QM10 — Quantum Scattering: The Lippmann-Schwinger Equation, S-Matrix, and Partial Waves

in Scalar–Conformal NUVO Systems *Preprint, Version 1.0*[*]

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Notation and Conventions

- $\mathcal{M}$ denotes the spacetime manifold.
- η denotes the reference Lorentzian metric (typically Minkowski in a global chart).
- g denotes the physical metric.
- The scalar field $\Lambda : \mathcal{M} \to \mathbb{R}_{>0}$ is the NUVO modulation field.
- The physical metric is scalar–conformal:

$$g_{\mu\nu} = \Lambda^2 \, \eta_{\mu\nu}.$$

- $\Lambda_0 > 0$ denotes the baseline scalar availability level supported by the intrinsic delivery structure of the underlying field. In the absence of localized structural occupation the scalar field satisfies $\Lambda(x) = \Lambda_0$.
- The dimensionless scalar diagnostic is

$$\lambda(x) \coloneqq \frac{\Lambda(x)}{\Lambda_0}.$$

- The scalar field represents the *locally available structural capacity* of the underlying delivery field. Localized structures may reduce this availability through occupation or transport, but the intrinsic delivery baseline Λ_0 remains fixed.
- Greek indices $\mu, \nu, \ldots$ range over spacetime coordinates $0, 1, 2, 3$.
- We use the Einstein summation convention unless explicitly stated otherwise.

Remark 0.1. *Unless otherwise stated, the background signature is* $(-, +, +, +)$.

[*]Bibliography is provisional. Cross-references to companion NUVO-series papers (M-, SR-, Q-, QB-, QM-series) will be updated with Zenodo DOIs in subsequent versions.

Program scope.

Abstract

The transport closure framework of the NUVO program, having established the single-particle Hilbert space in QM1–QM6, the multi-particle tensor product in QM7, the spin degree of freedom in QM8, and the entanglement structure in QM9, is now applied to the problem of two transport closure configurations interacting through a potential. The present paper, QM10, develops the complete theory of non-relativistic quantum scattering within this framework.

The two-body problem is first reduced to the one-body problem in the relative coordinate $\hat{r} = \hat{x}_1 - \hat{x}_2$ with reduced mass $\mu = m_1 m_2/(m_1 + m_2)$, using the center-of-mass separation of QM7. The scattering states $|\mathbf{k}^+\rangle$ and $|\mathbf{k}^-\rangle$ are defined as non-normalizable continuum eigenstates of the full Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$ at positive energy $E > 0$, requiring the rigged Hilbert space extension of QM1 to accommodate generalized eigenstates.

The *Lippmann-Schwinger equation*

$$|\mathbf{k}^+\rangle = |\mathbf{k}\rangle + \hat{G}_0^+ \, \hat{V} \, |\mathbf{k}^+\rangle, \qquad \hat{G}_0^+ = \lim_{\epsilon \to 0^+} (E - \hat{H}_0 + \mathrm{i}\epsilon)^{-1},$$

defines the out-going scattering state as the free momentum eigenstate $|\mathbf{k}\rangle$ corrected by the interaction $\hat{V}$ through the free resolvent $\hat{G}_0^+$. The scattering state $|\mathbf{k}^+\rangle$ has the asymptotic form $\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} + f(\theta,\varphi)\mathrm{e}^{\mathrm{i}kr}/r$ at large r, with $f(\theta,\varphi)$ the scattering amplitude.

The *S-matrix* $\hat{S}$ maps in-states to out-states: $S_{\mathbf{k}'\mathbf{k}} = \langle \mathbf{k}'^- | \mathbf{k}^+ \rangle$. Unitarity of the S-matrix $\hat{S}^\dagger \hat{S} = \hat{\mathbf{1}}$ follows from the conservation laws of QM4 and is equivalent to the *optical theorem* $\sigma_{\mathrm{tot}} = (4\pi/k)\,\mathrm{Im}\, f(0)$. The *T-matrix* relates to the S-matrix by $\hat{S} = \hat{\mathbf{1}} - 2\pi\mathrm{i}\delta(E' - E)\hat{T}$ and satisfies the Lippmann-Schwinger equation for the T-matrix: $\hat{T} = \hat{V} + \hat{V}\hat{G}_0^+\hat{T}$.

The *Born approximation* is the leading-order expansion $\hat{T} \approx \hat{V}$, giving the Born scattering amplitude $f^{(1)}(\mathbf{q}) = -(\mu/2\pi\Phi_0^2)\tilde{V}(\mathbf{q})$ where $\tilde{V}(\mathbf{q}) = \int V(\mathbf{r})\mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{r}}\mathrm{d}^3 r$ is the Fourier transform of the potential at momentum transfer $\mathbf{q} = \mathbf{k}' - \mathbf{k}$.

Partial wave analysis decomposes the scattering amplitude for a central potential $V = V(r)$ as $f(\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)f_\ell P_\ell(\cos\theta)$, where the partial wave amplitude $f_\ell = (\mathrm{e}^{2\mathrm{i}\delta_\ell} - 1)/(2\mathrm{i}k)$ is determined by the phase shift δ_ℓ — the solution of the radial Schrödinger equation in the ℓ-th angular momentum channel.

Spin-dependent scattering on $\mathcal{H}_{\mathrm{full}} = \mathcal{H} \otimes \mathbb{C}^2$ is analyzed via the 2×2 spin amplitude matrix $M(\theta)$, whose elements are the non-flip amplitude $A(\theta)$ and the spin-flip amplitude $B(\theta)$. The post-scattering reduced density matrix $\hat{\rho}_A = \mathrm{Tr}_{\mathrm{spatial}}(\hat{\rho}_{\mathrm{out}})$ gives the spin state of the scattered particle after tracing out the spatial degree of freedom.

No new postulates are introduced. All results follow from the QM4 dynamical framework, the QM5 angular momentum algebra, the QM7 center-of-mass separation, the QM8 full Hilbert space, and the QM9 density matrix formalism.

1 Introduction

1.1 Position Within the QM-Series

The NUVO QM-series has, through QM9, developed two parallel arcs of structure. The single-particle arc (QM1–QM6) established the Hilbert space, the observable and dynamical framework, the angular momentum algebra and its holonomy-derived spectrum, and the harmonic oscillator with its algebraic Fock space structure. The multi-particle arc (QM7–QM9) opened the tensor product sector, derived the spin degree of freedom from the double-cover holonomy, and established the complete theory of bipartite entanglement including the Bell inequality violation as a theorem. In both arcs, the energy of the systems under consideration was either discrete (bound state spectrum: $E_n < 0$ for the hydrogenic levels of QM5, $E_{n_1 n_2} > 0$ but quantized for the coupled oscillator of QM7) or zero (the ground state of the harmonic oscillator family). The present paper,

QM10, opens the *scattering sector*: the regime of positive energy $E > 0$ for two transport closure configurations that approach from large separation, interact through a potential $V(\hat{r})$ for a finite time, and separate again to large distance. The physical content that distinguishes scattering from the bound state problem is the *asymptotic* structure: the initial and final states are free, non-interacting plane waves, and all the information about the interaction is encoded in the change of direction and spin state between the initial and final free states.

The scattering problem reduces to a one-body problem in the relative coordinate through the center-of-mass separation of QM7. The Hamiltonian $\hat{H}^{(2)} = \hat{H}_{\mathrm{CM}} \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes \hat{H}_{\mathrm{rel}}$ factorizes into a free center-of-mass motion and a relative motion governed by $\hat{H}_{\mathrm{rel}} = \Phi_0^2/(2\mu)\nabla_r^2 + V(r)$, where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass of QM7 Theorem 6.1. All scattering observables — the differential cross section, the total cross section, the scattering amplitude — are determined by the relative Hamiltonian $\hat{H}_{\mathrm{rel}}$ alone. This reduction is the single most important simplification of the two-body scattering problem: the full two-body problem on $\mathcal{H}^{(2)} = \mathcal{H}_1 \otimes \mathcal{H}_2$ with its complicated tensor product structure reduces to a one-body problem on $\mathcal{H}_{\mathrm{rel}} = L^2(\mathbb{R}^3)$ with the effective Hamiltonian $\hat{H}_{\mathrm{rel}}$.

QM10 introduces two structural elements that are qualitatively new to the QM-series. The first is the *rigged Hilbert space*. The scattering states are non-normalizable continuum eigenstates $|\mathbf{k}\rangle$ of the free Hamiltonian $\hat{H}_0 = \Phi_0^2 k^2/(2\mu)$ at positive energy $E_k > 0$; they are not elements of $\mathcal{H} = L^2(\mathbb{R}^3)$ (since $\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}$ is not square-integrable) but of the larger space Φ^* of tempered distributions in the Gelfand triple $\Phi \subset \mathcal{H} \subset \Phi^*$. The generalized completeness relation $\int |\mathbf{k}\rangle\langle\mathbf{k}| \, \mathrm{d}^3k = \hat{\mathbf{1}}_{\mathcal{H}}$ and the generalized orthonormality $\langle\mathbf{k}|\mathbf{k}'\rangle = \delta^{(3)}(\mathbf{k} - \mathbf{k}')$ hold in this extended sense and are the continuum analogues of the discrete completeness and orthonormality relations of QM1. The second new element is the *S-matrix*. For bound states, the primary observable is the energy spectrum. For scattering states, the primary observable is the S-matrix $\hat{S}$, a unitary operator that maps the in-asymptotic free state $|\mathbf{k}\rangle$ to the out-asymptotic free state $|\mathbf{k}'\rangle$ with amplitude $S_{\mathbf{k}'\mathbf{k}} = \langle\mathbf{k}'^{-}|\mathbf{k}^{+}\rangle$. Unitarity of the S-matrix is a consequence of the conservation laws of QM4 (probability conservation under unitary time evolution) and has the immediate corollary — the optical theorem — that the total cross section equals $4\pi/k$ times the imaginary part of the forward scattering amplitude.

The density matrix formalism of QM9 makes its first appearance in a physical calculation in Sec. 8. For a spin-$\frac{1}{2}$ particle scattering from a target with spin-orbit coupling, the post-scattering state is an entangled spatial-spin state in $\mathcal{H}_{\mathrm{full}} = \mathcal{H} \otimes \mathbb{C}^2$. Tracing out the spatial degree of freedom yields the post-scattering spin density matrix $\hat{\rho}_{A\mathrm{out}} = \mathrm{Tr}_{\mathrm{spatial}}(M\hat{\rho}_{\mathrm{in}}M^\dagger)/(M\hat{\rho}_{\mathrm{in}}M^\dagger)$, from which the polarization of the scattered beam is computed via the Sherman function. This computation is the synthesis of all prior spin and entanglement structure: the full Hilbert space of QM8, the density matrix of QM9, and the scattering amplitude of the present paper combine in a single physical prediction. QM11 will extend the scattering framework to the relativistic sector, replacing the Pauli equation with the Dirac equation and deriving the g-factor $g = 2$ and the complete hydrogen fine structure as relativistic consequences.

1.2 Objective of the Present Work

The central objective of the present paper is to derive the complete non-relativistic scattering theory for two transport closure configurations interacting through a potential, using the two-body framework of QM7, the angular momentum algebra of QM5, the full spin Hilbert space of QM8, and the density matrix formalism of QM9. Specifically, the paper establishes six claims.

1. The free Hamiltonian $\hat{H}_0 = \Phi_0^2 k^2/(2\mu)$ on $\mathcal{H} = L^2(\mathbb{R}^3)$ has a purely continuous spectrum $[0, \infty)$ with generalized eigenstates $|\mathbf{k}\rangle \in \Phi^*$ (elements of the dual of the Schwartz space $\Phi =$

$\mathcal{S}(\mathbb{R}^3)$) satisfying $\langle \mathbf{r}|\mathbf{k}\rangle = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}/(2\pi)^{3/2}$, the generalized orthonormality $\langle \mathbf{k}|\mathbf{k}'\rangle = \delta^{(3)}(\mathbf{k}-\mathbf{k}')$, and the completeness $\int |\mathbf{k}\rangle\langle\mathbf{k}|\,\mathrm{d}^3k = \hat{\mathbf{1}}_{\mathcal{H}}$. The Rayleigh expansion of the plane wave in spherical harmonics and spherical Bessel functions connects the plane wave basis to the partial wave basis of claim (5).

2. The *Lippmann-Schwinger equation* $|\mathbf{k}^+\rangle = |\mathbf{k}\rangle + \hat{G}_0^+\hat{V}|\mathbf{k}^+\rangle$, where $\hat{G}_0^+ = \lim_{\epsilon\to 0^+}(E_k + \mathrm{i}\epsilon - \hat{H}_0)^{-1}$ is the retarded free resolvent, defines the out-going scattering state $|\mathbf{k}^+\rangle$ with position-space representation $\psi_{\mathbf{k}}^+(\mathbf{r}) \to (2\pi)^{-3/2}[\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} + f(\theta,\varphi)\mathrm{e}^{\mathrm{i}kr}/r]$ as $r\to\infty$, where $f(\theta,\varphi) = -(\mu/2\pi\Phi_0^2)\langle\mathbf{k}'|\hat{V}|\mathbf{k}^+\rangle$ is the scattering amplitude and the differential cross section is $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f|^2$.

3. The *S-matrix* $\hat{S} = \hat{\Omega}_-^\dagger\hat{\Omega}_+$ (where $\hat{\Omega}_\pm$ are the Møller wave operators) is unitary: $\hat{S}^\dagger\hat{S} = \hat{\mathbf{1}}$. The T-matrix is related by $S_{\mathbf{k}'\mathbf{k}} = \delta^{(3)}(\mathbf{k}'-\mathbf{k}) - 2\pi\mathrm{i}\delta(E'-E)T_{\mathbf{k}'\mathbf{k}}$, where $T_{\mathbf{k}'\mathbf{k}} = \langle\mathbf{k}'|\hat{V}|\mathbf{k}^+\rangle$, and the *optical theorem* $\sigma_{\mathrm{tot}} = (4\pi/k)\,\mathrm{Im}\,f(0)$ follows from S-matrix unitarity.

4. The *Born approximation* $\hat{T}\approx\hat{V}$ gives the first-order scattering amplitude $f^{(1)}(\mathbf{q}) = -(\mu/2\pi\Phi_0^2)\tilde{V}(\mathbf{q})$, where $\mathbf{q} = \mathbf{k}'-\mathbf{k}$ is the momentum transfer and $\tilde{V}(\mathbf{q}) = \int V(\mathbf{r})\mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{r}}\mathrm{d}^3r$ is the Fourier transform of the potential. For the Yukawa potential $V = -g^2\mathrm{e}^{-\lambda r}/r$, the Born amplitude gives $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = (\mu g^2/\Phi_0^2)^2(|\mathbf{q}|^2+\lambda^2)^{-2}$; in the Coulomb limit $\lambda\to 0$ this reduces to the Rutherford formula $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = (\mu e^2/2\Phi_0^2k^2)^2\sin^{-4}(\theta/2)$, which is exact for the Coulomb potential.

5. For a central potential $V = V(r)$, the scattering amplitude decomposes in *partial waves*: $f(\theta) = k^{-1}\sum_{\ell=0}^{\infty}(2\ell+1)\mathrm{e}^{\mathrm{i}\delta_\ell}\sin\delta_\ell\,P_\ell(\cos\theta)$, where the *phase shift* $\delta_\ell\in\mathbb{R}$ is determined by the asymptotic behavior $u_\ell(r)\to\sin(kr-\ell\pi/2+\delta_\ell)$ of the solution of the radial Schrödinger equation. The partial wave and total cross sections are $\sigma_\ell = (4\pi/k^2)(2\ell+1)\sin^2\delta_\ell$ and $\sigma_{\mathrm{tot}} = \sum_\ell\sigma_\ell$; Levinson's theorem $\delta_\ell(0) = n_\ell\pi$ relates the zero-energy phase shift to the number of bound states.

6. For a spin-$\frac{1}{2}$ particle scattering from a target with spin-orbit coupling on $\mathcal{H}_{\mathrm{full}} = \mathcal{H}\otimes\mathbb{C}^2$, the *spin scattering amplitude matrix* $M(\theta) = A(\theta)\sigma_0 + B(\theta)(\hat{n}\cdot\boldsymbol{\sigma})$ gives the differential cross section $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |A|^2+|B|^2$ for an unpolarized incident beam, the post-scattering spin density matrix $\hat{\rho}_{A\mathrm{out}} = M\hat{\rho}_{\mathrm{in}}M^\dagger/\mathrm{Tr}(M\hat{\rho}_{\mathrm{in}}M^\dagger)$, and the polarization of the scattered beam $P_{\hat{n}} = 2\,\mathrm{Im}(A^*B)/(|A|^2+|B|^2)$ (the Sherman function).

Claims (1) through (6) follow the logical chain of the paper. The rigged Hilbert space of claim (1) provides the mathematical setting; the Lippmann-Schwinger equation of claim (2) defines the scattering states and the scattering amplitude; the S-matrix of claim (3) encodes all scattering information and its unitarity gives the optical theorem; the Born approximation of claim (4) gives the leading-order amplitude explicitly in terms of the potential; the partial wave analysis of claim (5) provides the exact decomposition for central potentials; and the spin scattering of claim (6) combines all the preceding structure with the QM8 and QM9 frameworks in a single physical calculation.

1.3 What Is Not Assumed

The present work maintains without modification the interpretive discipline of the prior series. Five exclusions are of particular importance for QM10.

The rigged Hilbert space is not postulated as a new axiom of quantum mechanics. It is the correct mathematical extension of the QM1 Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)$ required to accommodate the generalized eigenstates of the free Hamiltonian at positive energy. Every physical quantity in QM10 is expressed as a matrix element $\langle\phi|\hat{O}|\psi\rangle$ with $\phi,\psi\in\mathcal{H}$ (normalizable states); the plane

wave states $|\mathbf{k}\rangle \in \Phi^*$ appear only in intermediate steps and in the completeness relation, never as physical states of the system. The rigged Hilbert space is a mathematical bookkeeping device that allows the Dirac notation $|\mathbf{k}\rangle$ to be used consistently; no new physical content is introduced.

Asymptotic completeness is asserted for short-range potentials but not proved. The statement that the Møller wave operators $\hat{\Omega}_\pm$ are partial isometries with range$(\hat{\Omega}_+)$ = range$(\hat{\Omega}_-)$ (which is what is needed for S-matrix unitarity via $\hat{S} = \hat{\Omega}_-^\dagger \hat{\Omega}_+$) is a non-trivial functional analytic theorem. For short-range potentials satisfying $|V(\mathbf{r})| \leq C(1+|\mathbf{r}|)^{-1-\epsilon}$ for some $C, \epsilon > 0$, asymptotic completeness was proved by Enss [1] and by Sigal and Soffer [5]. For the Coulomb potential $V = -e^2/r$ (which decays only as $1/r$), the standard Møller operators do not converge and the Dollard modification of the wave operators is required; the Coulomb case is noted but its detailed treatment is deferred.

The Coulomb scattering theory with the Dollard modification is not developed. The Coulomb potential requires a modified definition of the wave operators $\hat{\Omega}_\pm^{\text{Dollard}} = \lim_{t \to \mp\infty} \mathrm{e}^{\mathrm{i}\hat{H}t/\Phi_0} \hat{W}(t)$ where $\hat{W}(t)$ contains an additional Coulomb phase correction, because the ordinary Møller limits diverge for $1/r$ potentials. The Rutherford formula derived in Sec. 6.3 from the Born approximation is exact for the Coulomb potential (a special property of the Coulomb case) and so is the correct result despite the breakdown of the standard wave operator formalism; the rigorous Coulomb scattering theory confirming this is cited [3] rather than developed.

Multi-channel and inelastic scattering are not treated. In the present paper, the two transport closure configurations have fixed internal states before and after the collision (elastic scattering). The general case where the configurations can change their internal state during the collision (inelastic scattering, with a different final state from the initial) requires a coupled-channel formalism: a matrix-valued S-matrix with channel indices. This extension, together with the Breit-Wigner resonance formula for quasi-bound states that appear as poles of the S-matrix in the complex energy plane, is deferred.

Relativistic corrections to the scattering amplitude are not derived. The scattering amplitude of QM10 is the leading term in a v/c expansion; the corrections of order $(v/c)^2$ arise from the relativistic dispersion relation $E^2 = (pc)^2 + (mc^2)^2$, from magnetic interactions, and from radiative corrections in quantum electrodynamics. These corrections are derived in QM11 from the Dirac equation, making QM10 the non-relativistic foundation whose relativistic extension is the content of QM11.

1.4 Structure of the Paper

Sec. 2 recalls the center-of-mass and relative coordinate separation from QM7, the dynamical framework and conservation laws from QM4, the angular momentum algebra and partial wave prerequisites from QM5, and the full Hilbert space and density matrix formalisms from QM8 and QM9 that are used in the spin-dependent scattering analysis. Sec. 3 introduces the rigged Hilbert space as the mathematical setting for the scattering problem, defines the plane wave states $|\mathbf{k}\rangle$ as generalized eigenstates of $\hat{H}_0$ in the Gelfand triple $\Phi \subset \mathcal{H} \subset \Phi^*$, and derives the generalized completeness and orthonormality relations together with the Rayleigh plane wave expansion. Sec. 4 defines the retarded free resolvent $\hat{G}_0^+$, derives the Lippmann-Schwinger equation for the in- and out-scattering states, establishes their position-space form as an integral equation, and derives the asymptotic form and the connection between the Lippmann-Schwinger kernel and the scattering amplitude. Sec. 5 defines the Møller wave operators and the S-matrix, establishes the S-matrix unitarity (invoking asymptotic completeness), derives the S-matrix to T-matrix relation, and proves the optical theorem as a consequence of unitarity. Sec. 6 derives the Born approximation as the leading term in the iterative solution of the Lippmann-Schwinger equation for the T-matrix, com-

putes the Born amplitude for the Yukawa potential, and derives the Rutherford formula as the Coulomb limit. Sec. 7 derives the partial wave decomposition of the scattering amplitude for a central potential, defines the phase shifts from the asymptotic boundary condition on the radial wave function, derives the partial wave cross sections, and records the properties of the phase shifts including Levinson's theorem and the scattering length. Sec. 8 sets up the spin-dependent scattering problem on $\mathcal{H}_{\text{full}} = \mathcal{H} \otimes \mathbb{C}^2$, introduces the spin scattering amplitude matrix $M = A\sigma_0 + B(\hat{n} \cdot \boldsymbol{\sigma})$, derives the differential cross section and the post-scattering spin density matrix using the QM9 partial trace formalism, and derives the Sherman function for the polarization of the scattered beam. Sec. 9 records the derivational status of the rigged Hilbert space and asymptotic completeness, and the scope of the present construction. Sec. 10 summarizes the fourteen principal results, records the programmatic significance of the scattering sector, and prepares the transition to QM11.

2 Recalled Structure from Prior Papers

The present section collects the results from QM4, QM5, QM7, QM8, and QM9 that are directly required for the derivations of Secs. 3–8. Nothing in this section is new. The recalled material falls into four categories: the two-body reduction from QM7 that converts the scattering problem from a tensor product Hilbert space to a single Hilbert space; the dynamical framework and conservation laws from QM4 that underlie the S-matrix unitarity and the optical theorem; the angular momentum decomposition from QM5 that is the foundation of the partial wave analysis; and the spin and density matrix structures from QM8 and QM9 that enter the spin-dependent scattering analysis of Sec. 8.

2.1 The Two-Body Reduction from QM7

The following results from QM7 reduce the two-body scattering problem to a one-body problem in the relative coordinate.

Center-of-mass and relative coordinates (QM7 Theorem 6.1). For two transport closure configurations with masses m_1 and m_2 and position operators $\hat{x}_1$ and $\hat{x}_2$, the total mass $M = m_1 + m_2$ and reduced mass $\mu = m_1 m_2 / M$ define:

$$\hat{R} = \frac{m_1 \hat{x}_1 + m_2 \hat{x}_2}{M}, \qquad \hat{P} = \hat{p}_1 + \hat{p}_2, \tag{1}$$

$$\hat{r} = \hat{x}_1 - \hat{x}_2, \qquad \hat{p}_{\text{rel}} = \frac{m_2 \hat{p}_1 - m_1 \hat{p}_2}{M}. \tag{2}$$

These satisfy the canonical commutation relations: $[\hat{R}_j, \hat{P}_k] = \mathrm{i}\Phi_0 \delta_{jk}$, $[\hat{r}_j, \hat{p}_{\text{rel},k}] = \mathrm{i}\Phi_0 \delta_{jk}$, and all cross commutators vanish.

Factorization of the two-body Hamiltonian (QM7 Theorem 6.1). For a two-body Hamiltonian $\hat{H}^{(2)} = \hat{p}_1^2/(2m_1) + \hat{p}_2^2/(2m_2) + V(\hat{x}_1 - \hat{x}_2)$ with a translation-invariant potential:

$$\hat{H}^{(2)} = \hat{H}_{\text{CM}} \otimes \hat{\mathbf{1}}_{\text{rel}} + \hat{\mathbf{1}}_{\text{CM}} \otimes \hat{H}_{\text{rel}}, \tag{3}$$

where

$$\hat{H}_{\text{CM}} = \frac{\hat{P}^2}{2M}, \qquad \hat{H}_{\text{rel}} = \frac{\hat{p}_{\text{rel}}^2}{2\mu} + V(\hat{r}) = \hat{H}_0 + \hat{V}. \tag{4}$$

The center-of-mass motion is that of a free particle of mass M; the relative motion is governed by the one-body Hamiltonian $\hat{H}_{\text{rel}}$ on $\mathcal{H}_{\text{rel}} = L^2(\mathbb{R}^3)$ with effective mass μ. All scattering observables are determined by $\hat{H}_{\text{rel}}$ alone.

Remark 2.1. *The reduction of the two-body problem to the one-body problem is the structural content of QM7 Theorem 6.1 applied to the scattering regime. In the bound state context of QM7, the same reduction produced the coupled oscillator Hamiltonian with its discrete normal mode spectrum. In the scattering context of QM10, the same reduction produces the relative Hamiltonian* $\hat{H}_{\rm rel} = \hat{H}_0 + \hat{V}$ *whose continuous positive-energy spectrum* $\sigma(\hat{H}_{\rm rel}) \cap (0, \infty)$ *is the scattering sector. The two regimes are complementary: the bound state sector (*$E < 0$*, discrete spectrum) and the scattering sector (*$E > 0$*, continuous spectrum) together constitute the complete spectral data of* $\hat{H}_{\rm rel}$*, related by the Levinson theorem of Sec.* **??** *which connects the two.*

2.2 The Dynamical Framework and Conservation Laws from QM4

The following results from QM4 are the foundation of the S-matrix unitarity and the optical theorem.

Stone's theorem and unitary time evolution (QM4 Theorem 3.1). The self-adjoint Hamiltonian $\hat{H}_{\rm rel}$ on $\mathcal{H}_{\rm rel}$ generates a strongly continuous one-parameter unitary group $U(t) = \mathrm{e}^{-\mathrm{i}\hat{H}_{\rm rel}t/\Phi_0}$ satisfying $U(t)^\dagger U(t) = \hat{\mathbf{1}}$ for all $t \in \mathbb{R}$. This unitarity of time evolution is the primary input to the S-matrix unitarity of Theorem 5.6: the S-matrix $\hat{S} = \lim_{t\to+\infty} U_{\rm rel}(t)U_0(-t)^\dagger U_0(-t)U_{\rm rel}(-t)$ (schematically) inherits its unitarity from $U(t)^\dagger U(t) = \hat{\mathbf{1}}$.

Conservation laws (QM4 Theorem 5.2). An observable $\hat{A}$ is conserved by $\hat{H}_{\rm rel}$ if and only if $[\hat{H}_{\rm rel}, \hat{A}] = 0$. For a central potential $V = V(r)$:

$$[\hat{H}_{\rm rel}, \hat{L}^2] = 0, \tag{5}$$

$$[\hat{H}_{\rm rel}, \hat{L}_3] = 0. \tag{6}$$

These conservation laws have two consequences for QM10: (a) the scattering amplitude for a central potential depends only on the scattering angle θ (not on the azimuthal angle φ), enabling the partial wave decomposition of Sec. 7; and (b) the phase shift δ_ℓ in each angular momentum channel ℓ is a real number (conservation of probability in each channel separately), as established in Proposition 7.6.

The Heisenberg equation of motion and the self-adjoint domain (QM4 Theorem 4.2). The self-adjointness of $\hat{H}_{\rm rel}$ on the Sobolev domain $\mathcal{D}(\hat{H}_{\rm rel}) \subset \mathcal{H}_{\rm rel}$ for potentials in the Kato class is a hypothesis whose verification for specific potentials (Yukawa, Coulomb) is a standard result cited from the functional analysis literature [4]. The Heisenberg equation $\mathrm{d}\hat{A}/\mathrm{d}t = (\mathrm{i}/\Phi_0)[\hat{H}_{\rm rel}, \hat{A}]$ is used in the proof of the optical theorem (Theorem 5.8) through the probability conservation identity applied to the current operator.

Remark 2.2. *The Kato-Rellich theorem (QM4 Theorem 4.2) guarantees that* $\hat{H}_{\rm rel} = \hat{H}_0 + \hat{V}$ *is self-adjoint on* $\mathcal{D}(\hat{H}_0)$ *whenever* $\hat{V}$ *is* $\hat{H}_0$*-bounded with relative bound less than* 1*. For the Yukawa potential* $V(r) = -g^2\mathrm{e}^{-\lambda r}/r$*:* $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ *for any* $\lambda > 0$*, which is a sufficient condition. For the Coulomb potential* $V(r) = -e^2/r$*: by the Kato inequality* $|-e^2/r| \leq \epsilon(-\nabla^2) + C_\epsilon$ *for any* $\epsilon > 0$ *and a constant* C_ϵ *depending on* ϵ*, the Coulomb potential is* $\hat{H}_0$*-bounded with relative bound* 0 *and* $\hat{H}_{\rm rel}$ *is self-adjoint on* $\mathcal{D}(\hat{H}_0) = H^2(\mathbb{R}^3)$ *for* $d = 3$ *[4].*

2.3 The Angular Momentum Structure from QM5

The following results from QM5 are the direct inputs to the partial wave analysis of Sec. 7.

The spherical harmonic expansion (QM5 Theorems 6.2 and 7.1). The joint eigenstates of $\hat{L}^2$ and $\hat{L}_3$ on $L^2(S^2)$ are the spherical harmonics $Y_\ell^m(\theta, \varphi)$, orthonormal with respect to the solid

angle measure: $\int_{S^2} Y_\ell^{m*}(\hat{r}) Y_{\ell'}^{m'}(\hat{r})\,\mathrm{d}\Omega = \delta_{\ell\ell'}\delta_{mm'}$. The resolution of the identity on $L^2(S^2)$ is $\sum_{\ell,m} Y_\ell^m(\hat{r}) Y_\ell^{m*}(\hat{r}') = \delta(\hat{r}-\hat{r}')$.

The radial decomposition of $L^2(\mathbb{R}^3)$ (QM5 Proposition 6.3). Any state $\Psi \in \mathcal{H}_{\mathrm{rel}} = L^2(\mathbb{R}^3)$ decomposes as $\Psi(\mathbf{r}) = \sum_{\ell,m} r^{-1} u_{\ell m}(r) Y_\ell^m(\hat{r})$, with radial functions $u_{\ell m}(r) \in L^2(\mathbb{R}_{>0})$. Substituting into the Schrödinger equation for $\hat{H}_{\mathrm{rel}} = \hat{H}_0 + \hat{V}$ with central $\hat{V} = V(r)$ separates the angular and radial parts: the angular part gives $\hat{L}^2 Y_\ell^m = \ell(\ell+1)\Phi_0^2 Y_\ell^m$, and the radial part gives the radial Schrödinger equation for $u_\ell(r)$:

$$-\frac{\Phi_0^2}{2\mu} u_\ell''(r) + \left[\frac{\Phi_0^2 \ell(\ell+1)}{2\mu r^2} + V(r)\right] u_\ell(r) = E\, u_\ell(r), \tag{7}$$

with boundary condition $u_\ell(0) = 0$ (regularity at the origin). Equation (7) is the one-dimensional Schrödinger equation with an effective potential $V_{\mathrm{eff}}(r) = V(r) + \Phi_0^2 \ell(\ell+1)/(2\mu r^2)$ that includes the centrifugal barrier.

Spherical Bessel functions (QM5 Proposition 6.4 and the asymptotic analysis). For $V \equiv 0$ (the free radial equation), the two linearly independent solutions are $j_\ell(kr)$ (spherical Bessel, regular at $r = 0$) and $n_\ell(kr)$ (spherical Neumann, irregular at $r = 0$), with asymptotic behaviors:

$$j_\ell(x) \xrightarrow{x\to\infty} \frac{\sin(x - \ell\pi/2)}{x}, \tag{8}$$

$$h_\ell^{(1)}(x) \xrightarrow{x\to\infty} \frac{\mathrm{e}^{\mathrm{i}(x-\ell\pi/2-\pi/2)}}{\mathrm{i}x} = \frac{(-\mathrm{i})^{\ell+1}\mathrm{e}^{\mathrm{i}x}}{\mathrm{i}x}, \tag{9}$$

where $h_\ell^{(1)}(x) = j_\ell(x) + \mathrm{i} n_\ell(x)$ is the outgoing spherical Hankel function. The asymptotic form Eq. (8) determines the free wave behavior at large r in the partial wave analysis, and the deviation from this form in the presence of V is captured by the phase shift δ_ℓ.

Remark 2.3. *The conservation law* $[\hat{H}_{\mathrm{rel}}, \hat{L}^2] = 0$ *(Eq. (5)) has a direct consequence for the partial wave decomposition: since* $\hat{L}^2$ *is conserved, each angular momentum channel* ℓ *scatters independently. The S-matrix is block-diagonal in the angular momentum basis, with the* ℓ*-th block being the* 1×1 *complex number* $S_\ell = \mathrm{e}^{2\mathrm{i}\delta_\ell}$ *(a pure phase, since unitarity in each channel requires* $|S_\ell| = 1$*). The total S-matrix is therefore characterized by the countable set of real numbers* $\{\delta_\ell\}_{\ell=0}^\infty$*, one per angular momentum channel. This is the content of the partial wave decomposition of Theorem 7.4.*

2.4 The Full Hilbert Space, Pauli Algebra, and Density Matrix from QM8 and QM9

The following results from QM8 and QM9 enter the spin-dependent scattering analysis of Sec. 8.

The full spin-$\frac{1}{2}$ *Hilbert space* (QM8 Definition 5.1). The full single-particle Hilbert space for a spin-$\frac{1}{2}$ configuration is $\mathcal{H}_{\mathrm{full}} = \mathcal{H} \otimes \mathbb{C}^2 = L^2(\mathbb{R}^3, \mathbb{C}^2)$, with elements represented as two-component spinors $\Psi(\mathbf{r}) = (\Psi_\uparrow(\mathbf{r}), \Psi_\downarrow(\mathbf{r}))^{\mathsf{T}}$. The observable commutation $[\hat{A} \otimes \sigma_0, \hat{\mathbf{1}} \otimes \hat{B}] = 0$ (QM8 Proposition 5.3) ensures that spatial observables (including the scattering Hamiltonian) and spin observables commute on $\mathcal{H}_{\mathrm{full}}$.

The Pauli algebra and spin-orbit coupling (QM8 Theorems 4.2 and 7.1). The spin operators $\hat{\mathbf{S}}_j = (\Phi_0/2)\boldsymbol{\sigma}_j$ on $\mathbb{C}^2 = \mathbb{C}^2$ satisfy $\boldsymbol{\sigma}_j \boldsymbol{\sigma}_k = \delta_{jk}\sigma_0 + \mathrm{i}\epsilon_{jkl}\boldsymbol{\sigma}_l$. For spin-dependent scattering, the interaction Hamiltonian on $\mathcal{H}_{\mathrm{full}}$ includes the spin-orbit term $\xi(r)\hat{L} \otimes \hat{\mathbf{S}}$ (QM8 Definition 7.1):

$$\hat{V}_{\mathrm{full}} = V(r) \otimes \sigma_0 + \xi(r)\,(\hat{L} \otimes \sigma_0) \cdot (\hat{\mathbf{1}} \otimes \hat{\mathbf{S}}), \tag{10}$$

where $\xi(r) = (1/2\mu^2c^2)(1/r)(\mathrm{d}V/\mathrm{d}r)$ is the spin-orbit coupling function of QM8 Eq. (**??**). The eigenvalues of $\hat{L}\cdot\hat{\mathbf{S}}$ in the coupled basis $|j, m_j\rangle$ are $(\Phi_0^2/2)[j(j+1)-\ell(\ell+1)-\frac{3}{4}]$ (QM8 Proposition 7.2), and the spin-orbit interaction splits each angular momentum channel ℓ into two sub-channels with $j = \ell \pm \frac{1}{2}$ and different phase shifts $\delta_{\ell,+}$ and $\delta_{\ell,-}$.

The density matrix and partial trace (QM9 Definitions 4.1–4.2 and Theorem 4.2). For the post-scattering analysis of Sec. 8, the full post-scattering state $|\Psi_{\mathrm{out}}\rangle \in \mathcal{H}_{\mathrm{full}}$ is a spatial-spin entangled state. The spin state of the scattered particle is described by the reduced density matrix:

$$\hat{\rho}_{A\mathrm{out}} = \mathrm{Tr}_{\mathrm{spatial}}\big(|\Psi_{\mathrm{out}}\rangle\langle\Psi_{\mathrm{out}}|\big), \tag{11}$$

where the trace is over the spatial (position) degree of freedom, leaving a 2×2 density matrix on $\mathbb{C}^2$. The Born rule for the spin degree of freedom gives the polarization expectation $\langle\hat{n}\cdot\boldsymbol{\sigma}\rangle_{\mathrm{out}} = \mathrm{Tr}_{\mathbb{C}^2}(\hat{\rho}_{A\mathrm{out}}(\hat{n}\cdot\boldsymbol{\sigma}))$ (QM9 Theorem 4.2), the primary observable in spin-dependent scattering experiments.

Remark 2.4. *The spin-orbit coupling Eq.* (10) *modifies the partial wave analysis of Sec. 7 by splitting each ℓ-channel into two sub-channels. For $\ell \geq 1$, the coupled basis $|j, m_j\rangle$ with $j = \ell \pm \frac{1}{2}$ diagonalizes the interaction Hamiltonian Eq.* (10) *in each angular momentum channel (QM8 Proposition 7.2 with $\xi(r)$ replacing $\xi_{n\ell}$). The radial Schrödinger equation in the $j = \ell \pm \frac{1}{2}$ channel has an effective potential that differs by the spin-orbit eigenvalue, giving phase shifts $\delta_{\ell,\pm}$ that are generally different. The spin amplitude matrix $M = A\sigma_0 + B(\hat{n}\cdot\boldsymbol{\sigma})$ of Definition 8.2 encodes this splitting: the non-flip amplitude A is a symmetric combination of $\delta_{\ell,+}$ and $\delta_{\ell,-}$, while the spin-flip amplitude B is proportional to their difference, and vanishes in the absence of spin-orbit coupling (when $\delta_{\ell,+} = \delta_{\ell,-}$ for all ℓ).*

Remark 2.5. *Section 8 is the first section in the QM-series where the results of QM4, QM5, QM7, QM8, and QM9 are all used simultaneously in a single physical calculation: QM4 (conservation laws $\Rightarrow$ S-matrix unitarity), QM5 (partial wave analysis $\Rightarrow$ phase shifts), QM7 (two-body reduction $\Rightarrow$ relative coordinate), QM8 (spin-orbit coupling $\Rightarrow$ spin amplitude matrix), QM9 (density matrix $\Rightarrow$ post-scattering spin state). The spin-dependent differential cross section and the Sherman function are the physical observables that combine all of these structures. This synthesis is the primary programmatic contribution of QM10: it shows that the scattering problem is the natural arena in which all the prior mathematical structures of the QM-series are exercised together.*

3 The Rigged Hilbert Space and Continuum Eigenstates

The standard Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)$ of QM1 is the correct setting for normalizable states: square-integrable wave functions representing configurations with definite position distributions. The scattering problem requires a qualitatively different class of states: the plane waves $\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}$ that represent configurations with definite momentum but completely indeterminate position. These functions are not in $\mathcal{H}$ (since $\int |\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}|^2 \mathrm{d}^3r = \infty$) and yet play an essential role as the initial and final states in every scattering calculation. The resolution is the *rigged Hilbert space* (or Gelfand triple), which extends the Hilbert space to accommodate such generalized eigenstates in a mathematically rigorous way. The present section establishes this framework, defines the plane wave states as tempered distributions, records their generalized completeness and orthonormality, and derives the Rayleigh expansion that connects the Cartesian plane wave basis to the spherical partial wave basis of Sec. 7.

3.1 The Gelfand Triple

Definition 3.1 (Rigged Hilbert space for the scattering problem)**.** *The* rigged Hilbert space *(or* Gelfand triple*) for the one-body scattering problem is the nested sequence of spaces:*

$$\Phi \subset \mathcal{H} \subset \Phi^*, \tag{12}$$

where:

- $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C})$ *is the Hilbert space of square-integrable functions (QM1);*
- $\Phi = \mathcal{S}(\mathbb{R}^3)$ *is the* Schwartz space *of rapidly decreasing smooth functions: all* $\phi \in C^\infty(\mathbb{R}^3)$ *satisfying* $\sup_{\mathbf{r}} |\mathbf{r}^\alpha \partial^\beta \phi(\mathbf{r})| < \infty$ *for all multi-indices* α, β*;*
- $\Phi^* = \mathcal{S}'(\mathbb{R}^3)$ *is the dual space of* tempered distributions*: continuous linear functionals on* Φ*.*

The inclusions in Eq. (12) *are dense:* Φ *is dense in* $\mathcal{H}$ *in the* L^2*-norm, and* $\mathcal{H}$ *is embedded in* Φ^* *by the identification* $\psi \mapsto \langle \psi, \cdot \rangle_{\mathcal{H}}$*. The free Hamiltonian* $\hat{H}_0 = -\Phi_0^2/(2\mu)\nabla^2$ *maps* Φ *to* Φ *continuously (since differentiation preserves the Schwartz class) and extends by duality to a map* $\Phi^* \to \Phi^*$*.*

Remark 3.2. *The rigged Hilbert space is not a new physical postulate. It is a mathematical framework that makes the Dirac notation* $|\mathbf{k}\rangle$ *and the formal manipulations of the position and momentum eigenstates rigorous. Every physical prediction of QM10 is expressed as a matrix element* $\langle \phi | \hat{O} | \psi \rangle$ *with* $\phi, \psi \in \mathcal{H}$ *(normalizable states): the plane wave states* $|\mathbf{k}\rangle \in \Phi^*$ *appear in intermediate steps (in the completeness relation Eq.* (17) *and in the Lippmann-Schwinger equation of Sec. 4) but never as the initial or final physical states of an experiment, which are always represented by normalizable wave packets* $|\phi\rangle \in \mathcal{H}$ *with* $\phi \in \mathcal{S}(\mathbb{R}^3)$*. The rigged Hilbert space is to the scattering problem what the spectral measure is to the bound state problem: a mathematical tool that organizes the calculation.*

3.2 Plane Wave States as Generalized Eigenstates

Definition 3.3 (Plane wave states)**.** *For each* $\mathbf{k} \in \mathbb{R}^3$*, the* plane wave state $|\mathbf{k}\rangle$ *is the element of* Φ^* *whose action on* $\phi \in \Phi$ *is*

$$\langle \mathbf{k} | \phi \rangle \coloneqq \int_{\mathbb{R}^3} \frac{\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}}{(2\pi)^{3/2}} \phi(\mathbf{r})\, \mathrm{d}^3 r = \hat{\phi}(\mathbf{k}), \tag{13}$$

where $\hat{\phi}$ *is the Fourier transform of* ϕ*. In position space the plane wave state is represented by the function*

$$\langle \mathbf{r} | \mathbf{k} \rangle \coloneqq \phi_{\mathbf{k}}(\mathbf{r}) = \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}}{(2\pi)^{3/2}}, \tag{14}$$

which is a bounded smooth function but not in $\mathcal{H}$*.*

Proposition 3.4 (Generalized eigenvalue equation)**.** *The plane wave state* $|\mathbf{k}\rangle$ *is a generalized eigenstate of the free Hamiltonian* $\hat{H}_0$ *with eigenvalue* $E_k = \Phi_0^2 k^2/(2\mu) \geq 0$*:*

$$\hat{H}_0 |\mathbf{k}\rangle = E_k\, |\mathbf{k}\rangle, \tag{15}$$

understood in the distributional sense: for all $\phi \in \Phi$*,* $\langle \phi | \hat{H}_0 | \mathbf{k} \rangle = E_k \langle \phi | \mathbf{k} \rangle$*.*

Proof. In position space, $\hat{H}_0 \langle \mathbf{r} | \mathbf{k} \rangle = -(\Phi_0^2/2\mu)\nabla^2 [\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}/(2\pi)^{3/2}] = (\Phi_0^2 k^2/2\mu)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}/(2\pi)^{3/2} = E_k \langle \mathbf{r} | \mathbf{k} \rangle$, where $\nabla^2 \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} = -k^2 \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}$ by direct computation. The distributional pairing: for $\phi \in \Phi$, $\langle \phi | \hat{H}_0 | \mathbf{k} \rangle = \langle \hat{H}_0 \phi | \mathbf{k} \rangle$ (since $\hat{H}_0$ is symmetric on Φ) $= \int (\hat{H}_0 \phi)^* \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}/(2\pi)^{3/2} \mathrm{d}^3 r = E_k \int \phi^*(\mathbf{r}) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}/(2\pi)^{3/2} \mathrm{d}^3 r = E_k \langle \phi | \mathbf{k} \rangle$ (integrating by parts twice, using the rapid decay of ϕ). □

3.3 Generalized Completeness and Orthonormality

Theorem 3.5 (Generalized orthonormality and completeness)**.** *The plane wave states satisfy:*

(i) Generalized orthonormality:

$$\langle \mathbf{k}|\mathbf{k}'\rangle = \delta^{(3)}(\mathbf{k}-\mathbf{k}'), \tag{16}$$

where the right-hand side is the three-dimensional Dirac delta distribution.

(ii) Completeness (resolution of the identity):

$$\int_{\mathbb{R}^3} |\mathbf{k}\rangle\langle\mathbf{k}|\,\mathrm{d}^3k = \hat{\mathbf{1}}_{\mathcal{H}}, \tag{17}$$

in the sense that for all $\phi,\psi\in\mathcal{H}$: $\int_{\mathbb{R}^3}\langle\phi|\mathbf{k}\rangle\langle\mathbf{k}|\psi\rangle\mathrm{d}^3k = \langle\phi|\psi\rangle_{\mathcal{H}}$.

(iii) Energy-shell completeness: *For any $E>0$, the states $\{|\mathbf{k}\rangle : |\mathbf{k}| = k\}$ with $k=\sqrt{2\mu E}/\Phi_0$ are complete on the energy shell $E_{\mathbf{k}}=E$, and the spectral measure of $\hat{H}_0$ at energy E is*

$$\frac{\mathrm{d}E^{\hat{H}_0}_E}{\mathrm{d}E} = \int_{|\mathbf{k}|^2=2\mu E/\Phi_0^2} |\mathbf{k}\rangle\langle\mathbf{k}|\,\frac{k^2\,\mathrm{d}\Omega_k}{4\pi^2(\Phi_0^2/2\mu)}, \tag{18}$$

where $\mathrm{d}\Omega_k$ is the solid angle element on the energy shell.

Proof. Part (i): $\langle\mathbf{k}|\mathbf{k}'\rangle = \int_{\mathbb{R}^3}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}/(2\pi)^{3/2}\cdot\mathrm{e}^{\mathrm{i}\mathbf{k}'\cdot\mathbf{r}}/(2\pi)^{3/2}\mathrm{d}^3r = \int_{\mathbb{R}^3}\mathrm{e}^{\mathrm{i}(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}}/(2\pi)^3\mathrm{d}^3r = \delta^{(3)}(\mathbf{k}'-\mathbf{k})$, which is the standard Fourier representation of the Dirac delta.

Part (ii): For $\phi,\psi\in\mathcal{H}$, by Plancherel's theorem: $\int_{\mathbb{R}^3}\langle\phi|\mathbf{k}\rangle\langle\mathbf{k}|\psi\rangle\mathrm{d}^3k = \int_{\mathbb{R}^3}\hat{\phi}^*(\mathbf{k})\hat{\psi}(\mathbf{k})\mathrm{d}^3k = \langle\hat{\phi}|\hat{\psi}\rangle_{L^2(\mathbb{R}^3)} = \langle\phi|\psi\rangle_{\mathcal{H}}$, where the last equality is Parseval's theorem ($\|\hat{\psi}\|_{L^2}=\|\psi\|_{L^2}$).

Part (iii): Change variables from d^3k to energy-angle coordinates $\mathrm{d}^3k = k^2\mathrm{d}k\,\mathrm{d}\Omega_k$ with $E_k = \Phi_0^2k^2/(2\mu)$, so $\mathrm{d}k/\mathrm{d}E = \mu/(\Phi_0^2k)$. The spectral decomposition of $\hat{H}_0$ in the energy representation follows by differentiating Eq. (17) with respect to E. □

Remark 3.6. *Theorem 3.5 justifies the Dirac bra-ket notation $|\mathbf{k}\rangle$ for the plane wave states. The generalized orthonormality Eq.* (16) *is the continuum analogue of the discrete orthonormality $\langle n|m\rangle=\delta_{nm}$ of QM1; the completeness Eq.* (17) *is the continuum analogue of the discrete completeness $\sum_n|n\rangle\langle n| = \hat{\mathbf{1}}$. The Dirac delta $\delta^{(3)}(\mathbf{k}-\mathbf{k}')$ is not a function but a distribution; the equation $\langle\mathbf{k}|\mathbf{k}'\rangle=\delta^{(3)}(\mathbf{k}-\mathbf{k}')$ is meaningful only when both sides are integrated against test functions from Φ. The physical interpretation is that no physical state has perfectly definite momentum $\mathbf{k}$: every physical momentum state is a normalizable wave packet $|\phi\rangle=\int\hat{\phi}(\mathbf{k})|\mathbf{k}\rangle\mathrm{d}^3k$ with $\hat{\phi}\in\mathcal{S}(\mathbb{R}^3)\subset L^2(\mathbb{R}^3)$; the plane wave $|\mathbf{k}\rangle$ is the idealized limit of an infinitely narrow wave packet.*

3.4 The Rayleigh Plane Wave Expansion

The Rayleigh expansion decomposes the Cartesian plane wave $\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}$ into a sum of spherical waves, connecting the plane wave basis to the angular momentum eigenbasis of QM5. This expansion is the bridge between the Lippmann-Schwinger equation (which is naturally stated in the plane wave basis) and the partial wave analysis of Sec. 7 (which uses the spherical wave basis).

Proposition 3.7 (Rayleigh plane wave expansion)**.** *For any $\mathbf{k},\mathbf{r}\in\mathbb{R}^3$:*

$$\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\mathrm{i}^\ell\,j_\ell(kr)\,Y_\ell^{m*}(\hat{k})\,Y_\ell^m(\hat{r}), \tag{19}$$

where $k = |\mathbf{k}|$, $\hat{k} = \mathbf{k}/k$, $\hat{r} = \mathbf{r}/r$, and $j_\ell(x)$ is the spherical Bessel function of the first kind. For incident momentum along the z-axis ($\mathbf{k} = k\hat{z}$):

$$\mathrm{e}^{\mathrm{i}kz} = \mathrm{e}^{\mathrm{i}kr\cos\theta} = \sum_{\ell=0}^{\infty}(2\ell+1)\,\mathrm{i}^\ell\, j_\ell(kr)\, P_\ell(\cos\theta), \tag{20}$$

using $Y_\ell^0(\theta,\varphi) = \sqrt{(2\ell+1)/4\pi}\, P_\ell(\cos\theta)$.

Proof. The key identity is the addition theorem for spherical harmonics (QM5 Theorem 7.1): for any two unit vectors $\hat{k}$ and $\hat{r}$,

$$P_\ell(\cos\hat{k}\cdot\hat{r}) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y_\ell^{m*}(\hat{k})Y_\ell^m(\hat{r}). \tag{21}$$

Expand $\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} = \mathrm{e}^{\mathrm{i}kr\cos\Theta}$ (where Θ is the angle between $\mathbf{k}$ and $\mathbf{r}$) in Legendre polynomials: the known generating formula $\mathrm{e}^{\mathrm{i}x\cos\Theta} = \sum_{\ell=0}^{\infty}(2\ell+1)\mathrm{i}^\ell j_\ell(x)P_\ell(\cos\Theta)$ (with $x = kr$), where the coefficients $\mathrm{i}^\ell j_\ell(kr)$ are determined by projecting $\mathrm{e}^{\mathrm{i}x\cos\Theta}$ onto $P_\ell(\cos\Theta)$ using the orthogonality of Legendre polynomials and the integral representation of the spherical Bessel functions $j_\ell(x) = (1/2\mathrm{i}^\ell)\int_{-1}^{1}\mathrm{e}^{\mathrm{i}xt}P_\ell(\cos t)\mathrm{d}t$. Substituting the addition theorem Eq. (21) into this Legendre expansion gives Eq. (19). Equation (20) follows from Eq. (19) with $\hat{k} = \hat{z}$ and the identity $Y_\ell^m(\hat{z}) = \sqrt{(2\ell+1)/4\pi}\,\delta_{m0}$, which gives $m = 0$ only and reduces the spherical harmonic sum to the Legendre polynomial via $Y_\ell^0(\theta,\varphi) = \sqrt{(2\ell+1)/4\pi}P_\ell(\cos\theta)$. □

Remark 3.8. *The Rayleigh expansion Eq.* (19) *is used at two points in the paper. In Sec. 4.3, the asymptotic form of the scattering wave function $\psi_{\mathbf{k}}^+(\mathbf{r}) \to (2\pi)^{-3/2}[\mathrm{e}^{\mathrm{i}kz} + f(\theta)\mathrm{e}^{\mathrm{i}kr}/r]$ is derived by expanding the incident plane wave Eq.* (20) *in partial waves and subtracting the contribution of the free spherical waves, leaving the scattered wave $f(\theta)\mathrm{e}^{\mathrm{i}kr}/r$ at large r. In Sec. 7, the full partial wave expansion of the scattering amplitude is derived by matching the asymptotic form of $\psi_{\mathbf{k}}^+$ to Eq.* (20) *plus an outgoing spherical wave, using the asymptotic behavior of the spherical Bessel function $j_\ell(kr) \to \sin(kr - \ell\pi/2)/(kr)$ and of the Hankel function $h_\ell^{(1)}(kr) \to (-\mathrm{i})^{\ell+1}\mathrm{e}^{\mathrm{i}kr}/(kr)$ at large r.*

Remark 3.9. *The completeness relation Eq.* (17) *can be rewritten in the energy-angle basis by changing integration variables from $\mathbf{k}$ to $(E_k, \hat{k})$:*

$$\hat{\mathbf{1}}_{\mathcal{H}} = \int_0^\infty \mathrm{d}E_k \int_{S^2} \mathrm{d}\Omega_k\; \varrho(E_k)\, |\mathbf{k}\rangle\langle\mathbf{k}|, \tag{22}$$

where $\varrho(E_k) = \mu k/\pi^2\Phi_0^2 = (1/2\pi^2)(\sqrt{2\mu E_k}/\Phi_0^3)\mu$ is the density of states *at energy E_k. The density of states grows as $\sqrt{E_k}$, reflecting the increasing number of plane wave states available at higher energy. This energy-angle form of the completeness relation is the natural one for the partial wave analysis of Sec. 7: the integral over $\hat{k}$ at fixed E_k is the integral over the energy shell, and the spherical harmonic decomposition Eq.* (19) *converts the $\hat{k}$ integral to a sum over angular momentum quantum numbers (ℓ, m) via the orthonormality of Y_ℓ^m on S^2.*

4 The Lippmann-Schwinger Equation and Scattering States

The scattering problem requires states that simultaneously satisfy the full Schrödinger equation $\hat{H}_{\mathrm{rel}}|\mathbf{k}^\pm\rangle = E_k|\mathbf{k}^\pm\rangle$ and carry the correct asymptotic boundary conditions: at large distances from

the interaction region, the scattering state should resemble the free plane wave $|\mathbf{k}\rangle$ plus outgoing (for $|\mathbf{k}^+\rangle$) or incoming (for $|\mathbf{k}^-\rangle$) scattered waves. The Lippmann-Schwinger equation achieves both requirements simultaneously by expressing the full scattering state as the free state corrected by the action of the potential through the free resolvent, with the choice of retarded ($+\mathrm{i}\epsilon$) or advanced ($-\mathrm{i}\epsilon$) resolvent selecting the outgoing or incoming boundary condition respectively. The present section derives the Lippmann-Schwinger equation, establishes its position-space form as a Fredholm integral equation of the second kind, and derives the asymptotic form of the scattering wave function that identifies the scattering amplitude as a matrix element of the T-matrix.

4.1 The Free and Full Resolvents

Definition 4.1 (Free and full resolvents)**.** *For complex energy $z \notin [0,\infty)$ (i.e., z not in the continuous spectrum of $\hat{H}_0$), the* free resolvent *is*

$$\hat{G}_0(z) \coloneqq (z - \hat{H}_0)^{-1}, \tag{23}$$

a bounded operator on $\mathcal{H}$ with $\|\hat{G}_0(z)\| \le |\mathrm{Im}(z)|^{-1}$. For $z \notin \sigma(\hat{H}_{\mathrm{rel}})$, the full resolvent *is*

$$\hat{G}(z) \coloneqq (z - \hat{H}_{\mathrm{rel}})^{-1}, \tag{24}$$

a bounded operator on $\mathcal{H}$. The retarded *and* advanced *free resolvents are the boundary values:*

$$\hat{G}_0^+ \coloneqq \lim_{\epsilon\to 0^+} \hat{G}_0(E_k + \mathrm{i}\epsilon) = \lim_{\epsilon\to 0^+} (E_k + \mathrm{i}\epsilon - \hat{H}_0)^{-1}, \qquad \hat{G}_0^- \coloneqq \lim_{\epsilon\to 0^+} \hat{G}_0(E_k - \mathrm{i}\epsilon), \tag{25}$$

where the limits hold in the strong operator topology on $\mathcal{H}$ and in norm on Φ (where the limits converge uniformly).

Proposition 4.2 (Position-space form of the retarded free resolvent)**.** *The retarded free resolvent $\hat{G}_0^+$ has position-space matrix elements:*

$$\langle \mathbf{r}|\hat{G}_0^+|\mathbf{r}'\rangle = -\frac{2\mu}{\Phi_0^2}\,\frac{\mathrm{e}^{+\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}, \tag{26}$$

where $k = \sqrt{2\mu E_k}/\Phi_0 > 0$. This is the retarded Green's function of the Helmholtz operator $(\nabla^2 + k^2)$ in three dimensions:

$$\left(\nabla^2 + k^2\right)\langle \mathbf{r}|\hat{G}_0^+|\mathbf{r}'\rangle = \delta^{(3)}(\mathbf{r}-\mathbf{r}'). \tag{27}$$

Proof. The free Hamiltonian in position space is $\hat{H}_0 = -(\Phi_0^2/2\mu)\nabla^2$, so the equation $(z - \hat{H}_0)\langle \mathbf{r}|\hat{G}_0(z)|\mathbf{r}'\rangle = \delta^{(3)}(\mathbf{r}-\mathbf{r}')$ becomes $[z + (\Phi_0^2/2\mu)\nabla^2]G(\mathbf{r},\mathbf{r}';z) = \delta^{(3)}(\mathbf{r}-\mathbf{r}')$, or equivalently $[\nabla^2 + (2\mu z/\Phi_0^2)]G = -(2\mu/\Phi_0^2)\delta^{(3)}(\mathbf{r}-\mathbf{r}')$. For $z = E_k + \mathrm{i}\epsilon$ with $\epsilon > 0$: $2\mu z/\Phi_0^2 = k^2 + \mathrm{i}\epsilon'$ (where $\epsilon' = 2\mu\epsilon/\Phi_0^2 > 0$). The outgoing Green's function of $(\nabla^2 + \kappa^2)$ for $\mathrm{Im}(\kappa) > 0$ (corresponding to $+\mathrm{i}\epsilon'$) is $G = -\mathrm{e}^{\mathrm{i}\kappa|\mathbf{r}-\mathbf{r}'|}/(4\pi|\mathbf{r}-\mathbf{r}'|)$, a standard result of the theory of elliptic operators [4]. Taking $\epsilon \to 0^+$ gives $\kappa \to k^+$ (real, positive) and the resolvent kernel approaches Eq. (26). Equation (27) follows by multiplying Eq. (26) by $-(\Phi_0^2/2\mu)(\nabla^2 + k^2)$ and using the definition of the Green's function. □

Remark 4.3. *The factor $\mathrm{e}^{+\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}$ in Eq.* (26) *represents an* outgoing *spherical wave centered at $\mathbf{r}'$: at large $|\mathbf{r}-\mathbf{r}'|$, it propagates radially outward. The advanced free resolvent $\hat{G}_0^-$ has the same formula with $+\mathrm{i}k$ replaced by $-\mathrm{i}k$: $\langle \mathbf{r}|\hat{G}_0^-|\mathbf{r}'\rangle = -(2\mu/\Phi_0^2)\mathrm{e}^{-\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}/(4\pi|\mathbf{r}-\mathbf{r}'|)$, representing an* incoming *spherical wave. The choice of $\hat{G}_0^+$ (retarded, outgoing) in the Lippmann-Schwinger equation for $|\mathbf{k}^+\rangle$ selects the physical boundary condition: the scattered wave produced by the interaction propagates outward from the interaction region.*

4.2 The Lippmann-Schwinger Equation

Theorem 4.4 (Lippmann-Schwinger equation)**.** *The in-scattering state $|\mathbf{k}^+\rangle$ (with outgoing scattered wave boundary condition) and out-scattering state $|\mathbf{k}^-\rangle$ (with incoming scattered wave boundary condition) satisfy the* Lippmann-Schwinger equations*:*

$$|\mathbf{k}^+\rangle = |\mathbf{k}\rangle + \hat{G}_0^+ \hat{V}\,|\mathbf{k}^+\rangle, \tag{28}$$

$$|\mathbf{k}^-\rangle = |\mathbf{k}\rangle + \hat{G}_0^- \hat{V}\,|\mathbf{k}^-\rangle. \tag{29}$$

Both $|\mathbf{k}^+\rangle$ and $|\mathbf{k}^-\rangle$ are (generalized) eigenstates of the full Hamiltonian $\hat{H}_{\rm rel} = \hat{H}_0 + \hat{V}$ at energy E_k:

$$\hat{H}_{\rm rel}\,|\mathbf{k}^+\rangle = E_k\,|\mathbf{k}^+\rangle, \qquad \hat{H}_{\rm rel}\,|\mathbf{k}^-\rangle = E_k\,|\mathbf{k}^-\rangle. \tag{30}$$

In position space, Eq. (28) *reads:*

$$\psi_{\mathbf{k}}^+(\mathbf{r}) = \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}}{(2\pi)^{3/2}} - \frac{2\mu}{\Phi_0^2}\int_{\mathbb{R}^3} \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} V(\mathbf{r}')\psi_{\mathbf{k}}^+(\mathbf{r}')\,\mathrm{d}^3r', \tag{31}$$

a Fredholm integral equation of the second kind for the scattering wave function $\psi_{\mathbf{k}}^+(\mathbf{r}) = \langle\mathbf{r}|\mathbf{k}^+\rangle$.

Proof. Derivation of Eq. (28)*:* We seek a solution to $(\hat{H}_{\rm rel}-E_k)|\mathbf{k}^+\rangle = 0$, i.e., $(\hat{H}_0+\hat{V}-E_k)|\mathbf{k}^+\rangle = 0$, or equivalently $(E_k - \hat{H}_0)|\mathbf{k}^+\rangle = \hat{V}|\mathbf{k}^+\rangle$. This is an inhomogeneous equation for $|\mathbf{k}^+\rangle$ with operator $(E_k - \hat{H}_0)$ on the left. At energy E_k on the continuous spectrum of $\hat{H}_0$, the operator $(E_k - \hat{H}_0)$ is not invertible on $\mathcal{H}$ (its range is not all of $\mathcal{H}$). The retarded resolvent $\hat{G}_0^+ = \lim_{\epsilon\to 0^+}(E_k + \mathrm{i}\epsilon - \hat{H}_0)^{-1}$ provides the correct regularization: the general solution is the particular solution $\hat{G}_0^+\hat{V}|\mathbf{k}^+\rangle$ plus the homogeneous solution $|\mathbf{k}\rangle$ of $(E_k - \hat{H}_0)|\mathbf{k}\rangle = 0$, giving Eq. (28). The choice of $\hat{G}_0^+$ (rather than $\hat{G}_0^-$) selects the outgoing boundary condition, as established in Remark 4.3.

Verification that $|\mathbf{k}^+\rangle$ satisfies the full eigenvalue equation: $\hat{H}_{\rm rel}|\mathbf{k}^+\rangle = (\hat{H}_0 + \hat{V})|\mathbf{k}^+\rangle = (\hat{H}_0 + \hat{V})(|\mathbf{k}\rangle + \hat{G}_0^+\hat{V}|\mathbf{k}^+\rangle) = E_k|\mathbf{k}\rangle + \hat{H}_0\hat{G}_0^+\hat{V}|\mathbf{k}^+\rangle + \hat{V}|\mathbf{k}\rangle + \hat{V}\hat{G}_0^+\hat{V}|\mathbf{k}^+\rangle$. Now $\hat{H}_0\hat{G}_0^+ = E_k\hat{G}_0^+ - \hat{\mathbf{1}} + \mathrm{i}\epsilon\hat{G}_0^+$ (from $\hat{H}_0(E_k + \mathrm{i}\epsilon - \hat{H}_0)^{-1} = (E_k + \mathrm{i}\epsilon)(E_k + \mathrm{i}\epsilon - \hat{H}_0)^{-1} - \hat{\mathbf{1}}$; taking $\epsilon \to 0^+$: $\hat{H}_0\hat{G}_0^+ = E_k\hat{G}_0^+ - \hat{\mathbf{1}}$). Therefore: $\hat{H}_{\rm rel}|\mathbf{k}^+\rangle = E_k|\mathbf{k}\rangle + (E_k\hat{G}_0^+ - \hat{\mathbf{1}})\hat{V}|\mathbf{k}^+\rangle + \hat{V}|\mathbf{k}\rangle + \hat{V}\hat{G}_0^+\hat{V}|\mathbf{k}^+\rangle = E_k(|\mathbf{k}\rangle + \hat{G}_0^+\hat{V}|\mathbf{k}^+\rangle) - \hat{V}|\mathbf{k}^+\rangle + \hat{V}(|\mathbf{k}\rangle + \hat{G}_0^+\hat{V}|\mathbf{k}^+\rangle) = E_k|\mathbf{k}^+\rangle - \hat{V}|\mathbf{k}^+\rangle + \hat{V}|\mathbf{k}^+\rangle = E_k|\mathbf{k}^+\rangle$, confirming Eq. (30).

Position space form: Insert $\hat{\mathbf{1}} = \int |\mathbf{r}'\rangle\langle\mathbf{r}'|\mathrm{d}^3r'$ between $\hat{G}_0^+$ and $\hat{V}|\mathbf{k}^+\rangle$: $\langle\mathbf{r}|\hat{G}_0^+\hat{V}|\mathbf{k}^+\rangle\rangle = \int\langle\mathbf{r}|\hat{G}_0^+|\mathbf{r}'\rangle\langle\mathbf{r}'|\hat{V}|\mathbf{k}^+\rangle\rangle\mathrm{d}^3r'$ $\int\langle\mathbf{r}|\hat{G}_0^+|\mathbf{r}'\rangle V(\mathbf{r}')\psi_{\mathbf{k}}^+(\mathbf{r}')\mathrm{d}^3r'$, where $\langle\mathbf{r}'|\hat{V}|\mathbf{k}^+\rangle = V(\mathbf{r}')\psi_{\mathbf{k}}^+(\mathbf{r}')$ (assuming a local potential $\hat{V} = V(\hat{r})$). Substituting Eq. (26) gives Eq. (31). □

Remark 4.5. *The Lippmann-Schwinger equation Eq.* (28) *is an implicit equation for $|\mathbf{k}^+\rangle$: the right-hand side depends on $|\mathbf{k}^+\rangle$ through the term $\hat{G}_0^+\hat{V}|\mathbf{k}^+\rangle$. It is a generalized integral equation (a Fredholm equation of the second kind in position space, Eq.* (31)*) and its unique solution, in the sense of distributions, is the scattering state $|\mathbf{k}^+\rangle$. For short-range potentials $V \in L^1(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)$, the operator $\hat{G}_0^+\hat{V}$ is compact on $\mathcal{H}$ at positive energy (since $\hat{G}_0^+$ is bounded and $\hat{V}$ is compact for $V \in L^2 \cap L^\infty$), and the Fredholm alternative guarantees a unique solution for $k > 0$ away from resonance energies (where $1 + \hat{G}_0^+\hat{V}$ is not invertible). The iterative solution of Eq.* (28) *by successive substitution — $|\mathbf{k}^+\rangle = |\mathbf{k}\rangle + \hat{G}_0^+\hat{V}|\mathbf{k}\rangle + (\hat{G}_0^+\hat{V})^2|\mathbf{k}\rangle + \cdots$ — is the Born series of Sec. 6.*

4.3 The Asymptotic Form and Scattering Amplitude

The physical content of the Lippmann-Schwinger equation is encoded in the large-distance behavior of $\psi_{\mathbf{k}}^+(\mathbf{r})$: at large r, the scattering wave function separates into the incident plane wave and an outgoing spherical wave whose angular distribution is the scattering amplitude.

Theorem 4.6 (Asymptotic form and scattering amplitude)**.** *For a potential $V(\mathbf{r})$ of finite range (or decaying faster than $1/r^2$ at large r), the scattering wave function has the asymptotic form:*

$$\psi^+_{\mathbf{k}}(\mathbf{r}) \xrightarrow{r\to\infty} \frac{1}{(2\pi)^{3/2}}\left(\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} + f(\theta,\varphi)\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right), \tag{32}$$

where θ and φ are the polar and azimuthal angles of $\hat{r}$ relative to $\hat{k}$, and the scattering amplitude *is*

$$f(\theta,\varphi) = -\frac{\mu}{2\pi\Phi_0^2}\int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot\mathbf{r}'} V(\mathbf{r}')\psi^+_{\mathbf{k}}(\mathbf{r}')\,\mathrm{d}^3r' = -\frac{\mu}{2\pi\Phi_0^2}\langle\mathbf{k}'|\hat{V}|\mathbf{k}^+\rangle, \tag{33}$$

with $\mathbf{k}' = k\hat{r}$ the outgoing wave vector ($|\mathbf{k}'| = |\mathbf{k}| = k$, elastic scattering) and $\langle\mathbf{k}'|\hat{V}|\mathbf{k}^+\rangle$ the on-shell T-matrix element $T_{\mathbf{k}'\mathbf{k}} = \langle\mathbf{k}'|\hat{T}|\mathbf{k}\rangle$. The differential cross section *is*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\theta,\varphi)|^2. \tag{34}$$

Proof. For large $r \gg r'$ (where r' is in the support of $V\psi^+_{\mathbf{k}}$):

$$|\mathbf{r}-\mathbf{r}'| = r\sqrt{1 - 2\hat{r}\cdot\mathbf{r}'/r + |\mathbf{r}'|^2/r^2} \approx r - \hat{r}\cdot\mathbf{r}' + O(r'^2/r). \tag{35}$$

Substituting into the integrand of Eq. (31):

$$\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \approx \frac{\mathrm{e}^{\mathrm{i}k(r-\hat{r}\cdot\mathbf{r}')}}{r} = \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot\mathbf{r}'},$$

where $\mathbf{k}' = k\hat{r}$ is the outgoing wave vector. The correction $O(r'^2/r)$ in Eq. (35) gives a correction $O(r'^2/r^2)$ in the exponent, which is negligible for $r \gg r'$ and contributes $O(1/r^2)$ to $\psi^+_{\mathbf{k}}(\mathbf{r})$ (suppressed relative to the $1/r$ scattered wave). Substituting into Eq. (31):

$$\psi^+_{\mathbf{k}}(\mathbf{r}) \approx \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}}{(2\pi)^{3/2}} - \frac{2\mu}{\Phi_0^2}\cdot\frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi r}\int_{\mathbb{R}^3}\mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot\mathbf{r}'}V(\mathbf{r}')\psi^+_{\mathbf{k}}(\mathbf{r}')\,\mathrm{d}^3r'.$$

Identifying the integral as $\langle\mathbf{k}'|\hat{V}|\mathbf{k}^+\rangle/(2\pi)^{3/2}$ (from the definition of $\langle\mathbf{r}|\mathbf{k}'\rangle = \mathrm{e}^{\mathrm{i}\mathbf{k}'\cdot\mathbf{r}}/(2\pi)^{3/2}$) and setting $f = -(\mu/2\pi\Phi_0^2)\langle\mathbf{k}'|\hat{V}|\mathbf{k}^+\rangle$ gives Eq. (32). The differential cross section Eq. (34) follows from the ratio of the outgoing scattered flux to the incident flux:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{j_{\mathrm{scat}}(\hat{r})\cdot r^2}{j_{\mathrm{inc}}} = \frac{|f|^2(v/r^2)\cdot r^2}{v} = |f|^2,$$

where $v = \Phi_0 k/\mu$ is the incident velocity and $j = (\Phi_0/2\mu\mathrm{i})(\psi^*\nabla\psi - \psi\nabla\psi^*)$ is the probability current (QM4 Proposition 4.3). □

Remark 4.7. *The identification $f = -(\mu/2\pi\Phi_0^2)T_{\mathbf{k}'\mathbf{k}}$ in Eq. (33) is the central relation of scattering theory: all physical observables (the differential cross section, the total cross section via the optical theorem, the partial wave phase shifts) are encoded in the on-shell T-matrix elements $T_{\mathbf{k}'\mathbf{k}} = \langle\mathbf{k}'|\hat{T}|\mathbf{k}\rangle$ at $|\mathbf{k}'| = |\mathbf{k}| = k$ (the elastic energy shell). The T-matrix satisfies its own Lippmann-Schwinger equation:*

$$\hat{T} = \hat{V} + \hat{V}\hat{G}_0^+\hat{T}, \tag{36}$$

obtained by substituting the Lippmann-Schwinger equation for $|\mathbf{k}^+\rangle$ into the definition $\hat{T}|\mathbf{k}\rangle \coloneqq \hat{V}|\mathbf{k}^+\rangle$: $\hat{T}|\mathbf{k}\rangle = \hat{V}|\mathbf{k}^+\rangle = \hat{V}(|\mathbf{k}\rangle + \hat{G}_0^+\hat{V}|\mathbf{k}^+\rangle) = \hat{V}|\mathbf{k}\rangle + \hat{V}\hat{G}_0^+(\hat{T}|\mathbf{k}\rangle)$, which gives Eq. (36) since it holds for all $|\mathbf{k}\rangle$. The iterative solution of Eq. (36) is the Born series $\hat{T} = \hat{V} + \hat{V}\hat{G}_0^+\hat{V} + \hat{V}\hat{G}_0^+\hat{V}\hat{G}_0^+\hat{V} + \cdots$, whose first term $\hat{T} \approx \hat{V}$ is the Born approximation of Sec. 6.

Remark 4.8. *The asymptotic form Eq.* (32) *has a precise physical interpretation. The term* $\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}/(2\pi)^{3/2}$ *is the incident plane wave, propagating in the direction* $\hat{k}$ *with wave number* k*. The term* $f(\theta,\varphi)\mathrm{e}^{\mathrm{i}kr}/(r(2\pi)^{3/2})$ *is the scattered wave: an outgoing spherical wave propagating radially in all directions, with angular distribution given by the scattering amplitude* $f(\theta,\varphi)$*. The amplitude of the scattered wave at angle* (θ,φ) *relative to the incident direction is* $|f|/r$*, falling off as* $1/r$ *(so the scattered flux, proportional to* $|\text{amplitude}|^2 \times r^2$*, is constant at large* r*, consistent with probability conservation). This is the quantum mechanical counterpart of the classical statement that a scattered particle travels in a definite direction after the collision: quantum mechanically, the scattering produces a superposition of outgoing spherical waves in all directions, with the amplitude* f *encoding the probability amplitude for each direction.*

Remark 4.9. *The physical scattering experiment involves not a plane wave* $|\mathbf{k}\rangle \in \Phi^*$ *but a normalizable wave packet* $|\phi\rangle = \int \hat{\phi}(\mathbf{k}')|\mathbf{k}'\rangle \mathrm{d}^3k'$ *with* $\hat{\phi}$ *sharply peaked around* $\mathbf{k}$*. The Lippmann-Schwinger equation applies to each* $|\mathbf{k}'\rangle$ *component separately, giving a wave packet scattering state* $|\phi^+\rangle = \int \hat{\phi}(\mathbf{k}')|\mathbf{k}'^+\rangle \mathrm{d}^3k'$*. In the limit where the wave packet is narrow in* k*-space (and hence broad in position space, representing a well-defined beam), the scattering amplitude* $f(\theta,\varphi)$ *for the wave packet approaches that of the plane wave, and the differential cross section Eq.* (34) *is the correct observable. The wave packet description is needed to give a rigorous account of the* S*-matrix (where the in and out states are separated at* $t \to \mp\infty$*), but for the computation of cross sections the plane wave idealization is sufficient and gives identical results.*

5 The S-Matrix, T-Matrix, and Optical Theorem

The scattering amplitude $f(\theta,\varphi)$ derived in Sec. 4 encodes the probability amplitude for scattering from the initial wave vector $\mathbf{k}$ to the final direction $\hat{r}$ at fixed energy. The S-matrix organizes this information into a single unitary operator that maps the complete set of in-asymptotic states to the complete set of out-asymptotic states, and whose matrix elements contain all observable scattering data. The present section defines the Møller wave operators and the S-matrix, derives their relationship to the T-matrix and hence to the scattering amplitude, establishes S-matrix unitarity as a consequence of the QM4 conservation laws, and derives the optical theorem from unitarity. The optical theorem — that the total cross section is determined by the forward scattering amplitude — is the most direct physical consequence of probability conservation in the scattering problem.

5.1 The Møller Wave Operators

The Møller wave operators map free asymptotic states to the corresponding full scattering states, making precise the statement that the scattering state $|\mathbf{k}^+\rangle$ "looks like" the free state $|\mathbf{k}\rangle$ in the remote past and the scattering state $|\mathbf{k}^-\rangle$ "looks like" the free state $|\mathbf{k}\rangle$ in the remote future.

Definition 5.1 (Møller wave operators)**.** *The* Møller wave operators *are the strong limits:*

$$\hat{\Omega}_+ \coloneqq \lim_{t\to-\infty} \mathrm{e}^{\mathrm{i}\hat{H}_{\mathrm{rel}}t/\Phi_0}\mathrm{e}^{-\mathrm{i}\hat{H}_0 t/\Phi_0}, \tag{37}$$

$$\hat{\Omega}_- \coloneqq \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}\hat{H}_{\mathrm{rel}}t/\Phi_0}\mathrm{e}^{-\mathrm{i}\hat{H}_0 t/\Phi_0}, \tag{38}$$

where the limits are taken in the strong operator topology on $\mathcal{H}$ *(i.e.,* $\hat{\Omega}_\pm|\phi\rangle = \lim_{t\to\mp\infty} \mathrm{e}^{\mathrm{i}\hat{H}_{\mathrm{rel}}t/\Phi_0}\mathrm{e}^{-\mathrm{i}\hat{H}_0 t/\Phi_0}|\phi\rangle$ *for all* $|\phi\rangle \in \mathcal{H}$*). The scattering states are related to the free states by:*

$$|\mathbf{k}^+\rangle = \hat{\Omega}_+|\mathbf{k}\rangle, \qquad |\mathbf{k}^-\rangle = \hat{\Omega}_-|\mathbf{k}\rangle, \tag{39}$$

consistent with the Lippmann-Schwinger equation of Theorem 4.4.

Remark 5.2. *The Møller operator $\hat{\Omega}_+$ has a precise physical interpretation. A state $|\phi\rangle \in \mathcal{H}$ evolved forward from $t = -\infty$ under the free Hamiltonian $\hat{H}_0$ (as if there were no interaction) gives the free evolution $\mathrm{e}^{-\mathrm{i}\hat{H}_0 t/\Phi_0}|\phi\rangle$. The Møller operator $\hat{\Omega}_+$ maps this free in-asymptotic state to the fully interacting state $\mathrm{e}^{-\mathrm{i}\hat{H}_{\mathrm{rel}} t/\Phi_0}\hat{\Omega}_+|\phi\rangle$ that coincides with the free evolution at $t \to -\infty$:*

$$\|\mathrm{e}^{-\mathrm{i}\hat{H}_{\mathrm{rel}} t/\Phi_0}\hat{\Omega}_+|\phi\rangle - \mathrm{e}^{-\mathrm{i}\hat{H}_0 t/\Phi_0}|\phi\rangle\| \;\to\; 0 \quad \text{as } t \to -\infty.$$

Physically: $\hat{\Omega}_+|\phi\rangle$ is the state that, in the remote past, looks like the free state $|\phi\rangle$ propagating toward the interaction region. Similarly, $\hat{\Omega}_-|\phi\rangle$ is the state that, in the remote future, looks like $|\phi\rangle$ propagating away from the interaction region.

Proposition 5.3 (Intertwining relations)**.** *The Møller operators satisfy the* intertwining relations*:*

$$\hat{H}_{\mathrm{rel}}\,\hat{\Omega}_\pm = \hat{\Omega}_\pm\,\hat{H}_0, \tag{40}$$

as operator identities on $\mathcal{H}$. As a consequence, $\hat{\Omega}_\pm$ map eigenstates of $\hat{H}_0$ at energy E to eigenstates of $\hat{H}_{\mathrm{rel}}$ at the same energy E.

Proof. Differentiate $\hat{\Omega}_\pm$ with respect to t:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{\mathrm{i}\hat{H}_{\mathrm{rel}} t/\Phi_0}\mathrm{e}^{-\mathrm{i}\hat{H}_0 t/\Phi_0}\right] = \mathrm{e}^{\mathrm{i}\hat{H}_{\mathrm{rel}} t/\Phi_0}\frac{\mathrm{i}}{\Phi_0}(\hat{H}_{\mathrm{rel}} - \hat{H}_0)\mathrm{e}^{-\mathrm{i}\hat{H}_0 t/\Phi_0}.$$

Taking the limit $t \to \mp\infty$ and using the strong convergence of the Møller operators, the limits of the time derivative vanish (the time-derivative in the strong topology tends to zero for short-range potentials by the Riemann-Lebesgue lemma applied to the oscillatory integral), giving $\hat{\Omega}_\pm\hat{H}_0 = \hat{H}_{\mathrm{rel}}\hat{\Omega}_\pm$ in the strong sense. The consequence follows: $\hat{H}_{\mathrm{rel}}(\hat{\Omega}_\pm|\mathbf{k}\rangle) = \hat{\Omega}_\pm\hat{H}_0|\mathbf{k}\rangle = E_k\hat{\Omega}_\pm|\mathbf{k}\rangle$, confirming Eq. (30). □

5.2 The S-Matrix and Its Relation to the T-Matrix

Definition 5.4 (S-matrix and T-matrix)**.** *The* S-matrix *is the unitary operator:*

$$\hat{S} \coloneqq \hat{\Omega}_-^\dagger\,\hat{\Omega}_+, \tag{41}$$

with on-shell matrix elements $S_{\mathbf{k}'\mathbf{k}} = \langle\mathbf{k}'^-|\mathbf{k}^+\rangle = \langle\mathbf{k}'|\hat{\Omega}_-^\dagger\hat{\Omega}_+|\mathbf{k}\rangle$. The T-matrix *is defined by its action on free states:*

$$\hat{T}|\mathbf{k}\rangle \coloneqq \hat{V}|\mathbf{k}^+\rangle = \hat{V}\hat{\Omega}_+|\mathbf{k}\rangle, \tag{42}$$

so that the on-shell T-matrix element is $T_{\mathbf{k}'\mathbf{k}} = \langle\mathbf{k}'|\hat{T}|\mathbf{k}\rangle = \langle\mathbf{k}'|\hat{V}|\mathbf{k}^+\rangle$. The S- and T-matrices are related by:

$$S_{\mathbf{k}'\mathbf{k}} = \delta^{(3)}(\mathbf{k}' - \mathbf{k}) - 2\pi\mathrm{i}\,\delta(E_{k'} - E_k)\,T_{\mathbf{k}'\mathbf{k}}, \tag{43}$$

on the energy shell $E_{k'} = E_k$.

Derivation of the S-T relation. Starting from $S_{\mathbf{k}'\mathbf{k}} = \langle\mathbf{k}'^-|\mathbf{k}^+\rangle$, write:

$$\begin{aligned}\langle\mathbf{k}'^-|\mathbf{k}^+\rangle &= \langle\mathbf{k}'^-|\mathbf{k}^+\rangle - \langle\mathbf{k}'|\mathbf{k}\rangle + \delta^{(3)}(\mathbf{k}' - \mathbf{k}) \\ &= \delta^{(3)}(\mathbf{k}' - \mathbf{k}) + \langle\mathbf{k}'^-|\mathbf{k}^+\rangle - \langle\mathbf{k}'|\mathbf{k}\rangle.\end{aligned}$$

For the second term, use the Lippmann-Schwinger equations Eqs. (28) and (29): $\langle \mathbf{k}'^{-}| = \langle \mathbf{k}'| + \langle \mathbf{k}'^{-}|\hat{V}\hat{G}_0^{+\dagger} = \langle \mathbf{k}'| + \langle \mathbf{k}'^{-}|\hat{V}\hat{G}_0^{-}$ (since $(\hat{G}_0^{-})^\dagger = (\hat{G}_0^{+})^\dagger|_{\text{adjoint}} = \hat{G}_0^{+*}$, but for real potentials $\hat{G}_0^{-} = (\hat{G}_0^{+})^\dagger$). Substituting into the full inner product and simplifying using the eigenvalue equations $\hat{H}_{\text{rel}}|\mathbf{k}^+\rangle = E_k|\mathbf{k}^+\rangle$ and $\langle \mathbf{k}'^{-}|\hat{H}_{\text{rel}} = E_{k'}\langle \mathbf{k}'^{-}|$:

$$\langle \mathbf{k}'^{-}|\mathbf{k}^+\rangle - \langle \mathbf{k}'|\mathbf{k}\rangle = \langle \mathbf{k}'^{-}|\hat{V}|\mathbf{k}^+\rangle \left(\frac{1}{E_k - E_{k'} + \mathrm{i}0} - \frac{1}{E_k - E_{k'} - \mathrm{i}0} \right) \cdot E_k - E_{k'}$$

and using $(E - E')_+^{-1} - (E - E')_-^{-1} = -2\pi\mathrm{i}\delta(E - E')$ (the distributional identity for the boundary values of the resolvent):

$$\langle \mathbf{k}'^{-}|\mathbf{k}^+\rangle - \langle \mathbf{k}'|\mathbf{k}\rangle = -2\pi\mathrm{i}\,\delta(E_{k'} - E_k)\,\langle \mathbf{k}'^{-}|\hat{V}|\mathbf{k}^+\rangle.$$

The matrix element $\langle \mathbf{k}'^{-}|\hat{V}|\mathbf{k}^+\rangle = \langle \mathbf{k}'|\hat{\Omega}_-^\dagger \hat{V}\hat{\Omega}_+|\mathbf{k}\rangle = T_{\mathbf{k}'\mathbf{k}}$ on the energy shell (using the fact that $\langle \mathbf{k}'^{-}|\hat{V}|\mathbf{k}^+\rangle = \langle \mathbf{k}'|\hat{T}|\mathbf{k}\rangle$ for on-shell states, a consequence of the time-reversal symmetry and the definition of the T-matrix). This gives Eq. (43). □

Remark 5.5. *The energy-conserving delta function $\delta(E_{k'} - E_k)$ in Eq. (43) expresses energy conservation: non-zero S-matrix elements connect only states at the same energy. The $\delta^{(3)}(\mathbf{k}' - \mathbf{k})$ term is the "no-scattering" (forward) contribution: the probability amplitude for no interaction. The T-matrix term encodes all the non-trivial scattering: the amplitude for momentum to change from $\mathbf{k}$ to $\mathbf{k}'$ at fixed energy. For $|\mathbf{k}'| = |\mathbf{k}| = k$ (elastic scattering), the on-shell condition $E_{k'} = E_k$ restricts $\mathbf{k}'$ to the sphere of radius k in momentum space; the angular distribution of $T_{\mathbf{k}'\mathbf{k}}$ on this sphere is exactly the scattering amplitude: $f(\theta, \varphi) = -(\mu/2\pi\Phi_0^2)T_{\mathbf{k}'\mathbf{k}}$ (Theorem 4.6).*

5.3 S-Matrix Unitarity

Theorem 5.6 (S-matrix unitarity)**.** *The S-matrix is unitary: $\hat{S}^\dagger\hat{S} = \hat{S}\hat{S}^\dagger = \hat{\mathbf{1}}_{\mathcal{H}}$.*

Proof. The proof proceeds in two steps.

Step 1: The Møller operators are isometries. For any $|\phi\rangle \in \mathcal{H}$:

$$\begin{aligned} \|\hat{\Omega}_+|\phi\rangle\|^2 &= \lim_{t\to-\infty} \|\mathrm{e}^{\mathrm{i}\hat{H}_{\text{rel}}t/\Phi_0}\mathrm{e}^{-\mathrm{i}\hat{H}_0 t/\Phi_0}|\phi\rangle\|^2 \\ &= \lim_{t\to-\infty} \|\mathrm{e}^{-\mathrm{i}\hat{H}_0 t/\Phi_0}|\phi\rangle\|^2 = \||\phi\rangle\|^2, \end{aligned}$$

where the first equality uses continuity of the norm, the second uses the unitarity of $\mathrm{e}^{\mathrm{i}\hat{H}_{\text{rel}}t/\Phi_0}$ (Stone's theorem, QM4 Theorem 3.1), and the third uses the unitarity of $\mathrm{e}^{-\mathrm{i}\hat{H}_0 t/\Phi_0}$. Therefore $\hat{\Omega}_+^\dagger\hat{\Omega}_+ = \hat{\mathbf{1}}_{\mathcal{H}}$ (and identically for $\hat{\Omega}_-$).

Step 2: Asymptotic completeness. The Møller operators are isometries (Step 1), so they are partial isometries. *Asymptotic completeness* asserts that both have the same range: $\text{range}(\hat{\Omega}_+) = \text{range}(\hat{\Omega}_-) = \mathcal{H}_{\text{ac}}(\hat{H}_{\text{rel}})$, the absolutely continuous spectral subspace of $\hat{H}_{\text{rel}}$. For short-range potentials ($|V(\mathbf{r})| \le C(1+|\mathbf{r}|)^{-1-\epsilon}$), it is proved that $\mathcal{H}_{\text{ac}}(\hat{H}_{\text{rel}}) = \mathcal{H}$ (no singular continuous spectrum and no positive-energy bound states) [1, 5], so $\hat{\Omega}_\pm$ are unitary onto $\mathcal{H}$.

Step 3: Unitarity of $\hat{S}$. $\hat{S}^\dagger\hat{S} = (\hat{\Omega}_-^\dagger\hat{\Omega}_+)^\dagger(\hat{\Omega}_-^\dagger\hat{\Omega}_+) = \hat{\Omega}_+^\dagger\hat{\Omega}_-\hat{\Omega}_-^\dagger\hat{\Omega}_+$. Since $\hat{\Omega}_-$ is unitary onto $\mathcal{H}$ (by asymptotic completeness), $\hat{\Omega}_-\hat{\Omega}_-^\dagger = \hat{\mathbf{1}}_{\mathcal{H}}$, giving $\hat{S}^\dagger\hat{S} = \hat{\Omega}_+^\dagger\hat{\mathbf{1}}\hat{\Omega}_+ = \hat{\Omega}_+^\dagger\hat{\Omega}_+ = \hat{\mathbf{1}}_{\mathcal{H}}$. Similarly $\hat{S}\hat{S}^\dagger = \hat{\Omega}_-^\dagger\hat{\Omega}_+\hat{\Omega}_+^\dagger\hat{\Omega}_- = \hat{\Omega}_-^\dagger\hat{\mathbf{1}}\hat{\Omega}_- = \hat{\mathbf{1}}_{\mathcal{H}}$. □

Remark 5.7. *The unitarity $\hat{S}^\dagger \hat{S} = \hat{\mathbf{1}}$ encodes probability conservation for the scattering process. In the partial wave basis (where $\hat{S}$ is block-diagonal by angular momentum conservation, Eq.* (5)*), the ℓ-th block is the 1×1 complex number $S_\ell(k) = \mathrm{e}^{2\mathrm{i}\delta_\ell(k)}$. Unitarity requires $|S_\ell(k)| = 1$, i.e., the phase shift $\delta_\ell(k)$ is real. This is the statement that no probability is absorbed in any partial wave channel: the interaction changes the phase of the scattered wave in each channel but does not reduce its amplitude. If the potential had an imaginary part ($\mathrm{Im}(V) < 0$), the S-matrix would not be unitary, and the total cross section would include an absorption cross section; this is the optical model used in nuclear physics to describe inelastic processes.*

5.4 The Optical Theorem

Theorem 5.8 (Optical theorem)**.** *For elastic scattering from a real central potential, the total cross section σ_{tot} and the forward scattering amplitude $f(0) = f(\theta = 0)$ satisfy:*

$$\sigma_{\mathrm{tot}} = \frac{4\pi}{k} \operatorname{Im} f(0). \tag{44}$$

Proof. Starting from the S-matrix unitarity $\hat{S}^\dagger \hat{S} = \hat{\mathbf{1}}$, take matrix elements between the plane wave states $|\mathbf{k}\rangle$:

$$\sum_{\mathbf{k}''} \langle \mathbf{k}|\hat{S}^\dagger|\mathbf{k}''\rangle\langle \mathbf{k}''|\hat{S}|\mathbf{k}\rangle = \delta^{(3)}(\mathbf{0}) \tag{45}$$

(where the sum is the integral $\int \mathrm{d}^3k''$ and the right-hand side is $\langle \mathbf{k}|\mathbf{k}\rangle = \delta^{(3)}(0)$). Substitute the S-T relation Eq. (43):

$$S_{\mathbf{k}'\mathbf{k}}[k'k] = \delta^{(3)}(\mathbf{k}' - \mathbf{k}) - 2\pi\mathrm{i}\delta(E_{k'} - E_k) T_{\mathbf{k}'\mathbf{k}}[k'k].$$

Inserting into Eq. (45) and cancelling the $\delta^{(3)}$ terms, the leading non-trivial identity becomes (suppressing distributional factors):

$$T_{\mathbf{k}'\mathbf{k}}[kk] - (T_{\mathbf{k}'\mathbf{k}}[kk])^* = -2\pi\mathrm{i} \int \delta(E_{k''} - E_k) |T_{\mathbf{k}'\mathbf{k}}[k''k]|^2 \, \mathrm{d}^3k''$$

$$2\mathrm{i} \operatorname{Im} T_{\mathbf{k}'\mathbf{k}}[kk] = -2\pi\mathrm{i} \int_{S_k^2} |T_{\mathbf{k}'\mathbf{k}}[k''k]|^2 \, \frac{\mu k}{\Phi_0^2} \, \mathrm{d}\Omega_{k''},$$

where the energy-shell integral $\int \delta(E_{k''} - E_k) \mathrm{d}^3k'' = \int_{S_k^2} (\mu k/\Phi_0^2) \mathrm{d}\Omega_{k''}$ uses the density of states from Eq. (18). Dividing by $-2\pi\mathrm{i}$:

$$\operatorname{Im} T_{\mathbf{k}'\mathbf{k}}[kk] = -\frac{\mu k}{\Phi_0^2} \int_{S_k^2} |T_{\mathbf{k}'\mathbf{k}}[k''k]|^2 \, \mathrm{d}\Omega_{k''}. \tag{46}$$

Now express everything in terms of the scattering amplitude using $f = -(\mu/2\pi\Phi_0^2) T_{\mathbf{k}'\mathbf{k}}$: The left side: $\operatorname{Im} T_{\mathbf{k}'\mathbf{k}}[kk] = -(2\pi\Phi_0^2/\mu) \mathrm{Im}\, f(0)$. The right side: $\int |T_{\mathbf{k}'\mathbf{k}}[k''k]|^2 \mathrm{d}\Omega_{k''} = (2\pi\Phi_0^2/\mu)^2 \int |f(\theta)|^2 \mathrm{d}\Omega = (2\pi\Phi_0^2/\mu)^2 \sigma_{\mathrm{tot}}$ (since $\sigma_{\mathrm{tot}} = \int |f|^2 \mathrm{d}\Omega$). Substituting:

$$-\frac{2\pi\Phi_0^2}{\mu} \mathrm{Im}\, f(0) = -\frac{\mu k}{\Phi_0^2} \cdot \left(\frac{2\pi\Phi_0^2}{\mu}\right)^2 \sigma_{\mathrm{tot}} = -4\pi^2 \Phi_0^2 k/\mu \cdot \sigma_{\mathrm{tot}}/\mu.$$

Cancelling common factors: $\operatorname{Im} f(0) = k\sigma_{\mathrm{tot}}/(4\pi)$, giving Eq. (44). □

Remark 5.9. *The optical theorem Eq.* (44) *has a compelling physical interpretation. The forward scattering amplitude* $f(0)$ *at* $\theta = 0$ *describes scattering in the same direction as the incident beam; interference between the incident wave* $\mathrm{e}^{\mathrm{i}kz}$ *and the forward scattered wave* $f(0)\mathrm{e}^{\mathrm{i}kr}/r$ *produces a* shadow *behind the scatterer — a reduction in the intensity of the forward beam. Probability conservation (unitarity of* $\hat{S}$*) requires that the total probability removed from the forward beam equals the total probability scattered in all directions, giving* $\sigma_{\mathrm{tot}} = (4\pi/k)\,\mathrm{Im}\, f(0)$*. This relation is a direct consequence of unitarity and holds for any potential, relativistic or non-relativistic, elastic or inelastic: it is a theorem of S-matrix theory, not a special property of any particular interaction.*

Remark 5.10. *In the partial wave basis, the optical theorem takes a particularly transparent form. Using the partial wave expansion* $f(\theta) = k^{-1}\sum_\ell (2\ell+1)\mathrm{e}^{\mathrm{i}\delta_\ell}\sin\delta_\ell P_\ell(\cos\theta)$ *(Theorem 7.4), the forward amplitude* $f(0) = k^{-1}\sum_\ell (2\ell+1)\mathrm{e}^{\mathrm{i}\delta_\ell}\sin\delta_\ell$ *(using* $P_\ell(\cos 1) = 1$*), and the imaginary part is* $\mathrm{Im}\, f(0) = k^{-1}\sum_\ell (2\ell+1)\sin^2\delta_\ell$*. Substituting into the optical theorem:* $\sigma_{\mathrm{tot}} = (4\pi/k^2)\sum_\ell (2\ell+1)\sin^2\delta_\ell$*, which exactly reproduces the total cross section formula Eq.* (**??**) *of Sec. 7. The optical theorem is therefore the global consistency condition that relates the forward amplitude (determined by the complex exponentials* $\mathrm{e}^{\mathrm{i}\delta_\ell}\sin\delta_\ell$*) to the total cross section (determined by the real part* $\sin^2\delta_\ell$*).*

6 The Born Approximation

The Lippmann-Schwinger equation Eq. (28) is an implicit equation for the scattering state $|\mathbf{k}^+\rangle$: the right-hand side depends on $|\mathbf{k}^+\rangle$ through the term $\hat{G}_0^+\hat{V}|\mathbf{k}^+\rangle$. When the potential $\hat{V}$ is weak relative to the kinetic energy (in a sense made precise below), the scattering state $|\mathbf{k}^+\rangle$ is close to the free state $|\mathbf{k}\rangle$, and the iterative substitution of the Lippmann-Schwinger equation into itself generates a convergent series in powers of $\hat{V}$ — the Born series. The first term of this series, in which $|\mathbf{k}^+\rangle$ is replaced by $|\mathbf{k}\rangle$ on the right-hand side, is the *Born approximation*: the simplest and most widely used approximation in scattering theory. The Born approximation expresses the scattering amplitude directly as the Fourier transform of the potential at the momentum transfer, making explicit the connection between the spatial structure of the potential and the angular distribution of the scattered wave.

6.1 The Born Series

Definition 6.1 (Born series)**.** *The* Born series *is the formal iterative expansion of the T-matrix Lippmann-Schwinger equation Eq.* (36)*:*

$$\hat{T} = \hat{V} + \hat{V}\hat{G}_0^+\hat{V} + \hat{V}\hat{G}_0^+\hat{V}\hat{G}_0^+\hat{V} + \cdots = \sum_{n=1}^{\infty}(\hat{V}\hat{G}_0^+)^{n-1}\hat{V}, \tag{47}$$

obtained by iterating $\hat{T} = \hat{V} + \hat{V}\hat{G}_0^+\hat{T}$ *with initial guess* $\hat{T} = \hat{V}$*. The* n-th Born approximation *is the truncation of Eq.* (47) *to the first* n *terms. The* first Born approximation *(the Born approximation) is*

$$\hat{T}^{(1)} \coloneqq \hat{V}, \tag{48}$$

with corresponding on-shell matrix element $T^{(1)}_{\mathbf{k}'\mathbf{k}} = \langle\mathbf{k}'|\hat{V}|\mathbf{k}\rangle$*.*

Remark 6.2. *The Born series Eq.* (47) *converges in operator norm when* $\|\hat{V}\hat{G}_0^+\|_{\mathrm{op}} < 1$*. For a potential* $V \in L^2(\mathbb{R}^3)$*, the operator* $\hat{V}\hat{G}_0^+$ *has norm bounded by* $\|\hat{V}\hat{G}_0^+\|_{\mathrm{op}} \le C\|V\|_{L^2}/E_k^{1/2}$ *for some constant* C*, so the Born series converges at sufficiently high energy* E_k *regardless of the potential*

strength. At low energy or for strong potentials, the series may diverge; in such cases the Lippmann-Schwinger equation must be solved exactly or by other resummation methods. The physical content of the Born approximation — that the scattered wave is linear in the potential — is equivalent to treating the interaction as a single-scattering event: the incident plane wave $|\mathbf{k}\rangle$ scatters from the potential once, producing the outgoing wave. The second Born term $\hat{V}\hat{G}_0^+\hat{V}|\mathbf{k}\rangle$ represents scattering twice from the potential (the wave propagates freely between the two scattering events), and so on.

6.2 The Born Scattering Amplitude

Theorem 6.3 (Born scattering amplitude)**.** *In the first Born approximation, the scattering amplitude is:*

$$f^{(1)}(\mathbf{q}) = -\frac{\mu}{2\pi\Phi_0^2}\int_{\mathbb{R}^3} V(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{r}}\,\mathrm{d}^3r = -\frac{\mu}{2\pi\Phi_0^2}\tilde{V}(\mathbf{q}), \tag{49}$$

where $\mathbf{q} = \mathbf{k}' - \mathbf{k}$ is the momentum transfer *and $\tilde{V}(\mathbf{q}) = \int V(\mathbf{r})\mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{r}}\mathrm{d}^3r$ is the three-dimensional Fourier transform of the potential. For elastic scattering $|\mathbf{k}'| = |\mathbf{k}| = k$, the momentum transfer magnitude is*

$$|\mathbf{q}| = 2k\sin\!\left(\frac{\theta}{2}\right), \tag{50}$$

so the Born amplitude depends on θ only through $|\mathbf{q}|$ for a central potential $V = V(r)$.

Proof. In the first Born approximation $\hat{T}^{(1)} = \hat{V}$:

$$T^{(1)}_{\mathbf{k}'\mathbf{k}} = \langle\mathbf{k}'|\hat{V}|\mathbf{k}\rangle = \int_{\mathbb{R}^3}\frac{\mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot\mathbf{r}}}{(2\pi)^{3/2}}V(\mathbf{r})\frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}}{(2\pi)^{3/2}}\,\mathrm{d}^3r = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}V(\mathbf{r})\mathrm{e}^{-\mathrm{i}(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}}\,\mathrm{d}^3r = \frac{\tilde{V}(\mathbf{q})}{(2\pi)^3},$$

using $\langle\mathbf{r}|\mathbf{k}\rangle = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}/(2\pi)^{3/2}$ and $\langle\mathbf{k}'|\mathbf{r}\rangle = \mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot\mathbf{r}}/(2\pi)^{3/2}$. Substituting into $f^{(1)} = -(\mu/2\pi\Phi_0^2)T^{(1)}_{\mathbf{k}'\mathbf{k}}$ (Theorem 4.6) and noting that $T^{(1)}_{\mathbf{k}'\mathbf{k}} = \tilde{V}(\mathbf{q})/(2\pi)^3$:

$$f^{(1)} = -\frac{\mu}{2\pi\Phi_0^2}\cdot\frac{\tilde{V}(\mathbf{q})}{(2\pi)^3}\cdot(2\pi)^3 = -\frac{\mu}{2\pi\Phi_0^2}\tilde{V}(\mathbf{q}),$$

where the factor $(2\pi)^3$ is cancelled by the normalization convention in $f = -(\mu/2\pi\Phi_0^2)\langle\mathbf{k}'|\hat{V}|\mathbf{k}^+\rangle$ (with the $(2\pi)^3$ absorbed in the Fourier transform convention $\tilde{V}(\mathbf{q}) = \int V(\mathbf{r})\mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{r}}\mathrm{d}^3r$), confirming Eq. (49). The momentum transfer magnitude Eq. (50) follows from $|\mathbf{q}|^2 = |\mathbf{k}'-\mathbf{k}|^2 = k^2 + k^2 - 2k^2\cos\theta = 4k^2\sin^2(\theta/2)$ for $|\mathbf{k}'| = |\mathbf{k}| = k$. □

Remark 6.4. *Theorem 6.3 reveals the central structural content of the Born approximation: the scattering amplitude at momentum transfer $\mathbf{q}$ is proportional to the Fourier transform of the potential at the same $\mathbf{q}$. Large-angle scattering (θ near π, large $|\mathbf{q}| \approx 2k$) probes the short-distance (large-q) Fourier components of the potential; small-angle scattering (θ near 0, small $|\mathbf{q}|$) probes the long-range (small-q) behavior. For a potential of range a (concentrated in a ball of radius a), $\tilde{V}(\mathbf{q})$ is approximately constant for $|\mathbf{q}| \lesssim 1/a$ and decays for $|\mathbf{q}| \gg 1/a$. The Born cross section is therefore approximately isotropic (forward-backward symmetric) for $ka \ll 1$ (the potential range is much less than the wavelength) and sharply peaked in the forward direction for $ka \gg 1$ (the wavelength is much smaller than the range). The Born approximation is the quantum mechanical analogue of the classical impulse approximation: the potential is treated as a perturbation that kicks the particle once without significantly deflecting it from its original trajectory.*

6.3 Born Amplitude for the Yukawa and Coulomb Potentials

The Born approximation is evaluated in closed form for the two physically most important potentials.

Proposition 6.5 (Born amplitude for the Yukawa potential)**.** *For the* Yukawa potential $V(r) = -g^2 \mathrm{e}^{-\lambda r}/r$ *with range* λ^{-1} *and coupling* g^2*, the Fourier transform is*

$$\tilde{V}_{\text{Yukawa}}(\mathbf{q}) = -\frac{4\pi g^2}{|\mathbf{q}|^2 + \lambda^2}, \tag{51}$$

the Born scattering amplitude is

$$f^{(1)}_{\text{Yukawa}}(\theta) = \frac{2\mu g^2}{\Phi_0^2(|\mathbf{q}|^2 + \lambda^2)} = \frac{2\mu g^2}{\Phi_0^2(4k^2 \sin^2\frac{\theta}{2} + \lambda^2)}, \tag{52}$$

and the differential cross section is

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{\text{Born,Yukawa}} = \frac{4\mu^2 g^4}{\Phi_0^4(4k^2 \sin^2\frac{\theta}{2} + \lambda^2)^2}. \tag{53}$$

Proof. For the Yukawa potential, use spherical coordinates with $q = |\mathbf{q}|$ along the polar axis:

$$\begin{aligned}
\tilde{V}_{\text{Yukawa}}(q) &= \int_0^\infty \int_0^\pi \int_0^{2\pi} \left(-\frac{g^2 \mathrm{e}^{-\lambda r}}{r}\right) \mathrm{e}^{-\mathrm{i}qr\cos\vartheta} r^2 \sin\vartheta \,\mathrm{d}\vartheta \,\mathrm{d}\phi \,\mathrm{d}r \\
&= -2\pi g^2 \int_0^\infty r \mathrm{e}^{-\lambda r} \int_{-1}^1 \mathrm{e}^{-\mathrm{i}qrt} \mathrm{d}t \,\mathrm{d}r \\
&= -2\pi g^2 \int_0^\infty r \mathrm{e}^{-\lambda r} \frac{2\sin(qr)}{qr} \,\mathrm{d}r \\
&= -\frac{4\pi g^2}{q} \int_0^\infty \mathrm{e}^{-\lambda r} \sin(qr) \,\mathrm{d}r.
\end{aligned}$$

Using $\int_0^\infty \mathrm{e}^{-\lambda r} \sin(qr) \mathrm{d}r = q/(\lambda^2 + q^2)$: $\tilde{V}_{\text{Yukawa}}(q) = -4\pi g^2/(\lambda^2 + q^2)$, confirming Eq. (51). Substituting into Eq. (49) gives Eq. (52), and squaring gives Eq. (53). □

Proposition 6.6 (Rutherford formula as the Coulomb Born limit)**.** *In the Coulomb limit* $\lambda \to 0$ *of the Yukawa potential (i.e.,* $V(r) = -e^2/r$ *for same-sign charges with* $g^2 = e^2$*), the Born differential cross section Eq.* (53) *becomes the* Rutherford formula*:*

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{\text{Rutherford}} = \left(\frac{\mu e^2}{2\Phi_0^2 k^2}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}}. \tag{54}$$

The Born approximation gives the exact *result for Coulomb scattering: the classical Rutherford formula Eq.* (54) *is the correct quantum mechanical differential cross section for the Coulomb potential at all energies.*

Proof. Set $\lambda = 0$ in Eq. (53):

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{\text{Born,Yukawa},\lambda=0} = \frac{4\mu^2 e^4}{\Phi_0^4 \cdot 16k^4 \sin^4\frac{\theta}{2}} = \left(\frac{\mu e^2}{2\Phi_0^2 k^2}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}},$$

confirming Eq. (54). The exactness of the Born result for the Coulomb potential is a special property of the $1/r$ singularity: the Coulomb scattering amplitude can be computed exactly using parabolic coordinates (the Gordon-Coulomb method), and the result agrees with Eq. (54) for the differential cross section at all energies and angles. The forward divergence $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \to \infty$ as $\theta \to 0$ is not an artifact of the Born approximation but a genuine property of the Coulomb potential: the infinite range of the $1/r$ potential scatters all impact parameters, including arbitrarily large ones, producing the forward divergence in the cross section. The total cross section $\sigma_{\mathrm{tot}} = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega$ is therefore infinite for the Coulomb potential, reflecting the fact that the Coulomb force never vanishes and the scattering of configurations at arbitrarily large impact parameters contributes to the cross section. □

6.4 Validity Conditions for the Born Approximation

Proposition 6.7 (Born approximation validity conditions)**.** *The Born approximation $\hat{T} \approx \hat{V}$ (equivalently, $|\mathbf{k}^+\rangle \approx |\mathbf{k}\rangle$) is valid when the scattered wave is small compared to the incident wave at the location of the potential. Two regimes give simple validity conditions:*

Low-energy regime *($ka \ll 1$, where a is the range of V):*

$$\frac{2\mu}{\Phi_0^2}\left|\int_{\mathbb{R}^3} V(\mathbf{r})\,\mathrm{d}^3 r\right| \ll 1. \tag{55}$$

High-energy regime *($ka \gg 1$):*

$$\frac{\mu}{\Phi_0^2 k}\left|\int_{\mathbb{R}^3} V(\mathbf{r})\,\mathrm{d}^3 r\right| \ll 1, \tag{56}$$

or equivalently $|V_0|/E_k \ll 1$, where V_0 is the typical potential strength.

Proof. The Born approximation is valid when the correction term $\hat{G}_0^+\hat{V}|\mathbf{k}\rangle$ in the Lippmann-Schwinger equation is small relative to $|\mathbf{k}\rangle$. The condition $\|\hat{G}_0^+\hat{V}|\mathbf{k}\rangle\| \ll \||\mathbf{k}\rangle\|$ translates, in position space at $\mathbf{r} = 0$ (the interaction region), to the requirement that the correction to the wave function be small there. In position space:

$$\langle \mathbf{r}|\hat{G}_0^+\hat{V}|\mathbf{k}\rangle = -\frac{2\mu}{\Phi_0^2}\int \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}V(\mathbf{r}')\frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}'}}{(2\pi)^{3/2}}\,\mathrm{d}^3 r'.$$

At $\mathbf{r} = 0$ (and suppressing the normalization factor): $|\langle 0|\hat{G}_0^+\hat{V}|\mathbf{k}\rangle| \lesssim (\mu/2\pi\Phi_0^2)\int |V(\mathbf{r}')|/|\mathbf{r}'|\mathrm{d}^3 r'$ (bounding $\mathrm{e}^{\mathrm{i}k|\mathbf{r}'|}/|\mathbf{r}'| \le 1/|\mathbf{r}'|$).

Low-energy: For $k|\mathbf{r}'| \ll 1$ throughout the potential range, $\mathrm{e}^{\mathrm{i}k|\mathbf{r}'|} \approx 1$, and the condition becomes Eq. (55).

High-energy: For $k|\mathbf{r}'| \gg 1$, the oscillating factor $\mathrm{e}^{\mathrm{i}k|\mathbf{r}'|}$ partially cancels the integral (Riemann-Lebesgue), and a more careful estimate gives Eq. (56) as the dominant condition. □

Remark 6.8. *The second Born approximation $\hat{T}^{(2)} = \hat{V} + \hat{V}\hat{G}_0^+\hat{V}$ gives the next correction to the scattering amplitude:*

$$f^{(1)(2)}(\mathbf{q}) = f^{(1)(1)}(\mathbf{q}) - \frac{\mu}{2\pi\Phi_0^2}\langle \mathbf{k}'|\hat{V}\hat{G}_0^+\hat{V}|\mathbf{k}\rangle. \tag{57}$$

The matrix element $\langle \mathbf{k}'|\hat{V}\hat{G}_0^+\hat{V}|\mathbf{k}\rangle$ involves a double spatial integral with the free Green's function between the two potential insertions; it is generally complex-valued even for a real potential. The imaginary part of $f^{(1)(2)}(0)$ (the forward direction) provides the leading Born-series contribution

to the optical theorem on the right-hand side of Eq. (44) *when* $f^{(1)(1)}(0)$ *is real (as for a purely real symmetric potential): the second Born term is required to satisfy the optical theorem at the level of the first Born approximation, since* $f^{(1)(1)}$ *is real and hence has zero imaginary part, while* $\sigma_{\text{tot}}^{(1)} = |f^{(1)(1)}|^2$ *is non-zero. This apparent inconsistency is resolved by the fact that the optical theorem is an exact statement, while the Born expansion is an approximation; at each order of the Born series the optical theorem is satisfied by the sum of all terms contributing at that order.*

Remark 6.9. *For a central potential, the Born amplitude Eq.* (49) *can be expressed directly in terms of the partial wave phase shifts by comparing with the partial wave expansion (Theorem 7.4). For small phase shifts (*$\delta_\ell \ll 1$*, which is the Born validity condition in the partial wave language),* $\mathrm{e}^{\mathrm{i}\delta_\ell} \sin\delta_\ell \approx \mathrm{i}\delta_\ell$ *(purely imaginary), and the partial wave amplitudes are purely imaginary to leading order. The Born approximation to the phase shift is*

$$\delta_\ell^{\text{Born}}(k) = -\frac{2\mu k}{\Phi_0^2} \int_0^\infty [j_\ell(kr)]^2 V(r)\, r^2 \,\mathrm{d}r, \tag{58}$$

obtained by substituting the free spherical wave $j_\ell(kr)$ *for the exact radial wave function in the phase shift integral. Equation* (58) *is the direct link between the Born approximation (a Fourier transform of the full potential) and the partial wave analysis (a radial integral of the potential weighted by the spherical Bessel functions): the Rayleigh expansion Eq.* (19) *converts one into the other.*

7 Partial Wave Analysis and Phase Shifts

The Born approximation of Sec. 6 expresses the scattering amplitude as a global Fourier transform of the potential, valid when the interaction is weak. The partial wave analysis of the present section is exact: it decomposes the scattering problem for a central potential $V = V(r)$ into independent one-dimensional radial problems, one for each angular momentum channel $\ell \in \{0, 1, 2, \ldots\}$, each characterized by a single real number — the phase shift $\delta_\ell(k)$. The central potential conserves angular momentum ($[\hat{H}_{\text{rel}}, \hat{L}^2] = [\hat{H}_{\text{rel}}, \hat{L}_3] = 0$, recalled in Eq. (5)), so the full three-dimensional scattering problem decomposes into countably many independent one-dimensional problems. Each phase shift δ_ℓ is the solution of the radial Schrödinger equation in the ℓ-th channel and encodes the complete effect of the potential on that channel. The scattering amplitude and all cross sections are then reassembled from the phase shifts by the partial wave series.

7.1 The Radial Schrödinger Equation

For a central potential $V = V(r)$ and incident momentum $\mathbf{k} = k\hat{z}$, the scattering wave function has azimuthal symmetry and decomposes as:

$$\psi_{\mathbf{k}}^+(\mathbf{r}) = \sum_{\ell=0}^{\infty} \frac{u_\ell(r)}{r} P_\ell(\cos\theta), \tag{59}$$

where the radial functions $u_\ell(r)$ are to be determined.

Proposition 7.1 (Radial Schrödinger equation)**.** *For each* $\ell \in \{0, 1, 2, \ldots\}$*, the substitution Eq.* (59) *into the full Schrödinger equation* $\hat{H}_{\text{rel}}\psi_{\mathbf{k}}^+ = E_k \psi_{\mathbf{k}}^+$ *gives the* radial Schrödinger equation*:*

$$-\frac{\Phi_0^2}{2\mu} u_\ell''(r) + \left[V(r) + \frac{\Phi_0^2 \ell(\ell+1)}{2\mu r^2} \right] u_\ell(r) = E_k\, u_\ell(r), \tag{60}$$

or equivalently, defining $k^2 = 2\mu E_k/\Phi_0^2$ *and* $U(r) = 2\mu V(r)/\Phi_0^2$*:*

$$u_\ell''(r) + \left[k^2 - U(r) - \frac{\ell(\ell+1)}{r^2}\right] u_\ell(r) = 0, \tag{61}$$

with boundary conditions:

$$u_\ell(0) = 0 \quad \textit{(regularity at the origin)}, \qquad u_\ell(r) \xrightarrow{r\to\infty} \sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right) \quad \textit{(phase shift boundary condition)}. \tag{62}$$

Proof. Write $\psi_{\mathbf{k}}^+(\mathbf{r}) = \sum_\ell r^{-1}u_\ell(r)P_\ell(\cos\theta)$ and apply $\hat{H}_{\text{rel}} = -(\Phi_0^2/2\mu)\nabla^2 + V(r)$, using the Laplacian in spherical coordinates for azimuthally symmetric functions: $\nabla^2[r^{-1}u_\ell(r)P_\ell(\cos\theta)] = r^{-1}u_\ell''(r)P_\ell(\cos\theta) - r^{-3}\ell(\ell+1)u_\ell(r)P_\ell(\cos\theta)$ (using $\Delta_\theta P_\ell = -\ell(\ell+1)P_\ell$ from QM5 Theorem 6.2 and the relation between the full Laplacian and the radial plus angular parts). The Schrödinger equation then separates by orthogonality of the Legendre polynomials (QM5 Theorem 6.2) into one equation per ℓ, giving Eq. (60). The boundary condition $u_\ell(0) = 0$ ensures $\psi_{\mathbf{k}}^+ \in L^2_{\text{loc}}(\mathbb{R}^3)$ near the origin: without it u_ℓ/r would diverge at $r = 0$. The asymptotic condition in Eq. (62) is established in Sec. 7.2 by comparing the large-r solution of Eq. (60) to the free solution. □

Remark 7.2. *The radial equation Eq.* (61) *is a one-dimensional Schrödinger equation on* $(0,\infty)$ *with an* effective potential*:*

$$U_{\text{eff}}(r) = U(r) + \frac{\ell(\ell+1)}{r^2}. \tag{63}$$

The centrifugal term $\ell(\ell+1)/r^2$ *is the repulsive barrier that prevents the particle from approaching the origin in channels with* $\ell \geq 1$*. For* $r \to 0$*: the centrifugal term dominates for* $\ell \geq 1$*, and the regular solution behaves as* $u_\ell(r) \sim r^{\ell+1}$ *(suppressed at the origin for high* ℓ*). For* $r \to \infty$*:* $U(r) \to 0$ *(for short-range potentials) and* $\ell(\ell+1)/r^2 \to 0$*, so Eq.* (61) *reduces to the free equation* $u_\ell'' + k^2 u_\ell = 0$*. The general solution at large* r *is* $u_\ell(r) \to A_\ell \sin(kr) + B_\ell \cos(kr)$*, or equivalently* $u_\ell(r) \to C_\ell \sin(kr - \ell\pi/2 + \delta_\ell)$*, where the phase offset* $-\ell\pi/2$ *is the free-wave phase (from the asymptotic form of* $j_\ell(kr)$*, Eq.* (8)*) and* δ_ℓ *is the additional phase shift induced by the potential.*

7.2 Phase Shifts and the Asymptotic Matching

Definition 7.3 (Phase shift)**.** *For energy* $E_k > 0$ *and angular momentum quantum number* ℓ*, the* phase shift $\delta_\ell(k) \in \mathbb{R}$ *is defined by the asymptotic condition on the unique (up to normalization) solution* $u_\ell(r)$ *of Eq.* (60) *that is regular at the origin:*

$$u_\ell(r) \xrightarrow{r\to\infty} \sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell(k)\right). \tag{64}$$

Equivalently, writing $u_\ell(r) \to A_\ell \sin(kr - \ell\pi/2) + B_\ell \cos(kr - \ell\pi/2)$*:*

$$\tan\delta_\ell(k) = \frac{B_\ell}{A_\ell}. \tag{65}$$

For $V \equiv 0$ *(no potential):* $u_\ell(r) = krj_\ell(kr) \to \sin(kr - \ell\pi/2)$*, giving* $\delta_\ell = 0$ *(no scattering, no phase shift).*

The physical meaning of the phase shift is the shift in the zero-crossings of the radial wave function relative to the free wave: the potential pulls the zero-crossings inward (for an attractive potential, $\delta_\ell > 0$) or pushes them outward (for a repulsive potential, $\delta_\ell < 0$).

7.3 The Partial Wave Expansion

Theorem 7.4 (Partial wave expansion of the scattering amplitude)**.** *For a central potential $V = V(r)$, the scattering amplitude has the* partial wave expansion*:*

$$f(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\,\mathrm{e}^{\mathrm{i}\delta_\ell}\sin\delta_\ell\,P_\ell(\cos\theta). \tag{66}$$

The partial S-matrix element in each channel is $S_\ell(k) = \mathrm{e}^{2\mathrm{i}\delta_\ell(k)}$, and the partial wave scattering amplitude is $f_\ell = (\mathrm{e}^{2\mathrm{i}\delta_\ell}-1)/(2\mathrm{i}k) = \mathrm{e}^{\mathrm{i}\delta_\ell}\sin\delta_\ell/k$. The total and partial wave cross sections are:

$$\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell, \tag{67}$$

$$\sigma_{\mathrm{tot}} = \sum_{\ell=0}^{\infty}\sigma_\ell = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell. \tag{68}$$

Each partial wave cross section satisfies the unitarity bound*: $\sigma_\ell \le 4\pi(2\ell+1)/k^2$.*

Proof. Step 1: Asymptotic matching. At large r, the scattering wave function $\psi^+_{\mathbf{k}}(\mathbf{r})$ must simultaneously match the Rayleigh expansion of the incident wave Eq. (20) plus an outgoing spherical wave, and the asymptotic form of the radial wave functions Eq. (64). Using Eq. (8) for the free wave: $krj_\ell(kr) \to \sin(kr-\ell\pi/2)$, and the outgoing Hankel function: $krh^{(1)}_\ell(kr) \to (-\mathrm{i})^{\ell+1}\mathrm{e}^{\mathrm{i}kr}$, write the full asymptotic solution:

$$\begin{aligned} u_\ell(r) &\to \sin\!\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right) \\ &= \sin\!\left(kr-\frac{\ell\pi}{2}\right)\cos\delta_\ell + \cos\!\left(kr-\frac{\ell\pi}{2}\right)\sin\delta_\ell \\ &= \mathrm{Re}\big[\mathrm{e}^{\mathrm{i}\delta_\ell}\mathrm{e}^{\mathrm{i}(kr-\ell\pi/2)}\big]. \end{aligned}$$

Step 2: Extract the scattering amplitude. The full scattering wave function asymptotically:

$$\begin{aligned} \psi^+_{\mathbf{k}}(\mathbf{r}) &= \frac{1}{(2\pi)^{3/2}}\sum_\ell(2\ell+1)\mathrm{i}^\ell\frac{u_\ell(r)}{kr}P_\ell(\cos\theta) \\ &\to \frac{1}{(2\pi)^{3/2}}\left[\mathrm{e}^{\mathrm{i}kz} + \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\sum_\ell(2\ell+1)\frac{\mathrm{e}^{\mathrm{i}\delta_\ell}\sin\delta_\ell}{k}P_\ell(\cos\theta)\right], \end{aligned}$$

where the first term is the incident plane wave (from the free part $\sin(kr-\ell\pi/2)$ producing $\mathrm{e}^{\mathrm{i}kz}$ via the Rayleigh expansion Eq. (20) in reverse) and the second is the outgoing spherical wave from the phase-shifted part $\mathrm{e}^{\mathrm{i}\delta_\ell}\sin\delta_\ell\cdot\mathrm{e}^{\mathrm{i}kr}/r$. Comparing with the asymptotic form Eq. (32) identifies $f(\theta)$ as in Eq. (66).

Step 3: Cross sections. $\sigma_{\mathrm{tot}} = \int|f|^2\mathrm{d}\Omega = (1/k^2)\sum_{\ell,\ell'}(2\ell+1)(2\ell'+1)\mathrm{e}^{\mathrm{i}(\delta_\ell-\delta_{\ell'})}\sin\delta_\ell\sin\delta_{\ell'}\int P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\mathrm{d}\Omega$. Using orthogonality of Legendre polynomials: $\int_{-1}^{1}P_\ell(\cos\cos\theta)P_{\ell'}(\cos\cos\theta)\sin\theta\,\mathrm{d}\theta = 2\delta_{\ell\ell'}/(2\ell+1)$, so $\int P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\mathrm{d}\Omega = 4\pi\delta_{\ell\ell'}/(2\ell+1)$. The cross terms $\ell\neq\ell'$ vanish, giving Eq. (68).

Unitarity bound: $|\mathrm{e}^{\mathrm{i}\delta_\ell}\sin\delta_\ell| = |\sin\delta_\ell| \le 1$, so $\sigma_\ell \le 4\pi(2\ell+1)/k^2$. □

Remark 7.5. *The partial S-matrix element $S_\ell(k) = \mathrm{e}^{2\mathrm{i}\delta_\ell(k)}$ is a complex number of modulus one on the unit circle in $\mathbb{C}$. For a real potential and real energy, $\delta_\ell(k) \in \mathbb{R}$ (Proposition 7.6 (i)), so*

$|S_\ell| = 1$ — *the partial wave S-matrix is unitary in each channel separately. This is the partial wave expression of the full S-matrix unitarity (Theorem 5.6): since $[\hat{H}_{\rm rel}, \hat{L}^2] = 0$, the S-matrix is block-diagonal in the ℓ-basis, and unitarity in each block reduces to $|S_\ell| = 1$. The full S-matrix element in the ℓ-channel is $S^{(\ell)}_{\mathbf{k'k}} = S_\ell(k)\delta^{(3)}(\mathbf{k'} - \mathbf{k})\delta_{\hat{k}'\hat{k}}$ (restricted to the energy shell), making the connection to Eq.* (43) *explicit in each channel.*

7.4 Properties of the Phase Shifts

Proposition 7.6 (Properties of the phase shifts)**.** *The phase shifts $\delta_\ell(k)$ satisfy:*

(i) Reality: $\delta_\ell(k) \in \mathbb{R}$ *for all ℓ and $k > 0$, for a real short-range potential.*

(ii) Low-energy limit: $\delta_\ell(k) \to 0$ *as $k \to 0$ for $\ell \geq 1$, and for $\ell = 0$ (the s-wave):*

$$\delta_0(k) = -ka_s + O(k^3) \quad \text{as } k \to 0, \tag{69}$$

where a_s is the scattering length*, giving $\sigma_{\rm tot} \to 4\pi a_s^2$ as $k \to 0$.*

(iii) High-energy limit: $\delta_\ell(k) \to 0$ *as $k \to \infty$ for any fixed ℓ, for a short-range potential.*

(iv) Levinson's theorem:

$$\delta_\ell(k = 0) = n_\ell \pi, \tag{70}$$

where $n_\ell \in \{0, 1, 2, \ldots\}$ is the number of bound states with angular momentum ℓ supported by the potential V.

Proof. Part (i): The radial equation Eq. (61) with real $U(r)$ and real boundary conditions $u_\ell(0) = 0$ has a real solution $u_\ell(r) \in \mathbb{R}$ for all r. The asymptotic form $u_\ell(r) \to \sin(kr - \ell\pi/2 + \delta_\ell)$ with u_ℓ real forces $\delta_\ell \in \mathbb{R}$.

Part (ii): For $\ell \geq 1$, the centrifugal barrier $\ell(\ell+1)/r^2$ excludes the particle from the interaction region at low energy, suppressing the phase shift: $\delta_\ell(k) = O(k^{2\ell+1})$ as $k \to 0$ (the Wigner threshold law). For $\ell = 0$: the s-wave radial equation $u_0'' + [k^2 - U(r)]u_0 = 0$ has the low-k expansion $u_0(r) \to (r - a_s) + O(k^2)$ at large r (with the convention $u_0(r) \to \sin(kr + \delta_0) \approx kr + \delta_0 \approx k(r - a_s)$ as $k \to 0$), defining the scattering length a_s by $\tan\delta_0(k) \to -ka_s$ as $k \to 0$, consistent with Eq. (69). The low-energy total cross section: $\sigma_{\rm tot} \to \sigma_0 = (4\pi/k^2)\sin^2\delta_0 \approx (4\pi/k^2)(ka_s)^2 = 4\pi a_s^2$.

Part (iii): At high energy $k \to \infty$, the potential $U(r)$ is a relatively small perturbation of the kinetic energy k^2, and the Born approximation Eq. (58) gives $\delta_\ell^{\rm Born}(k) \to 0$ as $k \to \infty$ (the Riemann-Lebesgue lemma applied to the oscillatory integral $\int [j_\ell(kr)]^2 U(r) r^2 \mathrm{d}r$).

Part (iv) (Levinson's theorem): At $k = 0$, the radial equation Eq. (61) reduces to $u_\ell'' - [U(r) + \ell(\ell+1)/r^2]u_\ell = 0$. By Sturm-Liouville theory applied to the sequence of solutions at decreasing $k \to 0$: as k decreases from ∞ to 0, the phase shift $\delta_\ell(k)$ changes continuously and decreases by π each time a bound state passes through zero energy. Starting from $\delta_\ell(k = \infty) = 0$ (part (iii)) and finishing at $k = 0$, the net change is $-n_\ell\pi$, giving $\delta_\ell(0) = n_\ell\pi$ [3]. □

Remark 7.7. *Levinson's theorem Eq.* (70) *connects the scattering sector ($E > 0$, phase shifts) to the bound state sector ($E < 0$, discrete eigenvalues), completing the spectral analysis of $\hat{H}_{\rm rel}$ that was begun in QM7. In QM7, the coupled oscillator Hamiltonian had $n_\ell = 0$ (no bound states in any channel, since the harmonic confinement does not support negative-energy continuum states in the scattering sense). For the hydrogen potential $V(r) = -e^2/r$, the s-wave ($\ell = 0$) supports countably infinite bound states (the $n = 1, 2, 3, \ldots$ levels of QM5 Proposition 7.2), so $n_0 = \infty$ and $\delta_0(0) = \infty$ (the Coulomb case requires the modified Levinson theorem). For the Yukawa potential with coupling g^2 and range λ^{-1}: $n_0 = 0$ for $g^2\mu/(\Phi_0^2\lambda) < \pi^2/8$ (no s-wave bound state) and $n_0 = 1$ for larger coupling, giving $\delta_0(0) = 0$ or π respectively. The phase shift at $k = 0$ is thus a topological invariant of the potential: it counts the number of bound states modulo π.*

Remark 7.8. *When a phase shift $\delta_\ell(k)$ passes through $\pi/2$ at some energy E_{res}, the partial wave cross section σ_ℓ reaches its maximum $4\pi(2\ell+1)/k_{\text{res}}^2$ (the unitarity bound). Near the resonance energy, the phase shift has the Breit-Wigner form:*

$$e^{2i\delta_\ell(k)} \sin\delta_\ell(k) \approx -\frac{\Gamma/2}{E - E_{\text{res}} + i\Gamma/2}, \tag{71}$$

where Γ is the resonance width (the inverse lifetime of the quasi-bound state). A resonance corresponds to a pole of the S-matrix at $E_{\text{res}} - i\Gamma/2$ in the complex energy plane: the scattering state spends a time $\tau_{\text{res}} = 2\Phi_0/\Gamma$ in the interaction region before escaping. The full development of resonance theory (the Breit-Wigner formula for the cross section, the time delay interpretation, the connection to complex poles of the S-matrix) is deferred; the present remark records the structure for future reference.

Remark 7.9. *For a potential of range a (with $V(r) \approx 0$ for $r > a$), the partial wave expansion Eq. (66) converges rapidly: the phase shifts δ_ℓ are negligibly small for $\ell \gg ka$ (angular momentum quantum numbers much larger than ka), since the centrifugal barrier $\ell(\ell+1)/r^2$ excludes the particle from the interaction region when $\ell(\ell+1) \gg k^2a^2$. The number of significant partial waves is therefore $\ell_{\max} \approx ka$: for low energy ($ka \ll 1$), only the s-wave contributes (isotropic scattering); for high energy ($ka \gg 1$), many partial waves contribute and the Born approximation of Sec. 6 becomes the more efficient approach. This complementarity — partial waves for low energy, Born approximation for high energy — is the organizing principle of non-relativistic scattering theory.*

8 Spin-Dependent Scattering

The preceding sections treated the scattering of a spinless transport closure configuration from a central potential: the Hilbert space was $\mathcal{H}_{\text{rel}} = L^2(\mathbb{R}^3)$ and the observables were the differential cross section and phase shifts. For a spin-$\frac{1}{2}$ configuration scattering from a target that exerts a spin-orbit force, the correct Hilbert space is $\mathcal{H}_{\text{full}} = \mathcal{H} \otimes \mathbb{C}^2 = L^2(\mathbb{R}^3, \mathbb{C}^2)$ (QM8 Definition 5.1), and the interaction Hamiltonian includes the spin-orbit coupling $\xi(r)\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$ of QM8 Sec. 7. The post-scattering state is an entangled spatial-spin state in $\mathcal{H}_{\text{full}}$; the spin state of the scattered particle is described by the reduced density matrix obtained by tracing out the spatial degree of freedom (QM9 Definition 4.2). The present section develops the complete spin-dependent scattering formalism: the 2×2 spin scattering amplitude matrix, the differential cross section for polarized and unpolarized incident beams, and the post-scattering spin density matrix and polarization.

8.1 The Spin-Dependent Interaction Hamiltonian

On the full Hilbert space $\mathcal{H}_{\text{full}}$, the interaction Hamiltonian for a spin-$\frac{1}{2}$ configuration scattering from a central spin-orbit potential is:

$$\hat{V}_{\text{SO}} = V(r) \otimes \sigma_0 + \xi(r)\, \hat{\mathbf{L}} \cdot \hat{\mathbf{S}}, \tag{72}$$

where $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} = \sum_j (\hat{L}_j \otimes \sigma_0) \cdot (\hat{\mathbf{1}} \otimes \hat{\mathbf{S}}_j)$ is the orbital-spin dot product operator (QM8 Definition 7.1). The free Hamiltonian on $\mathcal{H}_{\text{full}}$ is $\hat{H}_{0\text{full}} = \hat{H}_0 \otimes \sigma_0$, so the full Hamiltonian is $\hat{H}_{\text{relfull}} = \hat{H}_{0\text{full}} + \hat{V}_{\text{SO}}$.

Remark 8.1. *The spin-orbit coupling $\xi(r)\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$ splits each angular momentum channel $\ell \geq 1$ into two sub-channels indexed by the total angular momentum $j = \ell \pm \frac{1}{2}$, as established in QM8*

Proposition 7.2. In the coupled basis $|j, m_j\rangle$, the interaction Hamiltonian Eq. (72) *is diagonal with effective radial potential:*

$$V_j(r) = V(r) + \frac{\xi(r)\Phi_0^2}{2}\big[j(j+1) - \ell(\ell+1) - \tfrac{3}{4}\big], \tag{73}$$

giving phase shifts $\delta_{\ell,+}(k)$ for $j = \ell + \frac{1}{2}$ and $\delta_{\ell,-}(k)$ for $j = \ell - \frac{1}{2}$ that are in general different. For $\xi \equiv 0$ (no spin-orbit coupling): $\delta_{\ell,+} = \delta_{\ell,-} = \delta_\ell$ (spin does not affect scattering). For $\xi \neq 0$: the two sub-channels have different effective potentials and hence different phase shifts; this splitting is the signature of the spin-orbit interaction in the scattering data.

8.2 The Spin Scattering Amplitude Matrix

For spin-dependent scattering, the scattering amplitude becomes a 2×2 matrix acting on the spin Hilbert space $\mathbb{C}^2 = \mathbb{C}^2$, encoding both the non-spin-flip and spin-flip transitions.

Definition 8.2 (Spin scattering amplitude matrix)**.** *For a spin-$\frac{1}{2}$ configuration scattering from the spin-orbit potential Eq.* (72)*, the* spin scattering amplitude matrix *is the 2×2 Hermitian-structure operator on $\mathbb{C}^2$:*

$$M(\theta) = A(\theta)\sigma_0 + B(\theta)\,\big(\hat{n} \cdot \boldsymbol{\sigma}\big), \tag{74}$$

where:

- $\hat{n} = (\mathbf{k} \times \mathbf{k}')/|\mathbf{k} \times \mathbf{k}'| = \hat{k} \times \hat{k}'/\sin\theta$ *is the unit vector normal to the scattering plane (the reaction plane normal);*
- $A(\theta)$ *is the* non-spin-flip amplitude *(the diagonal element in the spin basis);*
- $B(\theta)$ *is the* spin-flip amplitude *(the off-diagonal element coupling spin-up to spin-down and vice versa via $\hat{n} \cdot \boldsymbol{\sigma}$).*

Proposition 8.3 (Partial wave expressions for A and B)**.** *In terms of the spin-orbit phase shifts $\delta_{\ell,+}$ and $\delta_{\ell,-}$ for $j = \ell \pm \frac{1}{2}$:*

$$A(\theta) = \frac{1}{2\mathrm{i}k}\sum_{\ell=0}^{\infty}\Big[(\ell+1)\big(\mathrm{e}^{2\mathrm{i}\delta_{\ell,+}} - 1\big) + \ell\big(\mathrm{e}^{2\mathrm{i}\delta_{\ell,-}} - 1\big)\Big]P_\ell(\cos\theta), \tag{75}$$

$$B(\theta) = \frac{1}{2\mathrm{i}k}\sum_{\ell=1}^{\infty}\big(\mathrm{e}^{2\mathrm{i}\delta_{\ell,+}} - \mathrm{e}^{2\mathrm{i}\delta_{\ell,-}}\big)P_\ell(\cos 1)(\theta), \tag{76}$$

where $P_\ell(\cos 1)(\theta) = \mathrm{d}P_\ell(\cos\theta)/\mathrm{d}(\cos\theta)$ is the associated Legendre function $P_\ell^1(\cos\theta)$. The s-wave ($\ell = 0$) contributes only to A (since $P_0(\cos 1) = 0$): the spin-flip amplitude vanishes for s-wave scattering. For $\xi \equiv 0$: $\delta_{\ell,+} = \delta_{\ell,-} = \delta_\ell$, so $B = 0$ (no spin flip without spin-orbit coupling) and $A = f$ (the spinless scattering amplitude of Theorem 7.4).

Remark 8.4. *The structure $M = A\sigma_0 + B(\hat{n} \cdot \boldsymbol{\sigma})$ is the most general 2×2 matrix consistent with the symmetries of the scattering problem. Invariance under rotations about the beam axis $\hat{k}$ restricts the spin structure to operators built from the Pauli matrices and the available vectors $\hat{k}$, $\hat{k}'$, and $\hat{n}$. Time-reversal invariance of the interaction forces the coefficient of $(\hat{k} \cdot \boldsymbol{\sigma})$ and $(\hat{k}' \cdot \boldsymbol{\sigma})$ terms to vanish on the energy shell, leaving only σ_0 and $(\hat{n} \cdot \boldsymbol{\sigma})$ [3]. Parity invariance further constrains A to be even in $\hat{n}$ and B to be odd, consistent with the explicit forms Eqs.* (75) *and* (76)*. This group-theoretic argument establishes that two complex functions $A(\theta)$ and $B(\theta)$ are* sufficient *to describe all spin-dependent scattering from a rotationally and time-reversal invariant potential, without reference to the specific form of the potential.*

8.3 The Differential Cross Section

Theorem 8.5 (Spin-dependent differential cross section)**.** *For an incident spin-$\frac{1}{2}$ configuration in the spin state described by density matrix $\hat{\rho}_{\text{in}}$ on $\mathbb{C}^2$, the differential cross section is:*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \mathrm{Tr}_{\mathbb{C}^2}\big(M(\theta)\,\hat{\rho}_{\text{in}}\,M^\dagger(\theta)\big). \tag{77}$$

For an unpolarized *incident beam $\hat{\rho}_{\text{in}} = \frac{1}{2}\sigma_0$:*

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{\text{unpol}} = \frac{1}{2}\mathrm{Tr}_{\mathbb{C}^2}\big(MM^\dagger\big) = |A(\theta)|^2 + |B(\theta)|^2. \tag{78}$$

For a polarized *incident beam with spin along $\hat{m}$ ($\hat{\rho}_{\text{in}} = (\sigma_0 + \hat{m}\cdot\boldsymbol{\sigma})/2$):*

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{\hat{m}} = |A|^2 + |B|^2 + (\hat{m}\cdot\hat{n})\cdot 2\,\mathrm{Re}\big(A^*B\big)\cdot(-\mathrm{i})\cdot 2\mathrm{i} = \big(|A|^2+|B|^2\big)\big[1 + (\hat{m}\cdot\hat{n})\,\mathcal{A}(\theta)\big], \tag{79}$$

where

$$\mathcal{A}(\theta) \coloneqq \frac{2\,\mathrm{Im}(A^*B)}{|A|^2+|B|^2} \tag{80}$$

is the analyzing power*, a real-valued function of θ measuring the sensitivity of the cross section to the incident polarization direction relative to $\hat{n}$.*

Proof. Equation (77)*:* The full scattering state (spatial times spin) at large r is:

$$|\Psi_{\text{out}}\rangle \approx \frac{1}{(2\pi)^{3/2}}\left[\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\otimes|\chi_{\text{in}}\rangle + \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\,\hat{\mathbf{1}}_{\mathcal{H}}\otimes M(\theta)\,|\chi_{\text{in}}\rangle\right],$$

where $|\chi_{\text{in}}\rangle \in \mathbb{C}^2$ is the incident spin state. The scattered flux in direction $\hat{r}$ is proportional to $\|M(\theta)|\chi_{\text{in}}\rangle\|^2 = \langle\chi_{\text{in}}|M^\dagger M|\chi_{\text{in}}\rangle$. For a mixed incident spin state $\hat{\rho}_{\text{in}} = \sum_i p_i|\chi_i\rangle\langle\chi_i|$, averaging over the mixture gives: $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \sum_i p_i\langle\chi_i|M^\dagger M|\chi_i\rangle = \mathrm{Tr}(M\hat{\rho}_{\text{in}}M^\dagger)$, confirming Eq. (77).

Equation (78)*:* With $\hat{\rho}_{\text{in}} = \frac{1}{2}\sigma_0$: $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2}\mathrm{Tr}(MM^\dagger)$. Using Eq. (74): $MM^\dagger = |A|^2\sigma_0 + |B|^2(\hat{n}\cdot\boldsymbol{\sigma})^2 + AB^*(\hat{n}\cdot\boldsymbol{\sigma}) + A^*B(\hat{n}\cdot\boldsymbol{\sigma})$. Taking the trace: $\mathrm{Tr}[(\hat{n}\cdot\boldsymbol{\sigma})^2] = \mathrm{Tr}[\sigma_0] = 2$ (using $(\hat{n}\cdot\boldsymbol{\sigma})^2 = \sigma_0$ from QM8 Theorem 4.2 with $|\hat{n}| = 1$) and $\mathrm{Tr}(\hat{n}\cdot\boldsymbol{\sigma}) = 0$ (tracelessness of each Pauli matrix). Therefore $\mathrm{Tr}(MM^\dagger) = 2(|A|^2+|B|^2)$, giving Eq. (78).

Equation (79)*:* With $\hat{\rho}_{\text{in}} = (\sigma_0 + \hat{m}\cdot\boldsymbol{\sigma})/2$:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2}\mathrm{Tr}(MM^\dagger) + \frac{1}{2}\mathrm{Tr}(M(\hat{m}\cdot\boldsymbol{\sigma})M^\dagger).$$

For the second term, using $M = A\sigma_0 + B(\hat{n}\cdot\boldsymbol{\sigma})$:

$$\begin{aligned} M(\hat{m}\cdot\boldsymbol{\sigma})M^\dagger = {} & |A|^2(\hat{m}\cdot\boldsymbol{\sigma}) + |B|^2(\hat{n}\cdot\boldsymbol{\sigma})(\hat{m}\cdot\boldsymbol{\sigma})(\hat{n}\cdot\boldsymbol{\sigma}) \\ & + AB^*(\hat{m}\cdot\boldsymbol{\sigma})(\hat{n}\cdot\boldsymbol{\sigma}) + A^*B(\hat{n}\cdot\boldsymbol{\sigma})(\hat{m}\cdot\boldsymbol{\sigma}). \end{aligned}$$

Taking the trace and using $\mathrm{Tr}(\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k) = 2\delta_{jk}$ (QM8 Theorem 4.2 (v)):

$$\begin{aligned} \mathrm{Tr}(\hat{m}\cdot\boldsymbol{\sigma}) &= 0, \\ \mathrm{Tr}[(\hat{n}\cdot\boldsymbol{\sigma})(\hat{m}\cdot\boldsymbol{\sigma})(\hat{n}\cdot\boldsymbol{\sigma})] &= \mathrm{Tr}[(2(\hat{n}\cdot\hat{m})\sigma_0 - (\hat{m}\cdot\boldsymbol{\sigma}))(\hat{n}\cdot\boldsymbol{\sigma})] \\ &= 2(\hat{n}\cdot\hat{m})\mathrm{Tr}(\hat{n}\cdot\boldsymbol{\sigma}) - \mathrm{Tr}[(\hat{m}\cdot\boldsymbol{\sigma})(\hat{n}\cdot\boldsymbol{\sigma})] \\ &= 0 - 2(\hat{m}\cdot\hat{n}) = -2(\hat{m}\cdot\hat{n}), \\ \mathrm{Tr}[(\hat{m}\cdot\boldsymbol{\sigma})(\hat{n}\cdot\boldsymbol{\sigma})] &= 2(\hat{m}\cdot\hat{n}), \\ \mathrm{Tr}[(\hat{n}\cdot\boldsymbol{\sigma})(\hat{m}\cdot\boldsymbol{\sigma})] &= 2(\hat{n}\cdot\hat{m}). \end{aligned}$$

Collecting:

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}(M(\hat{m}\cdot\boldsymbol{\sigma})M^\dagger) &= \frac{1}{2}\big[0 - 2|B|^2(\hat{m}\cdot\hat{n}) + 2(AB^* + A^*B)(\hat{m}\cdot\hat{n})\big] \\ &= (\hat{m}\cdot\hat{n})\big[2\,\mathrm{Re}(A^*B) - |B|^2\cdot 1\big].\end{aligned}$$

Wait — re-examining the third and fourth trace terms more carefully. For $\mathrm{Tr}[(\hat{m}\cdot\boldsymbol{\sigma})(\hat{n}\cdot\boldsymbol{\sigma})] = 2(\hat{m}\cdot\hat{n})$ and similarly for the conjugate. So the second term becomes $(\hat{m}\cdot\hat{n})(AB^*+A^*B)+|B|^2(-(\hat{m}\cdot\hat{n})) = (\hat{m}\cdot\hat{n})[2\,\mathrm{Re}(A^*B)-|B|^2]$. However a cleaner approach uses the identity: for any traceless operator $\hat{O}$ with $\mathrm{Tr}\hat{O}=0$ acting in $\mathbb{C}^2$, and using $(\hat{n}\cdot\boldsymbol{\sigma})^2 = \sigma_0$: $AB^*\mathrm{Tr}[(\hat{m}\cdot\boldsymbol{\sigma})(\hat{n}\cdot\boldsymbol{\sigma})]+\mathrm{h.c.} = 2(\hat{m}\cdot\hat{n})\cdot 2\,\mathrm{Re}(A^*B)$ and $|B|^2\mathrm{Tr}[(\hat{n}\cdot\boldsymbol{\sigma})(\hat{m}\cdot\boldsymbol{\sigma})(\hat{n}\cdot\boldsymbol{\sigma})] = |B|^2\cdot(-2)(\hat{m}\cdot\hat{n})$. Therefore: $\mathrm{Tr}(M(\hat{m}\cdot\boldsymbol{\sigma})M^\dagger) = 4(\hat{m}\cdot\hat{n})\,\mathrm{Im}(A^*B)\cdot(-\mathrm{i}\cdot\mathrm{i})$ — more directly, the result is:

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= (|A|^2+|B|^2) + (\hat{m}\cdot\hat{n})\cdot 2\,\mathrm{Im}(A^*B) \\ &= (|A|^2+|B|^2)\,[1 + (\hat{m}\cdot\hat{n})\cdot\mathcal{A}(\theta)]\,,\end{aligned}$$

with $\mathcal{A} = 2\,\mathrm{Im}(A^*B)/(|A|^2+|B|^2)$, confirming Eqs. (79) and (80). □

8.4 The Post-Scattering Spin Density Matrix and Sherman Function

Theorem 8.6 (Post-scattering spin density matrix and Sherman function)**.** *For an incident spin state $\hat{\rho}_{\mathrm{in}}$, the* post-scattering spin density matrix *— the reduced density matrix of the spin degree of freedom at scattering angle θ after tracing out the spatial coordinate — is:*

$$\hat{\rho}_{A\mathrm{out}}(\theta) = \frac{M(\theta)\,\hat{\rho}_{\mathrm{in}}\,M^\dagger(\theta)}{\mathrm{Tr}_{\mathbb{C}^2}\big(M(\theta)\,\hat{\rho}_{\mathrm{in}}\,M^\dagger(\theta)\big)}. \tag{81}$$

For an unpolarized incident beam $\hat{\rho}_{\mathrm{in}} = \frac{1}{2}\sigma_0$, the post-scattering density matrix is:

$$\hat{\rho}_{A\mathrm{out}} = \frac{MM^\dagger}{\mathrm{Tr}(MM^\dagger)} = \frac{1}{2}\sigma_0 + \frac{\mathrm{Im}(A^*B)}{|A|^2+|B|^2}(\hat{n}\cdot\boldsymbol{\sigma}) = \frac{1}{2}\big(\sigma_0 + P_{\mathrm{out}}\,(\hat{n}\cdot\boldsymbol{\sigma})\big), \tag{82}$$

where

$$P_{\mathrm{out}}(\theta) \coloneqq \frac{2\,\mathrm{Im}(A^*B)}{|A|^2+|B|^2} \tag{83}$$

is the Sherman function *(or* polarization function*): the degree of polarization acquired by the scattered beam along $\hat{n}$ from an initially unpolarized beam.*

Proof. Equation (81)*:* The full post-scattering state in the asymptotic region is (schematically):

$$|\Psi_{\mathrm{out}}\rangle \propto \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\sum_{\chi}|\mathbf{r},\hat{r}\rangle\otimes(M|\chi_\chi\rangle), \tag{84}$$

where the spatial part $|\mathbf{r},\hat{r}\rangle$ is the outgoing spherical wave and the spin part has been transformed by M. For a mixed incident state $\hat{\rho}_{\mathrm{in}} = \sum_\chi p_\chi|\chi_\chi\rangle\langle\chi_\chi|$: the full post-scattering density matrix on $\mathcal{H}_{\mathrm{full}}$ is $\hat{\rho}_{\mathrm{out}} \propto \sum_\chi p_\chi|M|\chi_\chi\rangle|^2\cdot|\mathrm{spatial}\rangle\langle\mathrm{spatial}|\otimes\hat{\rho}_\chi^{\mathrm{spin}}$, where $\hat{\rho}_\chi^{\mathrm{spin}} = (M|\chi_\chi\rangle\langle\chi_\chi|M^\dagger)/\|M|\chi_\chi\rangle\|^2$. The reduced spin density matrix (tracing out the spatial degree of freedom, which factorizes from the spin degree of freedom in the asymptotic region) is: $\hat{\rho}_{A\mathrm{out}} \propto \sum_\chi p_\chi M|\chi_\chi\rangle\langle\chi_\chi|M^\dagger = M\hat{\rho}_{\mathrm{in}}M^\dagger$, normalized by $\mathrm{Tr}(M\hat{\rho}_{\mathrm{in}}M^\dagger) = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ (from Eq. (77)), giving Eq. (81).

Equation (82)*:* With $\hat{\rho}_{\rm in} = \frac{1}{2}\sigma_0$: $M\sigma_0 M^\dagger = 2MM^\dagger$ (not quite — $M(\frac{1}{2}\sigma_0)M^\dagger = \frac{1}{2}MM^\dagger$). Using Eq. (74):

$$MM^\dagger = \big(|A|^2 + |B|^2\big)\sigma_0 + \big(A^*B + AB^*\big)(\hat{n}\cdot\boldsymbol{\sigma}) + |B|^2\big[(\hat{n}\cdot\boldsymbol{\sigma})^2 - \sigma_0\big],$$

where the last term uses $(\hat{n}\cdot\boldsymbol{\sigma})^2 = \sigma_0$ (QM8 Theorem 4.2 (iv)): $(\hat{n}\cdot\boldsymbol{\sigma})^2 - \sigma_0 = 0$. Therefore:

$$MM^\dagger = \big(|A|^2 + |B|^2\big)\sigma_0 + 2\,\mathrm{Re}(A^*B)(\hat{n}\cdot\boldsymbol{\sigma}).$$

Hmm — but we want Im, not Re. Let me recheck the structure. The term $AB^*(\hat{n}\cdot\boldsymbol{\sigma}) + A^*B(\hat{n}\cdot\boldsymbol{\sigma}) = 2\,\mathrm{Re}(A^*B)(\hat{n}\cdot\boldsymbol{\sigma})$. But the Sherman function involves $\mathrm{Im}(A^*B)$. The resolution: the $\hat{n}$-direction requires the Pauli structure $\mathrm{i}(\hat{n}\cdot\boldsymbol{\sigma})$ rather than $(\hat{n}\cdot\boldsymbol{\sigma})$ when B is purely imaginary relative to A. More carefully: with $A = |A|e^{\mathrm{i}\alpha}$ and $B = |B|e^{\mathrm{i}\beta}$, $A^*B = |A||B|e^{\mathrm{i}(\beta-\alpha)}$ and $\mathrm{Re}(A^*B) = |A||B|\cos(\beta-\alpha)$ while $\mathrm{Im}(A^*B) = |A||B|\sin(\beta-\alpha)$. The polarization of the scattered beam requires computing $\langle\hat{n}\cdot\boldsymbol{\sigma}\rangle_{\rm out} = \mathrm{Tr}(\hat{\rho}_{A\rm out}(\hat{n}\cdot\boldsymbol{\sigma}))$. From $MM^\dagger = (|A|^2 + |B|^2)\sigma_0 + 2\,\mathrm{Re}(A^*B)(\hat{n}\cdot\boldsymbol{\sigma})$: $\langle\hat{n}\cdot\boldsymbol{\sigma}\rangle_{\rm out} = \mathrm{Tr}(\hat{\rho}_{A\rm out}(\hat{n}\cdot\boldsymbol{\sigma})) = [2\,\mathrm{Re}(A^*B)\cdot 2]/(|A|^2 + |B|^2)\cdot(1/2) = 2\,\mathrm{Re}(A^*B)/(|A|^2 + |B|^2)$. The standard convention for the Sherman function uses Im; this arises because in the standard choice where $\hat{n}$ is the y-axis (perpendicular to the scattering plane with k along z), the spin-flip amplitude B is defined with a factor of i relative to our convention. With the standard Wolfenstein convention [**?**] $B \to \mathrm{i}B_{\rm Wolfenstein}$, the Sherman function reads $P = 2\,\mathrm{Im}(A^*B_{\rm W})/(|A|^2 + |B_{\rm W}|^2)$, which is the standard form. In our convention:

$$\hat{\rho}_{A\rm out} = \frac{1}{2}\sigma_0 + \frac{\mathrm{Re}(A^*B)}{|A|^2 + |B|^2}(\hat{n}\cdot\boldsymbol{\sigma}), \tag{85}$$

and the Sherman function $P_{\rm out} = \mathrm{Re}(A^*B)/(|A|^2 + |B|^2)$ (real part in our convention, imaginary part in the Wolfenstein convention). □

Remark 8.7. *The Sherman function $P_{\rm out}(\theta)$ has a direct physical interpretation: it is the degree of polarization acquired by initially unpolarized spin-$\frac{1}{2}$ particles after scattering through angle θ. For $\xi \equiv 0$ (no spin-orbit coupling): $B = 0$, so $P_{\rm out} = 0$ — scattering from a spin-independent potential cannot polarize an initially unpolarized beam. For $\xi \neq 0$: $B \neq 0$ in general and $P_{\rm out} \neq 0$; the scattered beam acquires a net polarization perpendicular to the scattering plane (along $\hat{n}$). The degree of polarization is determined by the interference between the non-flip amplitude A (the part of the scattering that does not change the spin) and the spin-flip amplitude B (the part that does). Measurement of $P_{\rm out}(\theta)$ at different angles θ provides information about $B(\theta)$ relative to $A(\theta)$, and hence about the spin-orbit phase shift differences $\delta_{\ell,+} - \delta_{\ell,-}$ via Eq.* (76)*. The Sherman function was first derived in the context of electron scattering from atomic nuclei (Mott scattering), where the Coulomb potential plus relativistic spin-orbit coupling generates a non-zero B [2].*

Remark 8.8. *Theorem 8.6 is the synthesis of the QM8 and QM9 structures in a physical calculation, completing the program identified in Remark 2.5. The post-scattering density matrix Eq.* (81) *is derived using three prior results simultaneously: (a) the full Hilbert space $\mathcal{H}_{\rm full} = \mathcal{H}\otimes\mathbb{C}^2$ of QM8 Definition 5.1 (the spatial-spin entangled post-scattering state lives in $\mathcal{H}_{\rm full}$); (b) the partial trace of QM9 Definition 4.2 (tracing out the spatial degree of freedom gives the spin density matrix $\hat{\rho}_{A\rm out}$); (c) the Born rule for subsystem observables of QM9 Theorem 4.2 (the observable $\hat{n}\cdot\boldsymbol{\sigma}$ in the spin subsystem has expectation value $\mathrm{Tr}(\hat{\rho}_{A\rm out}(\hat{n}\cdot\boldsymbol{\sigma}))$). The Sherman function $P_{\rm out}$ is thus a physical observable that can be predicted entirely within the NUVO framework from the interplay of the scattering amplitude (QM10 Sec. 5), the Pauli algebra (QM8), and the density matrix (QM9). Its measurement in electron or neutron scattering experiments provides a direct test of the spin-orbit structure of the nuclear or atomic potential.*

9 Interpretive Clarifications and Scope

The present section collects the interpretive constraints governing the scattering analysis of the preceding sections and records the precise boundary between what the present paper establishes and what is deferred. Three items are addressed: the status of the rigged Hilbert space as a mathematical extension rather than a new physical postulate, the scope of the asymptotic completeness assertion and its consequences for S-matrix unitarity, and the complete inventory of what QM10 establishes and does not establish.

9.1 The Rigged Hilbert Space as a Mathematical Extension

The rigged Hilbert space $\Phi \subset \mathcal{H} \subset \Phi^*$ of Sec. 3 is not a new physical postulate of the NUVO program. It is the correct mathematical setting for the spectral theory of the free Hamiltonian $\hat{H}_0$ on $\mathcal{H} = L^2(\mathbb{R}^3)$, whose continuous spectrum $[0, \infty)$ does not support normalizable eigenstates. The Gelfand triple structure is a standard tool of functional analysis, established in the mathematical literature before its application to quantum mechanics, and its use here is no different in kind from the use of the Sobolev space $H^2(\mathbb{R}^3) \subset \mathcal{H}$ to define the domain of the kinetic energy operator in QM4: both are mathematical refinements of the basic Hilbert space structure that make precise statements possible about operators with continuous spectra or unbounded domains.

Every physical prediction of QM10 is expressed as a matrix element $\langle\phi|\hat{O}|\psi\rangle$ between normalizable states $\phi, \psi \in \mathcal{H}$. The experimental cross section $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f|^2$ is computed from the scattering amplitude f, which is the matrix element $\langle\mathbf{k}'|\hat{V}|\mathbf{k}^+\rangle$ (Theorem 4.6). In a physical experiment, the initial state is a normalized wave packet $|\phi\rangle = \int \hat{\phi}(\mathbf{k}')|\mathbf{k}'\rangle \mathrm{d}^3k'$ with $\hat{\phi} \in \mathcal{S}(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$, not a plane wave $|\mathbf{k}\rangle \in \Phi^*$. The plane wave idealization is the limit of an infinitely broad, spatially uniform beam; it gives the correct cross section formula in the limit where the wave packet is narrow in k-space (well-defined beam energy) and broad in position-space (beam diameter much larger than the target). No experiment operates with a literal plane wave, and no physical prediction of QM10 requires one.

9.2 Asymptotic Completeness and Its Scope

The unitarity proof of Theorem 5.6 invokes asymptotic completeness in Step 2: the assertion that $\mathrm{range}(\hat{\Omega}_+) = \mathrm{range}(\hat{\Omega}_-) = \mathcal{H}_{\mathrm{ac}}(\hat{H}_{\mathrm{rel}}) = \mathcal{H}$ for short-range potentials. This is a deep theorem of functional analysis, not a consequence of the algebraic structure of the Lippmann-Schwinger equation or the Møller operators. Two independent proofs exist for short-range potentials: Enss's phase-space analysis [1] and the Sigal-Soffer Mourre estimate method [5]. Both require the potential to satisfy $|V(\mathbf{r})| \leq C(1+|\mathbf{r}|)^{-1-\epsilon}$ for some $\epsilon > 0$, which excludes the Coulomb potential $V = -e^2/r$ (which decays only as $1/r$).

For the Coulomb potential, the Møller operators as defined in Eq. (37) do not converge: the long-range $1/r$ tail of the Coulomb potential generates a slowly varying phase that accumulates without bound as $t \to \pm\infty$. The Dollard modification replaces the free evolution $\mathrm{e}^{-\mathrm{i}\hat{H}_0 t/\Phi_0}$ with a modified evolution $\mathrm{e}^{-\mathrm{i}\hat{H}_0 t/\Phi_0}\hat{W}_C(t)$, where $\hat{W}_C(t)$ is a phase operator that compensates the long-range Coulomb phase; with this modification, the Coulomb Møller operators converge and asymptotic completeness holds. The Rutherford formula Eq. (54) derived from the Born approximation is nonetheless the correct differential cross section for Coulomb scattering at all energies (a special exactness of the Born approximation for the Coulomb potential confirmed by the exact Gordon-Coulomb solution); the Dollard modification affects the S-matrix definition but not the differential

cross section. QM10 takes the Rutherford formula as established by the Born calculation and notes the Dollard subtlety without developing the modified Coulomb S-matrix theory.

9.3 Scope of the Present Construction

The present paper establishes the following results, available as inputs to subsequent QM-series papers.

Rigged Hilbert space and continuum states: Definition 3.1 (Gelfand triple $\Phi \subset \mathcal{H} \subset \Phi^*$), Definition 3.3 (plane wave states $|\mathbf{k}\rangle$ as generalized eigenstates of $\hat{H}_0$), Proposition 3.4 (generalized eigenvalue equation in distributional sense), Theorem 3.5 (generalized orthonormality $\langle\mathbf{k}|\mathbf{k}'\rangle = \delta^{(3)}(\mathbf{k}-\mathbf{k}')$; completeness $\int |\mathbf{k}\rangle\langle\mathbf{k}|\mathrm{d}^3k = \hat{\mathbf{1}}_{\mathcal{H}}$; energy-shell completeness with density of states $\varrho(E) \propto \sqrt{E}$), and Proposition 3.7 (Rayleigh expansion of the plane wave in spherical harmonics and spherical Bessel functions).

Lippmann-Schwinger equation and scattering states: Definition 4.1 (free resolvent $\hat{G}_0(z)$ and its retarded/advanced boundary values $\hat{G}_0^+$, $\hat{G}_0^-$), Proposition 4.2 (position-space form of $\hat{G}_0^+$: outgoing Green's function of the Helmholtz operator), Theorem 4.4 (Lippmann-Schwinger equations for $|\mathbf{k}^+\rangle$ and $|\mathbf{k}^-\rangle$; verification of the eigenvalue equation; position-space Fredholm integral equation Eq. (31)), and Theorem 4.6 (asymptotic form $\psi_{\mathbf{k}}^+(\mathbf{r}) \to (2\pi)^{-3/2}[\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} + f(\theta,\varphi)\mathrm{e}^{\mathrm{i}kr}/r]$; scattering amplitude $f = -(\mu/2\pi\Phi_0^2)T_{\mathbf{k}'\mathbf{k}}$; differential cross section $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f|^2$).

S-matrix, T-matrix, and optical theorem: Definition 5.1 (Møller wave operators $\hat{\Omega}_\pm$; scattering states as $\hat{\Omega}_\pm|\mathbf{k}\rangle$), Proposition 5.3 (intertwining relation $\hat{H}_{\mathrm{rel}}\hat{\Omega}_\pm = \hat{\Omega}_\pm\hat{H}_0$), Definition 5.4 ($S$-matrix $\hat{S} = \hat{\Omega}_-^\dagger\hat{\Omega}_+$; T-matrix; S-T relation Eq. (43)), Theorem 5.6 (S-matrix unitarity in three steps: isometry of $\hat{\Omega}_\pm$, asymptotic completeness, unitarity of $\hat{S}$), and Theorem 5.8 (optical theorem $\sigma_{\mathrm{tot}} = (4\pi/k)\,\mathrm{Im}\,f(0)$ derived from $\hat{S}^\dagger\hat{S} = \hat{\mathbf{1}}$).

Born approximation: Definition 6.1 (Born series; n-th Born approximation; first Born $\hat{T}^{(1)} = \hat{V}$), Theorem 6.3 (Born scattering amplitude $f^{(1)}(\mathbf{q}) = -(\mu/2\pi\Phi_0^2)\tilde{V}(\mathbf{q})$; momentum transfer $|\mathbf{q}| = 2k\sin(\theta/2)$), Proposition 6.5 (Yukawa Born amplitude; differential cross section Eq. (53)), Proposition 6.6 (Rutherford formula as exact Coulomb Born limit; forward divergence for infinite-range potential), and Proposition 6.7 (Born validity conditions at low and high energy; connection to Born phase shifts Eq. (58)).

Partial wave analysis: Proposition 7.1 (radial Schrödinger equation; effective potential with centrifugal barrier; boundary conditions), Definition 7.3 (phase shift $\delta_\ell(k) \in \mathbb{R}$ from asymptotic boundary condition), Theorem 7.4 (partial wave expansion Eq. (66); partial S-matrix $S_\ell = \mathrm{e}^{2\mathrm{i}\delta_\ell}$; partial and total cross sections Eqs. (67)–(68); unitarity bound $\sigma_\ell \le 4\pi(2\ell+1)/k^2$), and Proposition 7.6 (reality; Wigner threshold law and scattering length a_s; high-energy vanishing; Levinson's theorem $\delta_\ell(0) = n_\ell\pi$).

Spin-dependent scattering: Definition 8.2 (spin scattering amplitude matrix $M = A\sigma_0 + B(\hat{n}\cdot\boldsymbol{\sigma})$), Proposition 8.3 (partial wave expressions for A and B in terms of spin-orbit phase shifts $\delta_{\ell,\pm}$; vanishing of B for $\xi \equiv 0$), Theorem 8.5 (spin-dependent differential cross section for general $\hat{\rho}_{\mathrm{in}}$; unpolarized result $|A|^2+|B|^2$; analyzing power $\mathcal{A}(\theta)$), and Theorem 8.6 (post-scattering spin density matrix Eq. (81); unpolarized result Eq. (85); Sherman function $P_{\mathrm{out}} = \mathrm{Re}(A^*B)/(|A|^2+|B|^2)$).

The following topics are outside the scope of the present paper.

Asymptotic completeness proof. The theorem that $\mathrm{range}(\hat{\Omega}_+) = \mathrm{range}(\hat{\Omega}_-) = \mathcal{H}$ for short-range potentials is invoked but not proved. The proof requires the Enss phase-space method [1] or the Sigal-Soffer Mourre estimate [5], which are beyond the scope of the present paper.

Coulomb scattering with Dollard modification. The standard Møller operators do not converge for the Coulomb $1/r$ potential; the Dollard modification is required for a rigorous S-matrix theory.

The Rutherford formula is established by the Born calculation and its exactness is noted; the full Coulomb scattering theory is deferred.

Resonances and Breit-Wigner formula. The structure of S-matrix poles in the complex energy plane, the Breit-Wigner resonance formula, and the time-delay interpretation of resonances are noted in Remark 7.8 but not developed.

Multi-channel and inelastic scattering. The coupled-channel S-matrix for processes where the internal states of the configurations change during scattering (inelastic scattering, reaction cross sections) is deferred.

Relativistic scattering. The Dirac equation replaces the Pauli equation for relativistic spin-$\frac{1}{2}$ scattering; the relativistic scattering amplitude, the Mott cross section (relativistic spin-orbit plus Coulomb), and the QED radiative corrections are deferred to QM11.

Three-body and N-body scattering. The Faddeev equations for three-body scattering and the general N-body asymptotic completeness problem (proved by Sigal and Soffer for N-body short-range systems) are beyond the scope of the present series.

10 Conclusion

10.1 Summary of Results

The present paper has developed the complete non-relativistic quantum scattering theory for two transport closure configurations interacting through a short-range potential, within the scalar–conformal NUVO framework. The fourteen principal results are as follows.

Rigged Hilbert space and plane waves (Theorem 3.5 and Proposition 3.7). The Gelfand triple $\Phi \subset \mathcal{H} \subset \Phi^*$ accommodates the generalized eigenstates $|\mathbf{k}\rangle$ of $\hat{H}_0$ with $\langle \mathbf{k}|\mathbf{k}'\rangle = \delta^{(3)}(\mathbf{k}-\mathbf{k}')$ and completeness $\int |\mathbf{k}\rangle\langle\mathbf{k}| \mathrm{d}^3k = \hat{\mathbf{1}}$. The Rayleigh expansion connects the plane wave and spherical wave bases via the spherical Bessel functions and spherical harmonics of QM5.

Free resolvent and its position-space form (Definition 4.1 and Proposition 4.2). $\langle \mathbf{r}|\hat{G}_0^+|\mathbf{r}'\rangle = -(2\mu/\Phi_0^2)\mathrm{e}^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}/(4\pi|\mathbf{r}-\mathbf{r}'|)$: the retarded Green's function of the Helmholtz operator, generating outgoing spherical waves.

Lippmann-Schwinger equation (Theorem 4.4). $|\mathbf{k}^+\rangle = |\mathbf{k}\rangle + \hat{G}_0^+\hat{V}|\mathbf{k}^+\rangle$ is the unique scattering state with outgoing boundary condition; its position-space form is a Fredholm integral equation of the second kind; the Born series is its iterative expansion.

Asymptotic form and scattering amplitude (Theorem 4.6). $\psi_{\mathbf{k}}^+(\mathbf{r}) \to (2\pi)^{-3/2}[\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} + f(\theta)\mathrm{e}^{\mathrm{i}kr}/r]$ as $r\to\infty$, with $f = -(\mu/2\pi\Phi_0^2)T_{\mathbf{k}'\mathbf{k}}$ and $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f|^2$.

Møller operators and S-matrix (Definitions 5.1 and 5.4). $\hat{S} = \hat{\Omega}_-^\dagger\hat{\Omega}_+$; S-T relation Eq. (43); intertwining $\hat{H}_{\mathrm{rel}}\hat{\Omega}_\pm = \hat{\Omega}_\pm\hat{H}_0$.

S-matrix unitarity (Theorem 5.6). $\hat{S}^\dagger\hat{S} = \hat{\mathbf{1}}$ from isometry of $\hat{\Omega}_\pm$ (unitarity of time evolution, QM4) and asymptotic completeness (cited for short-range potentials).

Optical theorem (Theorem 5.8). $\sigma_{\mathrm{tot}} = (4\pi/k)\,\mathrm{Im}\,f(0)$: total cross section equals $(4\pi/k)$ times the imaginary part of the forward scattering amplitude, derived from S-matrix unitarity.

Born approximation (Theorem 6.3). $f^{(1)}(\mathbf{q}) = -(\mu/2\pi\Phi_0^2)\tilde{V}(\mathbf{q})$: the Born amplitude is the Fourier transform of the potential at the momentum transfer $\mathbf{q} = \mathbf{k}'-\mathbf{k}$; large-angle scattering probes short-distance structure.

Yukawa and Rutherford cross sections (Propositions 6.5 and 6.6). $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}_{\mathrm{Yukawa}} = (2\mu g^2/\Phi_0^2)^2/(|\mathbf{q}|^2+\lambda^2)^2$; the Coulomb limit $\lambda\to 0$ gives the exact Rutherford formula $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = (\mu e^2/2\Phi_0^2k^2)^2\sin^{-4}(\theta/2)$.

Partial wave expansion (Theorem 7.4). $f(\theta) = k^{-1}\sum_\ell(2\ell+1)\mathrm{e}^{\mathrm{i}\delta_\ell}\sin\delta_\ell\,P_\ell(\cos\theta)$; partial cross sections $\sigma_\ell = (4\pi/k^2)(2\ell+1)\sin^2\delta_\ell$ summing to σ_{tot}.

Phase shift properties (Proposition 7.6). Reality; Wigner threshold law $\delta_\ell \to 0$ as $k \to 0$ for $\ell \geq 1$; scattering length a_s with $\sigma_{\text{tot}} \to 4\pi a_s^2$; Levinson's theorem $\delta_\ell(0) = n_\ell \pi$ connecting phase shifts to bound state count.

Spin scattering amplitude matrix (Definition 8.2 and Proposition 8.3). $M = A\sigma_0 + B(\hat{n}\cdot\boldsymbol{\sigma})$ with partial wave expressions in terms of spin-orbit phase shifts $\delta_{\ell,\pm}$; $B = 0$ for $\xi \equiv 0$.

Spin-dependent cross section and analyzing power (Theorem 8.5). $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \mathrm{Tr}(M\hat{\rho}_{\text{in}}M^\dagger)$; unpolarized result $|A|^2 + |B|^2$; analyzing power $\mathcal{A} = 2\,\mathrm{Im}(A^*B)/(|A|^2 + |B|^2)$ encoding sensitivity to incident polarization.

Post-scattering spin density matrix and Sherman function (Theorem 8.6). $\hat{\rho}_{A\text{out}} = M\hat{\rho}_{\text{in}}M^\dagger/\mathrm{Tr}(M\hat{\rho}_{\text{in}}M^\dagger)$ from QM9 partial trace; Sherman function $P_{\text{out}} = \mathrm{Re}(A^*B)/(|A|^2 + |B|^2)$ measuring the polarization acquired by an initially unpolarized beam.

10.2 Programmatic Significance

The results of the present paper are of broad programmatic significance on three grounds.

The first is the completion of the two-body spectral problem initiated in QM7. QM7 analyzed the two-body system in the bound state sector: the coupled oscillator Hamiltonian with discrete, quantized positive-energy spectrum and the normal mode Fock eigenstates. QM10 analyzes the same two-body framework in the scattering sector: the relative Hamiltonian with continuous positive-energy spectrum and the plane wave scattering eigenstates. The two sectors are complementary and together constitute the complete spectral data of the relative Hamiltonian $\hat{H}_{\text{rel}} = \hat{H}_0 + \hat{V}$: bound states ($E < 0$, discrete spectrum, QM7 Fock structure) and scattering states ($E > 0$, continuous spectrum, QM10 Lippmann-Schwinger structure). The connection between the two sectors is Levinson's theorem (Proposition 7.6 (iv)), which is the first explicit link in the QM-series between the bound state count n_ℓ of QM7 and the scattering phase shifts $\delta_\ell(0)$ of QM10. This structural complementarity is the organizing principle of the two-body spectral problem and will recur in the relativistic extension of QM11.

The second ground of significance is that QM10 is the first paper in the QM-series where all of QM4, QM5, QM7, QM8, and QM9 are used simultaneously in a single physical calculation. The spin-dependent scattering analysis of Sec. 8 requires: QM4 (unitarity of time evolution, conservation of angular momentum), QM5 (partial wave decomposition, spherical harmonics, phase shifts), QM7 (center-of-mass and relative coordinate separation), QM8 (full Hilbert space $\mathcal{H}_{\text{full}}$, Pauli algebra for $(\hat{n}\cdot\boldsymbol{\sigma})^2 = \sigma_0$, spin-orbit coupling eigenvalues), and QM9 (reduced density matrix $\hat{\rho}_{A\text{out}}$ from partial trace, Born rule for spin observables). The Sherman function $P_{\text{out}}(\theta)$ is the observable whose prediction requires all five inputs simultaneously: it is the synthesis of the NUVO QM-series in a single real-valued function that is directly measurable in electron and neutron scattering experiments. The derivation of the Sherman function from first principles within the NUVO framework — from the double-cover holonomy of QM8 through the Pauli algebra, the spin-orbit splitting of the phase shifts, the partial wave interference, and the QM9 density matrix — is the sharpest demonstration in the present series that the program produces concrete physical predictions from abstract geometric principles.

The third ground of significance is the Born approximation and the Rutherford formula. The derivation of the Rutherford formula (Proposition 6.6) from the Born approximation applied to the Coulomb potential reproduces the result that Rutherford used in 1911 to interpret the gold-foil experiment and infer the nuclear model of the atom. In the standard quantum mechanical treatment, the Rutherford formula is derived either from the exact Coulomb scattering solution (a technically involved calculation using parabolic coordinates) or from the Born approximation with the observation that the two agree exactly for the Coulomb case. In the NUVO framework, the

Born approximation is derived from the Lippmann-Schwinger equation (Theorem 6.3), which is itself derived from the QM4 dynamical framework applied to the scattering states; the Rutherford formula emerges as a theorem from this chain. The historical significance of the Rutherford formula in establishing the nuclear model of the atom makes it a natural calibration point for the NUVO scattering theory: the same formalism that gives the Rutherford cross section in the non-relativistic limit gives the Mott cross section (Rutherford plus spin-orbit, with the Sherman function) in the spin-orbit-coupled case, and the relativistic Mott cross section (derived from the Dirac equation) in QM11.

10.3 Transition to QM11

QM11 develops the relativistic extension of the spin-$\frac{1}{2}$ framework established in QM8, deriving the Dirac equation as the relativistic first-principles equation for a spin-$\frac{1}{2}$ transport closure configuration and completing several derivations deferred from QM8 and QM10.

The primary new structure of QM11 is the Dirac equation:

$$\big(\mathrm{i}\Phi_0\gamma^\mu\partial_\mu - mc\big)\Psi = 0, \tag{86}$$

a first-order relativistic wave equation on a four-component spinor $\Psi = (\Psi_L^\uparrow, \Psi_L^\downarrow, \Psi_S^\uparrow, \Psi_S^\downarrow)^\mathsf{T}$ (large and small components), where the Dirac matrices $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$ satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ (the Clifford algebra of Minkowski spacetime, extending the Pauli Clifford algebra $\{\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_k\} = 2\delta_{jk}\sigma_0$ of QM8 Theorem 4.2 to four dimensions). Three derivations deferred from QM8 are completed in QM11: the g-factor $g = 2$ for the electron spin magnetic moment emerges from the minimal coupling $\partial_\mu \to \partial_\mu - \mathrm{i}eA_\mu/(\Phi_0 c)$ in the Dirac equation; the non-relativistic spin-orbit Hamiltonian $\xi(r)\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}$ with the correct Thomas factor $\frac{1}{2}$ emerges from the Foldy-Wouthuysen reduction to order $(v/c)^2$; and the hydrogen fine structure energy $E_{nj} = E_n[1 + (\alpha^2/n^2)(n/(j+\frac{1}{2}) - \frac{3}{4})]$ is derived from the Dirac equation, completing the QM8 fine structure result with the Darwin term and the relativistic kinematic correction. The Dirac equation also provides the relativistic scattering formalism for QM10: the relativistic analogue of the Born amplitude for Coulomb scattering is the Mott cross section $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}_{\mathrm{Mott}} = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}_{\mathrm{Rutherford}} \cdot [1 - \beta^2\sin^2(\theta/2)]/(1-\beta^2)$ (where $\beta = v/c$), derived in QM11 from the Dirac-Coulomb scattering amplitude. Finally, and most significantly, QM11 establishes the spin-statistics theorem as the fifth holonomy quantization of the NUVO program: the relativistic holonomy of $\mathrm{SL}(2,\mathbb{C})$ (the double cover of the Lorentz group $\mathrm{SO}(3,1)$) connects the half-integer spin quantum number of QM8 to the fermionic exchange parity $\pi = -1$ of QM7 as a theorem derived from the CPT symmetry of the Dirac equation, not as a postulate.

References

[1] Volker Enss. Asymptotic completeness for quantum mechanical potential scattering. *Communications in Mathematical Physics*, 61(3):285–291, 1978. Proves asymptotic completeness for short-range potentials $|V(\mathbf{r})| \leq C(1+|\mathbf{r}|)^{-1-\epsilon}$ using phase-space analysis; the theorem asserts $\mathrm{range}(\Omega_+) = \mathrm{range}(\Omega_-) = \mathcal{H}_{\mathrm{ac}}(H) = \mathcal{H}$ (no singular continuous spectrum for short-range potentials), which is invoked in Step 2 of QM10 Theorem 5.1; the proof does not extend to Coulomb potentials (the Dollard modification is required for $1/r$).

[2] Nevill F. Mott. Scattering of fast electrons by atomic nuclei. *Proceedings of the Royal Society of London A*, 124(794):425–442, 1929. Derives the Mott scattering cross section for electrons from a Coulomb potential, including the spin-orbit (relativistic) correction; establishes that the

scattered beam becomes polarized (acquiring a non-zero Sherman function $S(\theta) \neq 0$) when relativistic spin-orbit effects are present; the spin scattering amplitude matrix $M = A\sigma_0 + B(\hat{n} \cdot \boldsymbol{\sigma})$ and the Sherman function of QM10 Definition 8.1 and Theorem 8.2 are the non-relativistic generalizations of the structures introduced in this paper; the relativistic Mott cross section is derived in QM11.

[3] Roger G. Newton. *Scattering Theory of Waves and Particles*. Springer-Verlag, New York, 2nd edition, 1982. Comprehensive reference for: the mathematical foundation of scattering theory including the rigged Hilbert space and spectral theory of the free Hamiltonian (Chapter 7); asymptotic completeness and its proof for short-range potentials (Chapter 7); the Coulomb scattering theory with the Dollard modification (Chapter 13); Levinson's theorem with the complete proof via Sturm-Liouville homotopy (Chapter 12, cited in QM10 Proposition 7.2 (iv)); the spin-dependent scattering amplitude matrix $M = AI + B(\hat{n} \cdot \boldsymbol{\sigma})$ and its symmetry derivation (Chapter 15, cited in QM10 Remark 8.2).

[4] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics, Vol. II: Fourier Analysis, Self-Adjointness*. Academic Press, New York, 1975. Reference for: the Kato-Rellich theorem for the self-adjointness of $H = T + V$ for Kato-class potentials (Theorem X.12, cited in QM10 Remark 2.2 for Yukawa and Coulomb potentials); the essential self-adjointness of $-\nabla^2$ on $\mathcal{S}(\mathbb{R}^3)$ (Theorem X.28); the Sobolev space $H^2(\mathbb{R}^3)$ as the natural domain of the kinetic energy operator; and the outgoing Green's function G_0^+ of the Helmholtz operator (cited in QM10 Proposition 4.1).

[5] Israel M. Sigal and Avy Soffer. The n-particle scattering problem: Asymptotic completeness for short-range systems. *Annals of Mathematics*, 126(1):35–108, 1987. Proves asymptotic completeness for N-body short-range scattering systems using the Mourre estimate and positive commutator methods; for $N = 2$ (the case relevant to QM10) the result specializes to the two-body asymptotic completeness invoked in QM10 Theorem 5.1 Step 2; the Mourre estimate approach provides an alternative to the Enss method and applies to a wider class of potentials including those with local singularities.