses $\langle \psi_m, \psi_n \rangle_{\mathcal{H}} = \delta_{mn}$ and $\langle \psi_k, \psi_{k'} \rangle_{\text{gen}} = \delta(k - k')$. Since $\|\Psi\|_{\mathcal{H}} = 1$ by assumption, Eq. (22) follows. □

Remark 7.9. *Equation* (22) *is the Parseval identity expressed in the spectral coefficient representation. In the NUVO framework it carries the interpretation of total-closure conservation in the spectral basis: the quantity $|c_n|^2$ measures the closure content of the state Ψ in the n-th bound mode, and $|c(k)|^2\, \mathrm{d}k$ measures the closure content in the continuous-transport channel with spectral parameter in $[k, k + \mathrm{d}k]$. The identity Eq.* (22) *states that these contributions sum to the total closure, which is unity by the normalization established in Sec. 3. This is the spectral-domain expression of Corollary 3.6: total closure is conserved not only in position space but in every spectral representation of the state.*

Remark 7.10. *The Parseval identity Eq.* (22) *also extends the Born frequency law of QB6 to the full Hilbert space setting. In QB6, the frequency of a coherence-gated interaction event associated with projector* $\hat{P}_n$ *was identified with* $|c_n|^2 = |\langle\psi_n, \Psi\rangle_{\mathcal{H}}|^2$ *in the discrete setting. Equation* (22) *confirms that these frequencies sum to unity across all spectral channels, consistent with the requirement that every interaction event is associated with exactly one spectral outcome. The extension to the continuous sector assigns frequency density* $|c(k)|^2$ *to the generalized channel* k*, in agreement with the standard quantum-mechanical Born rule for continuous observables. No new postulate is required; the extension follows from the Parseval identity and the identification of spectral coefficients with interaction amplitudes established in the QB-series.*

8 Interpretive Clarifications and Scope

The results of the preceding sections establish a complete Hilbert space framework for the scalar–conformal NUVO transport closure system. Before proceeding to the conclusion, it is appropriate to collect the interpretive boundaries that govern the present work and to record precisely what has and has not been established. This practice of explicit interpretive discipline has been maintained throughout the NUVO series and is continued here without modification.

8.1 Normalization as Structural Constraint, Not Probability Postulate

The normalization condition

$$\int_{\mathbb{R}^3} |\Psi(x,t)|^2 \, \mathrm{d}^3x = 1 \tag{23}$$

was established in Sec. 3 as a consequence of two results: the total-closure conservation law of Lemma 3.1, which follows from the divergence-form structure of the continuity relation Eq. (1), and the choice of closure units recorded in Definition 3.3. Neither step appeals to any probabilistic postulate.

In the standard quantum-mechanical formalism, Eq. (23) is introduced as a requirement of the probabilistic interpretation: the squared modulus $|\Psi|^2$ is declared to be a probability density, and normalization is the requirement that total probability equal unity. In the present framework the logical order is reversed. The condition Eq. (23) is derived first, from the geometry of the transport closure system, without any reference to probability. The probabilistic interpretation—the identification of $\int_B |\Psi|^2 \, \mathrm{d}^3x$ with the frequency of position interaction events localized in the region B—was established separately in QB6 as the Born frequency law, and was shown there to be a consequence of coherence-gated interaction dynamics rather than an assumed feature of the state.

The two results are numerically consistent: both assign the value $\int_B |\Psi|^2 \, \mathrm{d}^3x$ to position events in B. They are, however, logically independent within the NUVO framework. The normalization condition does not depend on the Born law, and the Born law does not depend on the present derivation of normalization. This logical independence is not a deficiency; it reflects the fact that the NUVO framework derives from a single geometric foundation two results that in the standard formalism are related only by interpretive convention.

8.2 The Role of $|\Psi|^2$

Throughout the present paper, the quantity $|\Psi(x,t)|^2$ has been identified with the normalized closure density $\tilde{\rho}(x,t)$, as established in QB1 and recalled in Sec. 2.2. This identification is exact and pointwise: the squared modulus of the complex state encoding equals the normalized closure density by the construction of the encoding Eq. (3), not by interpretation.

The identification of $|\Psi(x,t)|^2$ with a position-measurement probability density is a separate and distinct step. It follows from the Born frequency law of QB6, now extended to the full Hilbert space $\mathcal{H}$ by the results of the present paper. Specifically, the extension of the Born law to $\mathcal{H}$ is justified by the following chain of results established here: the inner product of Definition 4.3 extends the holonomic coherence functional of QB3 by Proposition 4.6; the projector-valued measure E_A of the spectral theorem generalizes the finite-dimensional projector algebra of QB4 by Remark 7.2; and the Parseval identity Eq. (22) confirms that the frequency weights $|c_n|^2$ and $|c(k)|^2$ sum to unity across all spectral channels, consistent with the QB6 frequency law, as recorded in Remark 7.10.

Two interpretations of $|\Psi(x,t)|^2$ therefore coexist within the NUVO framework. As a geometric quantity, it is the normalized closure density: a scalar field measuring the local distribution of admissible transport closure configurations, with no probabilistic content. As an operational quantity, it is the position-measurement frequency density: the expected relative frequency of position interaction events in an infinitesimal region around x, derived from the Born frequency law. These two interpretations assign the same numerical value to $|\Psi(x,t)|^2$ at every point and every time. They are distinguished not by their predictions but by their logical derivation: the first is a consequence of the encoding Eq. (3), and the second is a consequence of the coherence-gated interaction dynamics of QB6. This distinction is maintained throughout the QM-series.

8.3 Infinite-Dimensional Extension Without New Ontology

The construction of $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C})$ in Sec. 4 is a representational extension of the finite-dimensional pre-Hilbert space $\mathcal{H}^{\mathrm{fin}}$ of QB3. It is important to record explicitly what this extension does and does not introduce into the NUVO framework.

No new physical fields are introduced. The scalar capacity field Λ, the delivery substrate, the closure density ρ, and the transport phase ϕ are the same objects that have appeared throughout the M-, Q-, and QB-series. The Hilbert space $\mathcal{H}$ is not a new arena in which these objects act; it is the mathematical space in which the complex state encoding Ψ of these objects is represented.

No new substrates or interaction mechanisms are introduced. The extension from $\mathcal{H}^{\mathrm{fin}}$ to $\mathcal{H}$ does not posit any new physical degree of freedom. It is entirely analogous to the extension from a finite set of trigonometric polynomials to the full space of square-integrable periodic functions: the functions change, but the underlying domain and the integration measure do not.

No new ontological commitments are introduced. The Hilbert space $\mathcal{H}$, the rigged Hilbert space triple of Definition 6.3, the projection-valued measure of Definition 7.1, and the generalized eigenstates of Definition 6.5 are all mathematical representational objects. They encode existing transport closure structure in a form suited to spectral analysis and do not themselves carry physical content beyond what is already present in the transport closure geometry of the Q-series.

This principle—that representational extensions carry no additional ontological weight—has been applied consistently throughout the NUVO series. The introduction of the complex encoding Ψ in QB1 was explicitly presented as a representational step that altered no underlying physics. The present extension of that encoding from a finite-dimensional to an infinite-dimensional space is of the same character. The physical content of the framework resides in the transport closure geometry; the Hilbert space is the representational medium in which that content is expressed.

8.4 Scope of the Present Construction

The present paper establishes the Hilbert space framework that underlies all subsequent work in the QM-series. It is equally important to record what it does not establish, so that the logical dependencies of subsequent papers are transparent.

The paper does not introduce the superposition principle or derive interference phenomena. The linearity of the transport closure system implies that sums of admissible closure states are admissible, and that phase-coherent superpositions of states with different spatial support produce interference patterns in the closure density. These consequences are developed in QM2, which exploits the continuous superposition structure made available by $\mathcal{H}$ to treat the double-slit configuration as a two-path transport problem.

The paper does not derive the uncertainty relations. The canonical commutation relation Eq. (11), established in Proposition 5.6, is the algebraic foundation from which the Robertson–Schrödinger uncertainty inequality $\Delta\hat{x}^j \cdot \Delta\hat{p}_k \geq \frac{1}{2}\Phi_0\,\delta^j{}_k$ follows by an application of the Cauchy–Schwarz inequality on $\mathcal{H}$. This derivation is carried out in QM3, where the uncertainty relations are established as theorems rather than as principles.

The paper does not establish the full time-dependent Schrödinger equation. The transport closure system of the Q-series yields a Schrödinger-type representation, and Corollary 3.6 establishes that admissible transport preserves the $\mathcal{H}$-norm. The derivation of the time-dependent equation $\mathrm{i}\Phi_0\,\partial_t\Psi = \hat{H}\,\Psi$ as a theorem on $\mathcal{H}$, together with the associated symmetry and conservation structure, is undertaken in QM4.

The paper does not treat multi-particle states, tensor product structure, spin, or entanglement. These topics require the construction of multi-particle configuration spaces as tensor products of single-particle Hilbert spaces of the form established here. The multi-particle framework is developed in QM7, the treatment of spin—as a double-cover holonomy structure on the state space—follows in QM8, and entanglement as a non-factorizability condition on multi-particle closure states is addressed in QM9.

In each case, the complete separable Hilbert space $\mathcal{H}$ constructed in the present paper is the essential prerequisite. The results of QM2 through QM11 are logically dependent on the framework established here and could not be developed within the finite-dimensional setting of the QB-series.

9 Conclusion

9.1 Summary of Results

The present paper has established the complete Hilbert space framework for the scalar–conformal NUVO transport closure system, extending the finite-dimensional pre-Hilbert structure of the QB-series to the full infinite-dimensional setting required for the QM-series. The principal results are as follows.

Total-closure conservation (Lemma 3.1). The continuity relation governing the closure density ρ, recalled from the Q-series, implies that the total integrated closure $C_{\mathrm{tot}} = \int_{\mathbb{R}^3} \rho(x,t)\,\mathrm{d}^3x$ is time-invariant under all admissible transport evolutions. This is a consequence solely of the divergence-form structure of the continuity relation and does not depend on any probabilistic interpretation of ρ.

Normalization as a structural constraint (Definition 3.3 and Proposition 3.4). The conservation of total closure, combined with the linearity of the continuity relation and the freedom to choose closure units, implies that admissible closure states may be represented without loss of generality by a normalized closure density satisfying $\int_{\mathbb{R}^3} \tilde{\rho}(x,t)\,\mathrm{d}^3x = 1$. Via the complex state encoding of QB1, this yields the normalization condition $\int_{\mathbb{R}^3} |\Psi(x,t)|^2\,\mathrm{d}^3x = 1$ as a structural consequence of the transport geometry, not as a probabilistic postulate. Corollary 3.6 records that normalization is preserved for all time under admissible transport, identifying this preservation as the geometric precursor of the unitarity of Schrödinger evolution to be established in QM4.

The Hilbert space of closure states (Lemma 4.4 and Theorem 4.7). The space $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C})$, equipped with the closure inner product $\langle \Psi_1, \Psi_2 \rangle_{\mathcal{H}} = \int_{\mathbb{R}^3} \overline{\Psi_1}\,\Psi_2\,\mathrm{d}^3x$, is a separable complex Hilbert space. Proposition 4.6 confirms that this inner product extends the holonomic coherence functional of QB3 without modification, so that the orthogonality relations and observable structure of the QB-series are faithfully embedded in $\mathcal{H}$.

Essential self-adjointness of transport generators (Lemma 5.1 and Theorem 5.3). The momentum transport generators $\hat{p}_j = -\mathrm{i}\Phi_0\,\partial_j$ of QB2, defined on the dense Schwartz domain $\mathcal{S}(\mathbb{R}^3) \subset \mathcal{H}$, are essentially self-adjoint. Their unique self-adjoint closures have domains equal to the first-order Sobolev spaces $H^1(\mathbb{R}^3)$. Essential self-adjointness guarantees that each generator defines a unique, physically unambiguous observable on $\mathcal{H}$, with no freedom of boundary condition or extension choice. Proposition 5.6 promotes the canonical commutation relation of QB2 to the complete Hilbert space setting, establishing $[\hat{x}^j, \hat{p}_k] = \mathrm{i}\Phi_0\,\delta^j{}_k$ on the common dense domain $\mathcal{S}(\mathbb{R}^3)$.

Spectral theorem for transport generators (Definition 7.1 and Theorem 7.3). Each self-adjoint transport generator admits a unique projection-valued measure E_A on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $A = \int_{\mathbb{R}} a\,\mathrm{d}E_A(a)$, and every closure state $\Psi \in \mathcal{H}$ admits a spectral decomposition $\Psi = \int_{\sigma(A)} \mathrm{d}E_A(a)\,\Psi$ in the $\mathcal{H}$-norm. The projection-valued measure E_A is identified as the infinite-dimensional generalization of the projector algebra of the QB-series, embedding the finite-dimensional observable structure of QB4 through QB7 in the full spectral theory.

Resolution of the identity and Parseval identity (Propositions 7.5, 7.7, and 7.8). The generalized momentum eigenstates $\{\psi_p\}_{p\in\mathbb{R}^3}$, defined in the extended distributional space $\mathcal{S}'(\mathbb{R}^3)$ via the rigged Hilbert space triple of Definition 6.3, satisfy the completeness relation $\int_{\mathbb{R}^3} \psi_p(x)\,\overline{\psi_p(y)}\,\mathrm{d}^3p = \delta^{(3)}(x-y)$, so that every $\Psi \in \mathcal{H}$ expands in the momentum basis via the Fourier transform at scale Φ_0. For a Hamiltonian with both discrete and continuous spectrum, every normalized closure state decomposes as $\Psi = \sum_n c_n\,\psi_n + \int c(k)\,\psi_k\,\mathrm{d}k$, and the Parseval identity $\sum_n |c_n|^2 + \int |c(k)|^2\,\mathrm{d}k = 1$ expresses total-closure conservation in the spectral coefficient representation, extending the Born frequency law of QB6 to the continuous spectrum without new postulate.

9.2 Programmatic Significance

The Hilbert space $\mathcal{H}$ constructed in the present paper is the ambient mathematical setting for all subsequent work in the QM-series. Every paper from QM2 through QM11 operates within this framework and depends on the results established here. Without the complete separable Hilbert space of Theorem 4.7, the superposition of continuous-spectrum states required for the interference analysis of QM2 is not available; without the spectral theorem of Theorem 7.3, the uncertainty relations of QM3 and the angular momentum eigenvalue structure of QM5 cannot be derived; without the generalized eigenstate framework of Sec. 6, the scattering and tunneling analysis of QM10 has no representational foundation. The present paper is in this sense the single structural prerequisite on which the entire QM-series rests.

The essential self-adjointness result of Theorem 5.3 and the spectral theorem of Theorem 7.3 together establish that the transport generators of QB2 are well-defined observables in the full quantum-mechanical sense. In QB2, the operators $\hat{p}_j$ and $\hat{E}$ were identified as representations of transport generators on the finite-dimensional span $\mathcal{H}^{\mathrm{fin}}$. Their promotion to essentially self-adjoint operators on $\mathcal{H}$ confirms that this identification extends without modification to the complete Hilbert space, and that the operator structure derived from transport closure is not an artifact of the finite-dimensional setting but a genuine feature of the full quantum-mechanical framework. The bridge between the geometric NUVO transport closure system and the standard formalism of quantum mechanics is thereby established at the level of unbounded self-adjoint operators and their spectral theory, the most general level at which that formalism operates.

With the results of the present paper in place, the gap between the scalar–conformal NUVO framework and the standard quantum-mechanical state formalism is closed without postulate. The state space is the separable Hilbert space $\mathcal{H}$ of Theorem 4.7, derived as the natural completion of the transport closure state encoding. The normalization condition is Proposition 3.4, derived from closure conservation. The observable structure is the projection-valued measure of Theorem 7.3, derived from the self-adjoint transport generators of QB2. The statistical rule is the Born frequency law of QB6, extended to $\mathcal{H}$ by the Parseval identity of Proposition 7.8. Every element of the quantum-mechanical formalism that the QM-series will require is present in the framework, and each has been derived as a structural consequence of the scalar–conformal transport geometry rather than introduced as an independent assumption.

9.3 Transition to QM2

The complete Hilbert space $\mathcal{H}$ supports arbitrary superpositions of closure states, including superpositions of states whose closure densities have disjoint or overlapping spatial support and whose transport phases are independently defined. The linearity of the transport closure system, established in the Q-series, implies that any finite or norm-convergent linear combination of admissible closure states is itself an admissible closure state. The next paper in the series, QM2, develops the consequences of this linearity in the full Hilbert space setting. The superposition principle is established as a theorem rather than a postulate, following directly from the linear structure of the transport equations on $\mathcal{H}$. Two-path superpositions—configurations in which the closure transport proceeds simultaneously along two spatially separated channels—are then shown to produce interference patterns in the closure density: the phase-coherent sum of two transport modes yields a closure density that is not the sum of the individual closure densities but contains cross-terms determined by the relative phase accumulation along the two paths. This is the scalar–conformal NUVO account of the double-slit experiment, and its derivation depends essentially on the continuous superposition structure and the spectral expansion established in the present paper.

References

[1] Israel M. Gel fand and Naum Ya. Vilenkin. *Generalized Functions, Vol. 4: Applications of Harmonic Analysis*. Academic Press, New York, 1964. Primary reference for: the rigged Hilbert space (Gelfand triple) construction and the nuclear spectral theorem guaranteeing complete generalized eigenstate expansions for self-adjoint operators that stabilize the Schwartz space.

[2] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics, Vol. I: Functional Analysis*. Academic Press, New York, 1972. Primary reference for: Hilbert space theory, the spectral theorem for self-adjoint operators (Ch. VII–VIII), the Riesz–Fischer theorem, the Stone–von Neumann theorem, and the Fourier–Plancherel theory.

[3] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics, Vol. II: Fourier Analysis, Self-Adjointness*. Academic Press, New York, 1975. Primary reference for: essential self-adjointness of differential operators (Ch. X), the Kato–Rellich theorem, Sobolev space domains, and the Fourier-transform characterization of momentum operators on $L^2(\mathbb{R}^3)$.