QM3 — Uncertainty Relations and the Limits of Transport Resolution

in Scalar–Conformal NUVO Systems *Preprint, Version 1.0*[*]

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Notation and Conventions

- $\mathcal{M}$ denotes the spacetime manifold.
- η denotes the reference Lorentzian metric (typically Minkowski in a global chart).
- g denotes the physical metric.
- The scalar field $\Lambda : \mathcal{M} \to \mathbb{R}_{>0}$ is the NUVO modulation field.
- The physical metric is scalar–conformal:

$$g_{\mu\nu} = \Lambda^2 \eta_{\mu\nu}.$$

- $\Lambda_0 > 0$ denotes the baseline scalar availability level supported by the intrinsic delivery structure of the underlying field. In the absence of localized structural occupation the scalar field satisfies $\Lambda(x) = \Lambda_0$.
- The dimensionless scalar diagnostic is

$$\lambda(x) := \frac{\Lambda(x)}{\Lambda_0}.$$

- The scalar field represents the *locally available structural capacity* of the underlying delivery field. Localized structures may reduce this availability through occupation or transport, but the intrinsic delivery baseline Λ_0 remains fixed.
- Greek indices $\mu, \nu, \ldots$ range over spacetime coordinates $0, 1, 2, 3$.
- We use the Einstein summation convention unless explicitly stated otherwise.

Remark 0.1. *Unless otherwise stated, the background signature is* $(-, +, +, +)$.

[*]Bibliography is provisional. Cross-references to companion NUVO-series papers (M-, SR-, Q-, QB-, QM-series) will be updated with Zenodo DOIs in subsequent versions.

Program scope.

Abstract

The canonical commutation relation $[\hat{x}^j, \hat{p}_k] = \mathrm{i}\Phi_0\, \delta^j{}_k$ was established in QM1 as a representation identity of the scalar–conformal transport generators on the Hilbert space $\mathcal{H}$. The present paper derives the structural consequences of this relation for the simultaneous resolution of transport closure configurations with respect to pairs of non-commuting observables.

The Robertson uncertainty inequality $\Delta A \cdot \Delta B \geq \frac{1}{2}|\langle [A, B]\rangle|$ is established as a theorem for any two self-adjoint operators A and B on $\mathcal{H}$, derived from the Cauchy–Schwarz inequality on $\mathcal{H}$ alone. No measurement disturbance argument, no microscope thought experiment, and no physical postulate enters the derivation. The Schrödinger improvement of the Robertson bound, which includes an anti-commutator correction term, is derived as a corollary by the same method.

Applied to the position and momentum transport generators, the Robertson inequality yields the Heisenberg uncertainty relation $\Delta x^j \cdot \Delta p_k \geq (\Phi_0/2)\, \delta^j{}_k$. This bound is interpreted within the NUVO framework as a geometric constraint on the simultaneous spatial resolution and transport momentum resolution of the closure density: the product of the spatial spread and the momentum-space spread of any admissible closure state cannot be less than $\Phi_0/2$.

The energy-time uncertainty relation $\Delta E \cdot \Delta t \geq \Phi_0/2$ is derived separately from the phase coherence lifetime of the transport closure system. Since time is not an operator in the Hilbert space framework, the energy-time relation requires a distinct argument; it is established from the rate of change of expectation values under transport evolution and the coherence lifetime of the transport phase.

Minimum-uncertainty states are identified as those for which the Cauchy–Schwarz inequality is saturated. For the position-momentum pair, these are Gaussian closure configurations: states whose closure density is a Gaussian in position space and whose momentum-space transform is correspondingly Gaussian, with the product of widths equal to $\Phi_0/2$. These minimum-uncertainty Gaussian states are the single-particle precursors of the coherent states developed in QM6.

No new postulates are introduced. The uncertainty relations are structural theorems derived from the commutation relation of QM1 and the Hilbert space geometry of the closure state space.

1 Introduction

1.1 Position Within the QM-Series

The scalar–conformal NUVO program has now established, through the M-, Q-, QB-, and QM-series, a progressive derivation of quantum structure from transport closure geometry. QM1 constructed the complete separable Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C})$ as the natural completion of the space of transport closure encodings, derived normalization as a structural constraint from total-closure conservation, promoted the momentum and energy transport generators of QB2 to essentially self-adjoint operators on their Sobolev domains, and established the canonical commutation relation $[\hat{x}^j, \hat{p}_k] = \mathrm{i}\Phi_0\, \delta^j{}_k$ on the dense Schwartz domain $\mathcal{S}(\mathbb{R}^3) \subset \mathcal{H}$. QM2 derived the superposition principle as a theorem from the linearity of the transport closure operator in the integrable exchange-sector regime, analyzed two-path transport configurations, derived the interference fringe pattern from the transport phase difference $\Delta\phi(x) = \phi_B(x) - \phi_A(x)$, and established the complementarity relation $\mathcal{V}^2 + \mathcal{W}^2 \leq 1$ as a consequence of the Cauchy–Schwarz inequality on $\mathcal{H}$. The present paper, QM3, completes the algebraic foundation of the QM-series by deriving the uncertainty relations as structural theorems from the canonical commutation relation of QM1 and the same Cauchy–Schwarz inequality that yielded the complementarity relation in QM2.

The position of QM3 within the series is parallel to that of QM2. Neither paper requires the Schrödinger dynamics of QM4: the uncertainty relations, like the superposition principle and the complementarity relation, are algebraic consequences of the Hilbert space structure of QM1 and the commutation relation derived in QB2. QM1 provides the Hilbert space, the inner product, the Cauchy–Schwarz inequality, and the canonical commutation relation; QM2 and QM3 extract the physical content of these structures by two parallel applications of the same algebraic technique. In QM2 the technique yields the complementarity inequality from the coefficient structure of a two-path superposition; in QM3 the same technique yields the Robertson uncertainty inequality from the commutator of two transport observables. Together, QM1, QM2, and QM3 establish the complete algebraic and geometric structure of the quantum state space without reference to dynamics: the normalization, the superposition structure, the complementarity between path information and coherence, and the uncertainty bounds on simultaneous transport resolution are all present before any time evolution is introduced.

The uncertainty principle in the standard formulation of quantum mechanics is presented in two distinct ways that are often conflated. The first is the measurement-disturbance account, associated with Heisenberg's microscope thought experiment: measuring the position of a particle with a photon imparts a momentum kick whose magnitude is bounded below by $\hbar/\Delta x$, so that the precision of the position measurement limits the precision of any subsequent momentum determination. The second is the state-spread account, associated with the Robertson inequality: for any quantum state, the product of the standard deviations of position and momentum in that state is at least $\hbar/2$, regardless of any measurement. In the NUVO framework, only the second account is available and only the second account is derived in the present paper. The Robertson inequality follows directly from the canonical commutation relation and the Cauchy–Schwarz inequality on $\mathcal{H}$ and makes no reference to photons, measurement devices, disturbance, or any physical process. It is a statement about the transport closure state $\Psi \in \mathcal{H}$ itself: the spatial spread Δx^j of the closure density $|\Psi|^2$ and the momentum spread Δp_j of the momentum-space density $|\tilde{\Psi}|^2$ cannot simultaneously be made smaller than their product equals $\Phi_0/2$ for any admissible closure state. The measurement-disturbance account is a separate result that requires the coherence-gated interaction theory of QB5 and QB6 and lies outside the scope of the present paper.

The results established in QM3 propagate forward through the QM-series in two principal ways. First, the minimum-uncertainty Gaussian closure states identified in Sec. 6 are the structural precursors of the coherent states of QM6: states that minimize the position-momentum uncertainty product and additionally preserve their Gaussian profile under harmonic oscillator dynamics. The connection between the algebraic minimum-uncertainty property (established here) and the dynamical coherence property (established in QM6) is one of the program's key bridges between the algebraic and dynamical layers of the QM-series. Second, the Robertson inequality of Sec. 3, stated for general self-adjoint operators A and B on $\mathcal{H}$, is the template from which all uncertainty relations in the QM-series are derived: the angular momentum uncertainty relations of Sec. 7 and QM5, the spin uncertainty relations of QM8, and the spectral linewidth interpretation of resonance widths in QM10 all follow from the same inequality applied to the relevant commutation algebra.

1.2 Objective of the Present Work

The central objective of the present paper is to establish the uncertainty relations as structural theorems of the scalar–conformal NUVO transport closure system on $\mathcal{H}$, derived from the canonical commutation relation of QM1 and the Cauchy–Schwarz inequality, without invoking measurement disturbance, physical thought experiments, or new postulates. Specifically, the paper aims to establish five claims.

1. The Robertson uncertainty inequality $\Delta A \cdot \Delta B \geq \frac{1}{2}|\langle [A,\, B]\rangle|$ holds for any two self-adjoint

operators A and B on $\mathcal{H}$, for any normalized closure state Ψ in their common domain. The derivation proceeds in two steps: the cross inner product $\left\langle \widehat{\Delta A}\Psi, \widehat{\Delta B}\Psi \right\rangle_{\mathcal{H}}$ is decomposed into real and imaginary parts proportional to the anti-commutator and commutator of A and B respectively, and the Cauchy–Schwarz inequality then bounds the product of standard deviations by the modulus of this inner product. The Schrödinger improvement, which retains the anti-commutator contribution and yields the tighter bound $(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}|\langle\{\widehat{\Delta A}, \widehat{\Delta B}\}\rangle|^2 + \frac{1}{4}|\langle[A, B]\rangle|^2$, is established as a corollary.

2. Applied to the position and momentum transport generators with $[\hat{x}^j, \hat{p}_k] = \mathrm{i}\Phi_0\, \delta^j{}_k$ from QM1 Proposition 5.4, the Robertson inequality yields the Heisenberg uncertainty relation $\Delta x^j \cdot \Delta p_k \geq (\Phi_0/2)\, \delta^j{}_k$ for any normalized closure state. This is interpreted as a geometric constraint on the simultaneous spatial and momentum resolution of the closure density: the root-mean-square spatial spread and root-mean-square momentum spread of any admissible closure state cannot simultaneously be less than $\Phi_0/2$ in their product.

3. The energy-time uncertainty relation $\Delta E \cdot \Delta t_G \geq \Phi_0/2$, where Δt_G is the characteristic time over which the expectation value of an observable G changes by one standard deviation, is derived from the rate-of-change identity $|\mathrm{d}\langle G\rangle/\mathrm{d}t| = (1/\Phi_0)|\langle[\hat{H}, G]\rangle|$ and the Robertson inequality applied to $\hat{H}$ and G. Since time is a parameter rather than an operator in the Hilbert space framework, this derivation is necessarily different from the position-momentum argument and follows the route through the Heisenberg equation of motion established in QM4.

4. Minimum-uncertainty states—those for which the Cauchy–Schwarz inequality is saturated and the Robertson bound is achieved as an equality—are characterized for the position-momentum pair. They are precisely the Gaussian closure configurations of the form $\Psi(x) \propto \exp[-(x-\langle x\rangle)^2/(4\sigma^2)+\mathrm{i}\langle p\rangle x/\Phi_0]$, derived by solving the saturation condition $\widehat{\Delta p}\Psi = \mathrm{i}\mu\, \widehat{\Delta x}\Psi$ as a first-order differential equation in position space.

5. The angular momentum uncertainty relations $\Delta L_j \cdot \Delta L_k \geq (\Phi_0/2)|\langle \hat{L}_l\rangle|$ follow from the Robertson inequality and the commutation algebra $[\hat{L}_j, \hat{L}_k] = \mathrm{i}\Phi_0\, \epsilon_{jkl}\hat{L}_l$ introduced in QM4 and developed fully in QM5. These relations establish a template for all angular momentum uncertainty analysis in the QM-series.

Claims (1) through (5) are logically ordered: the Robertson inequality of claim (1) is the universal template; claims (2) and (5) are specializations to specific commutation algebras; claim (3) is a distinct derivation required by the non-operator character of time; and claim (4) is the saturation analysis of claim (1) applied to the position-momentum pair.

1.3 What Is Not Assumed

The present work maintains without modification the interpretive discipline of the Q-, QB-, QM1, and QM2 papers. Three exclusions are of particular importance for QM3 given the history of the uncertainty principle.

The uncertainty relations are not postulated. In the standard formulation of quantum mechanics, the Heisenberg uncertainty principle is sometimes stated as an independent axiom of the theory, alongside the state-space postulate, the Born rule, and the Schrödinger equation. In the NUVO framework it is a theorem, derived from the canonical commutation relation of QB2 and QM1 and the Cauchy–Schwarz inequality of QM1 Lemma 4.2. Neither of these inputs is a new postulate of the present paper: the canonical commutation relation was derived in QB2 from the representation of transport generators through phase gradients, and the Cauchy–Schwarz inequality is a consequence of the inner product axioms satisfied by the holonomic coherence functional of QB3.

The measurement-disturbance interpretation is not adopted. The quantities ΔA and ΔB in the Robertson inequality are the standard deviations of the observables A and B in the closure state Ψ, defined in Definition 2.2 as properties of Ψ and A (or B) independently of any interaction or measurement event. The Robertson inequality bounds their product; it makes no claim about what happens when a position measurement is performed and a subsequent momentum measurement is made. The measurement-disturbance account is a separate phenomenon that involves the coherence-gated interaction structure of QB5 and QB6 and is not derived in the present paper.

Time is not an operator in $\mathcal{H}$. The energy-time uncertainty relation of Sec. 5 is not derived by applying the Robertson inequality with $A = \hat{H}$ and $B = \hat{t}$ for some time operator $\hat{t}$, because no such self-adjoint operator exists in $\mathcal{H}$: the Pauli argument shows that a time operator satisfying $[\hat{H}, \hat{t}] = -\mathrm{i}\Phi_0$ would require the Hamiltonian to have spectrum all of $\mathbb{R}$, contradicting the physical requirement that the energy be bounded below. The energy-time relation is instead derived from the rate-of-change identity for expectation values and the Robertson inequality applied to $\hat{H}$ and a general observable G. The resulting relation involves a state-dependent characteristic time Δt_G rather than an operator, and its physical interpretation is accordingly different from the position-momentum relation.

1.4 Structure of the Paper

Sec. 2 recalls the canonical commutation relation from QM1, the definition of the standard deviation of a transport observable as a property of the closure state, the Cauchy–Schwarz inequality from QM1 Lemma 4.2 in the form used here, and the algebraic technique introduced in QM2 that is used again in the Robertson derivation. Sec. 3 derives the Robertson uncertainty inequality for general self-adjoint operators on $\mathcal{H}$ from the Cauchy–Schwarz inequality, establishes the Schrödinger improvement as a corollary, characterizes the saturation condition, and records the interpretive constraints on the result. Sec. 4 applies the Robertson inequality to the position and momentum transport generators, derives the Heisenberg uncertainty relation, interprets it as a geometric constraint on the simultaneous transport resolution of the closure density, and records the full three-dimensional structure. Sec. 5 derives the energy-time uncertainty relation from the rate-of-change identity for expectation values and the Robertson inequality, discusses the non-operator character of time and the Pauli argument, and interprets the relation in terms of the transport phase coherence lifetime. Sec. 6 identifies minimum-uncertainty states for the position-momentum pair as Gaussian closure configurations by solving the saturation condition as a differential equation, records their properties, and establishes their programmatic role as single-particle precursors of the coherent states of QM6. Sec. 7 applies the Robertson inequality to the angular momentum commutation algebra, establishes the angular momentum uncertainty relations, and records the scope of the angular momentum analysis relative to the full treatment in QM5. Sec. 8 collects interpretive clarifications, maintains the interpretive boundary conditions of the prior series, and records the scope of the present construction. Sec. 9 summarizes the results, records their programmatic significance for the QM-series, and prepares the transition to QM5.

2 Recalled Structure from QM1 and QM2

The present section collects the four inputs from QM1 and QM2 that are directly needed for the derivations of Secs. 3–7. The canonical commutation relation, the definition of standard deviation as a closure-state property, the Cauchy–Schwarz inequality, and the algebraic decomposition technique are all prior results; nothing in this section is new. Recording them together before the derivations serves two purposes: it makes the logical dependencies of the paper explicit, and it identifies the structural parallel between the Robertson derivation of Sec. 3 and the complementarity derivation of QM2, which both draw on the same two inputs from QM1.

2.1 The Canonical Commutation Relation

The foundational algebraic input to the uncertainty relations is the canonical commutation relation established in QB2 and promoted to the complete Hilbert space in QM1.

Origin. In QB2, the momentum transport generators $\hat{p}_j = -\mathrm{i}\Phi_0\,\partial_j$ and the position operators $\hat{x}^j$ (multiplication by x^j) were shown to satisfy the commutation identity

$$[\hat{x}^j, \hat{p}_k]\,\Psi = \mathrm{i}\Phi_0\,\delta^j{}_k\,\Psi \tag{1}$$

for Ψ in the finite representational span $\mathcal{H}^{\mathrm{fin}}$, derived as a direct consequence of the representation of spatial transport generators through phase gradients. QM1 Proposition 5.4 promoted Eq. (1) to the complete Hilbert space $\mathcal{H}$, establishing that it holds for all $\Psi \in \mathcal{S}(\mathbb{R}^3) \subset \mathcal{H}$.

Key algebraic property. The right-hand side of Eq. (1) is $\mathrm{i}\Phi_0\,\delta^j{}_k\,\Psi$: a scalar multiple of Ψ. This means the commutator $[\hat{x}^j, \hat{p}_k]$ acts as a scalar multiple of the identity operator on $\mathcal{S}(\mathbb{R}^3)$, so its expectation value in any normalized state Ψ is the same constant:

$$\langle[\hat{x}^j, \hat{p}_k]\rangle = \left\langle\Psi, [\hat{x}^j, \hat{p}_k]\Psi\right\rangle_{\mathcal{H}} = \mathrm{i}\Phi_0\,\delta^j{}_k \tag{2}$$

for all normalized $\Psi \in \mathcal{S}(\mathbb{R}^3)$ and all j, k. This state-independence of the commutator expectation value is what makes the Heisenberg bound $\Delta x^j \cdot \Delta p_k \geq (\Phi_0/2)\,\delta^j{}_k$ a universal constraint holding for every admissible closure state, not just for special states.

Remark 2.1. *Equation* (1) *is not a postulate of the NUVO framework. It was derived in QB2 from the differential operator representation of spatial transport: the momentum generator* $\hat{p}_k = -\mathrm{i}\Phi_0\,\partial_k$ *acts on the transport phase through differentiation, and the position operator* $\hat{x}^j$ *acts through multiplication by* x^j*; their commutator measures the failure of these two operations to commute, which is exactly* $-\mathrm{i}\Phi_0\,\partial_k(x^j\,\cdot) + \mathrm{i}\Phi_0\,x^j\partial_k(\,\cdot\,) = \mathrm{i}\Phi_0\,\delta^j{}_k$ *by the product rule. The present paper uses Eq.* (1) *as a recalled input; its derivation is complete in QB2 and QM1.*

2.2 Standard Deviation of a Transport Observable

The quantities ΔA and ΔB appearing in the Robertson inequality are defined here as properties of the closure state, prior to and independent of any interaction or measurement.

Definition 2.2 (Standard deviation of a transport observable)**.** *Let A be a self-adjoint operator on $\mathcal{H}$ and let $\Psi \in \mathcal{D}(A) \cap \mathcal{D}(A^2)$ be a normalized closure state. The* expectation value *of A in Ψ is*

$$\langle A\rangle \coloneqq \langle\Psi, A\,\Psi\rangle_{\mathcal{H}} \in \mathbb{R}, \tag{3}$$

which is real since A is self-adjoint. The dispersion operator *of A is*

$$\widehat{\Delta A} \coloneqq A - \langle A\rangle\,\hat{\mathbf{1}}, \tag{4}$$

a self-adjoint operator with the same domain as A. The variance *of A in Ψ is*

$$(\Delta A)^2 := \langle A^2 \rangle - \langle A \rangle^2 = \left\langle \Psi, \widehat{\Delta A}^2 \, \Psi \right\rangle_{\mathcal{H}} = \left\| \widehat{\Delta A} \, \Psi \right\|_{\mathcal{H}}^2, \tag{5}$$

and the standard deviation *(uncertainty) of A is $\Delta A = \left\| \widehat{\Delta A} \, \Psi \right\|_{\mathcal{H}} \geq 0$.*

Remark 2.3. *The identity $(\Delta A)^2 = \left\| \widehat{\Delta A} \, \Psi \right\|_{\mathcal{H}}^2$ in Eq. (5) is the key representation: the variance of A in Ψ is the squared $\mathcal{H}$-norm of the state $\widehat{\Delta A} \, \Psi = (A - \langle A \rangle)\Psi$. This representation is what makes the Cauchy–Schwarz inequality applicable in Sec. 3: the product of variances $(\Delta A)^2(\Delta B)^2 = \left\| \widehat{\Delta A}\Psi \right\|^2 \left\| \widehat{\Delta B}\Psi \right\|^2$ is immediately bounded below by $| \left\langle \widehat{\Delta A}\Psi, \widehat{\Delta B}\Psi \right\rangle_{\mathcal{H}} |^2$ via the Cauchy–Schwarz inequality of QM1 Lemma 4.2.*

Remark 2.4. *Definition 2.2 defines ΔA entirely in terms of the closure state Ψ and the operator A. It does not refer to any sequence of measurements, any ensemble of preparations, or any interaction event. In the NUVO framework, ΔA is a property of the transport closure configuration Ψ: it is the root-mean-square deviation of the closure density weighted by the operator A from the mean value $\langle A \rangle$. For $A = \hat{x}^j$ (multiplication by x^j), this is the root-mean-square spatial spread of the closure density $|\Psi(x)|^2$ in the j-th direction. For $A = \hat{p}_j$ (differentiation), this is the root-mean-square spread of the momentum-space density $|\tilde{\Psi}(p)|^2$ in the j-th direction. Both quantities are intrinsic geometric properties of the closure state; no measurement is implied.*

2.3 The Cauchy–Schwarz Inequality in the Form Used Here

The Cauchy–Schwarz inequality was established in QM1 Lemma 4.2 as a property of the closure inner product. It is recalled here in the specific form in which it enters the Robertson derivation.

General form (QM1 Lemma 4.2, property (iv)). For any $f, g \in \mathcal{H}$:

$$| \langle f, g \rangle_{\mathcal{H}} |^2 \leq \|f\|_{\mathcal{H}}^2 \, \|g\|_{\mathcal{H}}^2, \tag{6}$$

with equality if and only if f and g are proportional: $g = \lambda f$ for some $\lambda \in \mathbb{C}$.

Applied form for the Robertson derivation. Set $f = \widehat{\Delta A}\Psi$ and $g = \widehat{\Delta B}\Psi$ for normalized $\Psi \in \mathcal{D}(A) \cap \mathcal{D}(B)$. By Eq. (5): $\|f\|_{\mathcal{H}}^2 = \left\| \widehat{\Delta A}\Psi \right\|_{\mathcal{H}}^2 = (\Delta A)^2$ and $\|g\|_{\mathcal{H}}^2 = \left\| \widehat{\Delta B}\Psi \right\|_{\mathcal{H}}^2 = (\Delta B)^2$. The Cauchy–Schwarz inequality then reads:

$$| \left\langle \widehat{\Delta A}\Psi, \widehat{\Delta B}\Psi \right\rangle_{\mathcal{H}} |^2 \leq (\Delta A)^2(\Delta B)^2. \tag{7}$$

The Robertson inequality will follow by bounding the left-hand side of Eq. (7) below by a term involving the commutator $[\hat{x}^j, \hat{p}_k]$, completing the chain

$$(\Delta A)^2(\Delta B)^2 \geq | \left\langle \widehat{\Delta A}\Psi, \widehat{\Delta B}\Psi \right\rangle_{\mathcal{H}} |^2 \geq \tfrac{1}{4} |\langle [A, B] \rangle|^2.$$

The first inequality is Cauchy–Schwarz; the second is the decomposition derived in Lemma 3.1.

2.4 The Algebraic Technique from QM2

The derivation of the Robertson inequality in Sec. 3 uses an algebraic technique that appeared in a different form in QM2. Identifying the structural parallel here makes both derivations easier to follow and reinforces the unity of the algebraic approach across the two papers.

In QM2, the complementarity relation $\mathcal{V}^2+\mathcal{W}^2 \le 1$ was derived by two steps. First, the squared norm of the two-path superposition $\Psi_{AB} = c_A\Psi_A + c_B\Psi_B$ was expanded as

$$\|\Psi_{AB}\|_{\mathcal{H}}^2 = |c_A|^2 + |c_B|^2 + 2\,\mathrm{Re}[c_A\overline{c_B}\,\langle\Psi_A, \Psi_B\rangle_{\mathcal{H}}],$$

decomposing the norm into a sum of squared moduli and a cross-term. Second, the Cauchy–Schwarz inequality bounded the cross-term, yielding the complementarity inequality.

In QM3, the same two-step structure appears with different objects. First, the cross inner product $\left\langle\widehat{\Delta A}\Psi, \widehat{\Delta B}\Psi\right\rangle_{\mathcal{H}}$ will be decomposed into its real part (proportional to the anti-commutator expectation value) and its imaginary part (proportional to the commutator expectation value), in Lemma 3.1. Second, the Cauchy–Schwarz inequality Eq. (7) bounds the modulus of this inner product, yielding the Robertson inequality.

The structural parallel is:

	QM2	QM3
Objects	$c_A\Psi_A,\ c_B\Psi_B$	$\widehat{\Delta A}\Psi,\ \widehat{\Delta B}\Psi$
Decomposition	real + cross-term in norm	real + imaginary in inner product
Key identity	$\|\Psi_{AB}\|^2 = 1$ (normalization)	$\mathrm{Im}(\langle f, g\rangle) = \frac{1}{2}\langle[A, B]\rangle$
Bound applied	Cauchy–Schwarz on $\mathcal{H}$	Cauchy–Schwarz on $\mathcal{H}$
Result	$\mathcal{V}^2 + \mathcal{W}^2 \le 1$	$\Delta A \cdot \Delta B \ge \frac{1}{2}\|\langle[A, B]\rangle\|$

Both results are algebraic inequalities on $\mathcal{H}$ derived from the same Cauchy–Schwarz inequality; the difference is in what is decomposed and what the decomposition yields. In QM2 the decomposition is at the level of the coefficient structure of the superposition state; in QM3 the decomposition is at the level of the inner product between two operator-shifted states. The technique is the same; the objects are different.

Remark 2.5. *The structural parallel between QM2 and QM3 reflects the broader algebraic unity of the Hilbert space framework established in QM1. Three fundamental quantum constraints—the normalization condition $\|\Psi\|_{\mathcal{H}} = 1$ (from closure conservation, QM1), the complementarity relation $\mathcal{V}^2+\mathcal{W}^2 \le 1$ (from two-path superposition, QM2), and the Robertson uncertainty inequality $\Delta A \cdot \Delta B \ge \frac{1}{2}|\langle[A, B]\rangle|$ (from the present paper)—all arise from the inner product structure of $\mathcal{H}$ and the Cauchy–Schwarz inequality, without any postulate beyond the transport closure geometry. The Hilbert space $\mathcal{H}$ is not a formal device for packaging pre-existing quantum axioms; it is the natural representational structure in which the scalar–conformal transport geometry expresses its constraints, and those constraints emerge algebraically from the geometry rather than being imposed upon it.*

3 The Robertson and Schrödinger Uncertainty Inequalities

The present section derives the Robertson uncertainty inequality and its Schrödinger improvement as theorems on $\mathcal{H}$, using only the inner product structure of QM1 and the algebraic technique identified in Sec. 2.4. The derivation proceeds in three steps: the cross inner product $\left\langle\widehat{\Delta A}\Psi, \widehat{\Delta B}\Psi\right\rangle_{\mathcal{H}}$ is decomposed into real and imaginary parts, the Cauchy–Schwarz inequality bounds its modulus, and the Robertson inequality follows by retaining only the imaginary part while the Schrödinger improvement retains both. The saturation condition—the precise characterization of states for which equality holds—is then derived as a proposition, completing the algebraic analysis.

3.1 Decomposition of the Cross Inner Product

The key computational step is the identification of the imaginary part of the cross inner product $\left\langle \widehat{\Delta A}\Psi, \widehat{\Delta B}\Psi \right\rangle_{\mathcal{H}}$ with the commutator expectation value $\langle [A, B] \rangle$. This identification connects the abstract inner product structure of $\mathcal{H}$ to the specific algebraic structure of the observable algebra, making the commutation relation the driving input to the uncertainty bound.

Lemma 3.1 (Decomposition of the cross inner product)**.** *Let A, B be self-adjoint operators on $\mathcal{H}$, let $\Psi \in \mathcal{D}(A) \cap \mathcal{D}(B) \cap \mathcal{D}(AB) \cap \mathcal{D}(BA)$ be normalized, and let $f := \widehat{\Delta A}\Psi$, $g := \widehat{\Delta B}\Psi$. Then*

$$\langle f, g \rangle_{\mathcal{H}} = \frac{1}{2}\langle \{\widehat{\Delta A},\, \widehat{\Delta B}\} \rangle + \frac{\mathrm{i}}{2}\langle [A,\, B] \rangle, \tag{8}$$

where $\{\widehat{\Delta A},\, \widehat{\Delta B}\} = \widehat{\Delta A}\widehat{\Delta B} + \widehat{\Delta B}\widehat{\Delta A}$ is the anti-commutator and $[A,\, B] = AB - BA$ is the commutator. In particular:

$$\mathrm{Re}\big(\langle f, g \rangle_{\mathcal{H}}\big) = \frac{1}{2}\langle \{\widehat{\Delta A},\, \widehat{\Delta B}\} \rangle, \tag{9}$$

$$\mathrm{Im}\big(\langle f, g \rangle_{\mathcal{H}}\big) = \frac{1}{2}\langle [A,\, B] \rangle. \tag{10}$$

Proof. Compute $\langle f, g \rangle_{\mathcal{H}} + \langle g, f \rangle_{\mathcal{H}}$ and $\langle f, g \rangle_{\mathcal{H}} - \langle g, f \rangle_{\mathcal{H}}$ separately.

Sum: Using the self-adjointness of $\widehat{\Delta A}$ (since $\widehat{\Delta A} = A - \langle A \rangle \hat{\mathbf{1}}$ and A is self-adjoint):

$$\begin{aligned}
\langle f, g \rangle_{\mathcal{H}} + \langle g, f \rangle_{\mathcal{H}} &= \left\langle \widehat{\Delta A}\Psi, \widehat{\Delta B}\Psi \right\rangle_{\mathcal{H}} + \left\langle \widehat{\Delta B}\Psi, \widehat{\Delta A}\Psi \right\rangle_{\mathcal{H}} \\
&= \left\langle \Psi, \widehat{\Delta A}\widehat{\Delta B}\Psi \right\rangle_{\mathcal{H}} + \left\langle \Psi, \widehat{\Delta B}\widehat{\Delta A}\Psi \right\rangle_{\mathcal{H}} \\
&= \left\langle \Psi, (\widehat{\Delta A}\widehat{\Delta B} + \widehat{\Delta B}\widehat{\Delta A})\Psi \right\rangle_{\mathcal{H}} = \langle \{\widehat{\Delta A},\, \widehat{\Delta B}\} \rangle,
\end{aligned}$$

where in the second step we used the self-adjointness of $\widehat{\Delta A}$: $\left\langle \widehat{\Delta A}\Psi, \widehat{\Delta B}\Psi \right\rangle_{\mathcal{H}} = \left\langle \Psi, \widehat{\Delta A}\widehat{\Delta B}\Psi \right\rangle_{\mathcal{H}}$. Since $\langle g, f \rangle_{\mathcal{H}} = \overline{\langle f, g \rangle_{\mathcal{H}}}$, the sum equals $\langle f, g \rangle_{\mathcal{H}} + \overline{\langle f, g \rangle_{\mathcal{H}}} = 2\,\mathrm{Re}(\langle f, g \rangle_{\mathcal{H}})$, giving Eq. (9).

Difference: By the same self-adjointness argument:

$$\begin{aligned}
\langle f, g \rangle_{\mathcal{H}} - \langle g, f \rangle_{\mathcal{H}} &= \left\langle \Psi, \widehat{\Delta A}\widehat{\Delta B}\Psi \right\rangle_{\mathcal{H}} - \left\langle \Psi, \widehat{\Delta B}\widehat{\Delta A}\Psi \right\rangle_{\mathcal{H}} \\
&= \left\langle \Psi, (\widehat{\Delta A}\widehat{\Delta B} - \widehat{\Delta B}\widehat{\Delta A})\Psi \right\rangle_{\mathcal{H}} = \langle [\widehat{\Delta A},\, \widehat{\Delta B}] \rangle.
\end{aligned}$$

The key identity needed is $[\widehat{\Delta A},\, \widehat{\Delta B}] = [A,\, B]$. To verify: $\widehat{\Delta A}\widehat{\Delta B} - \widehat{\Delta B}\widehat{\Delta A} = (A - \langle A \rangle \hat{\mathbf{1}})(B - \langle B \rangle \hat{\mathbf{1}}) - (B - \langle B \rangle \hat{\mathbf{1}})(A - \langle A \rangle \hat{\mathbf{1}})$. Expanding both products and collecting terms: the constant terms $\langle A \rangle \langle B \rangle \hat{\mathbf{1}}$ cancel, the $\langle B \rangle A$ and $\langle A \rangle B$ terms cancel between the two products, and what remains is $AB - BA = [A,\, B]$. Therefore $\langle [\widehat{\Delta A},\, \widehat{\Delta B}] \rangle = \langle [A,\, B] \rangle$. Since $\langle f, g \rangle_{\mathcal{H}} - \overline{\langle f, g \rangle_{\mathcal{H}}} = 2\mathrm{i}\,\mathrm{Im}(\langle f, g \rangle_{\mathcal{H}})$, Eq. (10) follows.

Combining Eqs. (9) and (10) into $\langle f, g \rangle_{\mathcal{H}} = \mathrm{Re}(\langle f, g \rangle_{\mathcal{H}}) + \mathrm{i}\,\mathrm{Im}(\langle f, g \rangle_{\mathcal{H}})$ gives Eq. (8). ☐

Remark 3.2. *The identification of the commutator $[A,\, B]$ with the imaginary part of the cross inner product in Eq.* (10) *is the algebraic bridge between the abstract Hilbert space geometry and the observable algebra. The commutator is anti-Hermitian for self-adjoint A and B: $[A,\, B]^\dagger = (AB - BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = BA - AB = -[A,\, B]$, so its expectation value $\langle [A,\, B] \rangle$ is purely imaginary for all normalized Ψ. This is consistent with Eq.* (10)*: the imaginary part of $\langle f, g \rangle_{\mathcal{H}}$ equals $\frac{1}{2}\langle [A,\, B] \rangle$, which is real as required since $\langle [A,\, B] \rangle$ is purely imaginary and dividing by the explicit* i *in Eq.* (8) *recovers a real coefficient.*

3.2 The Robertson Uncertainty Inequality

The Robertson inequality follows by combining the Cauchy–Schwarz bound Eq. (7) with the lower bound on $|\langle f, g\rangle_{\mathcal{H}}|^2$ provided by the imaginary part of Lemma 3.1.

Theorem 3.3 (Robertson uncertainty inequality)**.** *Let A and B be self-adjoint operators on $\mathcal{H}$ and let $\Psi \in \mathcal{D}(A) \cap \mathcal{D}(B) \cap \mathcal{D}(AB) \cap \mathcal{D}(BA)$ be normalized. Then*

$$\Delta A \cdot \Delta B \geq \frac{1}{2} \left| \langle [A,\, B] \rangle \right|. \tag{11}$$

Proof. The Cauchy–Schwarz inequality Eq. (7) gives

$$(\Delta A)^2 (\Delta B)^2 = \left\| \widehat{\Delta A} \Psi \right\|_{\mathcal{H}}^2 \left\| \widehat{\Delta B} \Psi \right\|_{\mathcal{H}}^2 \geq \left| \left\langle \widehat{\Delta A} \Psi, \widehat{\Delta B} \Psi \right\rangle_{\mathcal{H}} \right|^2.$$

For any complex number $z = a + ib$ with $a, b \in \mathbb{R}$, $|z|^2 = a^2 + b^2 \geq b^2$. Applying this to $z = \langle f, g\rangle_{\mathcal{H}}$ with imaginary part $b = \operatorname{Im}(\langle f, g\rangle_{\mathcal{H}}) = \frac{1}{2}\langle [A,\, B]\rangle$ from Eq. (10):

$$\left| \left\langle \widehat{\Delta A} \Psi, \widehat{\Delta B} \Psi \right\rangle_{\mathcal{H}} \right|^2 \geq \left[\operatorname{Im}(\langle f, g\rangle_{\mathcal{H}}) \right]^2 = \frac{1}{4} |\langle [A,\, B] \rangle|^2.$$

Combining the two inequalities:

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} |\langle [A,\, B] \rangle|^2.$$

Taking the positive square root of both sides yields Eq. (11). □

Remark 3.4. *The proof of Theorem 3.3 uses two inputs: the Cauchy–Schwarz inequality on $\mathcal{H}$ (QM1 Lemma 4.2) and the decomposition of Lemma 3.1. No physical postulate, no measurement argument, and no reference to any specific physical system enters the derivation. The Robertson inequality is a theorem of Hilbert space theory applied to self-adjoint operators; it holds for any pair (A, B) regardless of their physical interpretation, provided only that the commutator $[A, B]$ is well-defined on the domain in question. The physical content enters when specific operators are substituted for A and B—as in the Heisenberg relation of Sec. 4—but the inequality itself is purely algebraic.*

3.3 The Schrödinger Uncertainty Improvement

The Robertson inequality retains only the imaginary part of the cross inner product and discards the real (anti-commutator) part. The Schrödinger improvement retains both parts, yielding a tighter bound whose additional term is state-dependent.

Theorem 3.5 (Schrödinger uncertainty inequality)**.** *Under the hypotheses of Theorem 3.3,*

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} \left| \langle \{ \widehat{\Delta A},\, \widehat{\Delta B} \} \rangle \right|^2 + \frac{1}{4} \left| \langle [A,\, B] \rangle \right|^2. \tag{12}$$

The Robertson inequality Eq. (11) *follows as the special case obtained by dropping the non-negative first term on the right.*

Proof. From the Cauchy–Schwarz inequality: $(\Delta A)^2(\Delta B)^2 \geq |\langle f, g\rangle_{\mathcal{H}}|^2$. By Lemma 3.1,

$$|\langle f, g\rangle_{\mathcal{H}}|^2 = \left|\tfrac{1}{2}\langle\{\widehat{\Delta A},\, \widehat{\Delta B}\}\rangle + \tfrac{\mathrm{i}}{2}\langle[A,\, B]\rangle\right|^2.$$

Writing $z = a + \mathrm{i}b$ with $a = \frac{1}{2}\langle\{\widehat{\Delta A},\, \widehat{\Delta B}\}\rangle \in \mathbb{R}$ and $b = \frac{1}{2}\langle[A,\, B]\rangle \in \mathbb{R}$ (both real since the anti-commutator is Hermitian and the commutator is anti-Hermitian for self-adjoint A, B):

$$|z|^2 = a^2 + b^2 = \frac{1}{4}|\langle\{\widehat{\Delta A},\, \widehat{\Delta B}\}\rangle|^2 + \frac{1}{4}|\langle[A,\, B]\rangle|^2.$$

Combining with the Cauchy–Schwarz bound gives Eq. (12). The Robertson inequality follows by omitting the first term, which is non-negative. □

Remark 3.6. *The Schrödinger bound Eq.* (12) *is strictly tighter than the Robertson bound Eq.* (11) *whenever* $\langle\{\widehat{\Delta A},\, \widehat{\Delta B}\}\rangle \neq 0$. *The anti-commutator term* $\frac{1}{4}|\langle\{\widehat{\Delta A},\, \widehat{\Delta B}\}\rangle|^2$ *is state-dependent: it vanishes for states in which the anti-commutator expectation value is zero and is positive otherwise. For the position-momentum pair with* $A = \hat{x}^j$ *and* $B = \hat{p}_j$*, the anti-commutator term is zero precisely for the Gaussian minimum-uncertainty states of Sec. 6: the saturation condition requires both the commutator bound to be saturated (Cauchy-Schwarz tight) and the anti-commutator term to vanish, as shown in Proposition 3.7 below. For states that are not minimum-uncertainty states, the Schrödinger bound may provide a significantly tighter lower bound on the uncertainty product than Robertson alone.*

3.4 Saturation of the Robertson Bound

The characterization of states for which the Robertson bound is achieved as an equality is important both for identifying the minimum-uncertainty states of Sec. 6 and for the physical interpretation of the bound.

Proposition 3.7 (Saturation condition for the Robertson inequality)**.** *Equality holds in the Robertson inequality Eq.* (11) *if and only if the following two conditions are simultaneously satisfied:*

(i) Cauchy–Schwarz saturated: $\widehat{\Delta B}\,\Psi = \lambda\,\widehat{\Delta A}\,\Psi$ *for some* $\lambda \in \mathbb{C}$*;*

(ii) Anti-commutator term vanishes: $\mathrm{Re}(\left\langle\widehat{\Delta A}\Psi, \widehat{\Delta B}\Psi\right\rangle_{\mathcal{H}}) = \frac{1}{2}\langle\{\widehat{\Delta A},\, \widehat{\Delta B}\}\rangle = 0$*, which is equivalent to* $\lambda \in \mathrm{i}\mathbb{R}$ *(i.e.,* $\lambda = \mathrm{i}\mu$ *for some* $\mu \in \mathbb{R}$*).*

The combined saturation condition is therefore

$$(B - \langle B\rangle)\,\Psi = \mathrm{i}\mu\,(A - \langle A\rangle)\,\Psi, \qquad \mu \in \mathbb{R}. \tag{13}$$

Proof. Necessity. If equality holds in Eq. (11), then $(\Delta A)^2(\Delta B)^2 = \frac{1}{4}|\langle[A, B]\rangle|^2$. From the Schrödinger inequality Eq. (12), this requires both $|\langle f, g\rangle_{\mathcal{H}}|^2 = (\Delta A)^2(\Delta B)^2$ (Cauchy–Schwarz saturated, condition (i)) and $\frac{1}{4}|\langle\{\widehat{\Delta A},\, \widehat{\Delta B}\}\rangle|^2 = 0$ (the anti-commutator term absent, condition (ii)). Cauchy–Schwarz is saturated if and only if $g = \lambda f$, i.e., $\widehat{\Delta B}\Psi = \lambda\,\widehat{\Delta A}\Psi$ for some $\lambda \in \mathbb{C}$.

Condition on λ. If $\widehat{\Delta B}\Psi = \lambda\,\widehat{\Delta A}\Psi$, then

$$\mathrm{Re}(\langle f, g\rangle_{\mathcal{H}}) = \mathrm{Re}(\lambda\,\|f\|^2_{\mathcal{H}}) = \mathrm{Re}(\lambda)\cdot(\Delta A)^2.$$

For condition (ii) to hold, $\mathrm{Re}(\lambda) = 0$, so $\lambda = \mathrm{i}\mu$ for $\mu \in \mathbb{R}$. Substituting $\lambda = \mathrm{i}\mu$ into $\widehat{\Delta B}\Psi = \lambda\,\widehat{\Delta A}\Psi$ gives Eq. (13).

Sufficiency. If Eq. (13) holds, then $\widehat{\Delta B}\Psi = \mathrm{i}\mu\,\widehat{\Delta A}\Psi$, so $\langle f, g\rangle_{\mathcal{H}} = \mathrm{i}\mu\,\|f\|^2_{\mathcal{H}}$ is purely imaginary, giving $\mathrm{Re}(\langle f, g\rangle_{\mathcal{H}}) = 0$ and $|\langle f, g\rangle_{\mathcal{H}}| = |\mu| \cdot (\Delta A)^2$. The left side of the Cauchy–Schwarz inequality becomes $(\Delta A)^2(\Delta B)^2 = (\Delta A)^2 \cdot |\mu|^2(\Delta A)^2 = \mu^2(\Delta A)^4$ and the right side equals $|\langle f, g\rangle_{\mathcal{H}}|^2 = \mu^2(\Delta A)^4$, confirming Cauchy–Schwarz is saturated. The anti-commutator term vanishes since $\mathrm{Re}(\langle f, g\rangle_{\mathcal{H}}) = 0$. Both conditions are satisfied, and equality holds in Eq. (11). □

Remark 3.8. *The saturation condition Eq.* (13) *is a first-order eigenvalue equation for the operator* $\widehat{\Delta B} - \mathrm{i}\mu\,\widehat{\Delta A}$*:*

$$(\widehat{\Delta B} - \mathrm{i}\mu\,\widehat{\Delta A})\Psi = 0.$$

For $A = \hat{x}^j$ *(multiplication by* x^j*) and* $B = \hat{p}_j = -\mathrm{i}\Phi_0\,\partial_j$*, this becomes the differential equation*

$$\left[-\mathrm{i}\Phi_0\,\partial_j - \langle p_j\rangle - \mathrm{i}\mu(x^j - \langle x^j\rangle)\right]\Psi = 0,$$

which is the first-order ODE solved in Sec. 6 to yield the Gaussian minimum-uncertainty state. The eigenvalue equation formulation makes it clear that the set of minimum-uncertainty states $\mathcal{M}$ *for a given pair* (A, B) *and given mean values* $(\langle A\rangle, \langle B\rangle)$ *is determined by the kernel of the operator* $\widehat{\Delta B} - \mathrm{i}\mu\,\widehat{\Delta A}$*, which for the position-momentum pair is one-dimensional (up to overall phase) and corresponds to the Gaussian family of Sec. 6.*

Remark 3.9. *Theorem 3.3 establishes the Robertson inequality as a theorem of the scalar–conformal NUVO transport closure system. Its derivation from the Cauchy–Schwarz inequality and Lemma 3.1 is complete and self-contained within the Hilbert space framework of QM1. The result is general: it holds for any two self-adjoint operators on* $\mathcal{H}$ *with well-defined commutator on a common domain, and its content is entirely determined by the algebraic structure of the observable pair* (A, B) *and the state* Ψ*. No measurement process, no physical disturbance, and no additional postulate is invoked at any point in the derivation. In the standard formulation of quantum mechanics, Robertson [4] derived this inequality in 1929 as a general consequence of the Hilbert space formalism. In the present framework, it is a consequence of the scalar–conformal transport geometry, the Hilbert space structure that geometry generates, and nothing more.*

4 The Heisenberg Position-Momentum Uncertainty Relation

The Robertson inequality of Theorem 3.3 is a universal algebraic result holding for any pair of self-adjoint operators on $\mathcal{H}$. Its physical content in the scalar–conformal NUVO framework is unlocked when specific transport observables are substituted for A and B. The most fundamental substitution is $A = \hat{x}^j$ and $B = \hat{p}_k$: the position and momentum transport generators whose commutation relation was derived in QB2 and promoted to $\mathcal{H}$ in QM1. The present section carries out this substitution, derives the Heisenberg uncertainty relation as a corollary of Robertson, and establishes its interpretation within the NUVO framework as a geometric constraint on the simultaneous transport resolution of the closure density in position and momentum space.

4.1 Application to the Transport Generators

The substitution $A = \hat{x}^j$, $B = \hat{p}_k$ into the Robertson inequality is immediate once the commutator expectation value is identified from the canonical commutation relation.

Theorem 4.1 (Heisenberg uncertainty relation)**.** *For any normalized closure state* $\Psi \in \mathcal{S}(\mathbb{R}^3) \subset \mathcal{H}$ *and spatial indices* $j, k \in \{1, 2, 3\}$*,*

$$\Delta x^j \cdot \Delta p_k \geq \frac{\Phi_0}{2}\,\delta^j{}_k. \tag{14}$$

In particular, for each spatial direction j,

$$\Delta x_j \cdot \Delta p_j \geq \frac{\Phi_0}{2}, \tag{15}$$

where $\Delta x_j := \Delta x^j$ is the root-mean-square spatial spread of the closure density $|\Psi(x)|^2$ in the j-th direction and $\Delta p_j := \Delta p_j$ is the root-mean-square spread of the momentum-space density $|\tilde{\Psi}(p)|^2$ in the j-th direction.

Proof. Apply Theorem 3.3 with $A = \hat{x}^j$ and $B = \hat{p}_k$. Both operators are self-adjoint on $\mathcal{H}$ by QM1 Theorem 5.2: $\hat{x}^j$ is multiplication by x^j (symmetric and self-adjoint on its natural domain) and $\hat{p}_k = -\mathrm{i}\Phi_0\,\partial_k$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^3)$ with closure domain $H^1(\mathbb{R}^3)$. The domain condition $\Psi \in \mathcal{S}(\mathbb{R}^3) \subset \mathcal{D}(\hat{x}^j) \cap \mathcal{D}(\hat{p}_k)$ is satisfied. By the canonical commutation relation, recalled in Eq. (1):

$$[\hat{x}^j,\, \hat{p}_k]\Psi = \mathrm{i}\Phi_0\,\delta^j{}_k\,\Psi.$$

Since this is a scalar multiple of Ψ, the expectation value is

$$\langle[\hat{x}^j,\, \hat{p}_k]\rangle = \left\langle \Psi, \mathrm{i}\Phi_0\,\delta^j{}_k\,\Psi\right\rangle_{\mathcal{H}} = \mathrm{i}\Phi_0\,\delta^j{}_k\,\|\Psi\|^2_{\mathcal{H}} = \mathrm{i}\Phi_0\,\delta^j{}_k,$$

using normalization $\|\Psi\|_{\mathcal{H}} = 1$. Therefore

$$\left|\langle[\hat{x}^j,\, \hat{p}_k]\rangle\right| = \Phi_0\,\delta^j{}_k.$$

Substituting into the Robertson inequality Eq. (11):

$$\Delta x^j \cdot \Delta p_k \geq \frac{1}{2}\cdot\Phi_0\,\delta^j{}_k = \frac{\Phi_0}{2}\,\delta^j{}_k,$$

which is Eq. (14). Equation (15) is the special case $j = k$. □

Remark 4.2. *The bound $\Phi_0/2$ in Eq.* (15) *is independent of the state Ψ, the spatial direction j, and any physical parameters of the transport system. This universality follows directly from the state-independence of the commutator expectation value $\langle[\hat{x}^j, \hat{p}_j]\rangle = \mathrm{i}\Phi_0$: since the commutator acts as a scalar multiple of the identity on $\mathcal{S}(\mathbb{R}^3)$, every normalized closure state yields the same commutator expectation value and hence the same lower bound. The bound $\Phi_0/2 = \hbar/2$ is a universal geometric feature of the scalar–conformal transport system, fixed by the phase constant $\Phi_0 = \hbar$ identified through the hydrogenic correspondence of the Q-series.*

4.2 Geometric Interpretation as Transport Resolution

The quantities Δx_j and Δp_j appearing in Eq. (15) have precise geometric interpretations within the NUVO transport closure framework that are worth recording explicitly.

The position uncertainty $\Delta x_j = \Delta x^j$ is the root-mean-square spatial spread of the closure density $\rho(x) = |\Psi(x)|^2$ in the j-th direction:

$$(\Delta x_j)^2 = \int_{\mathbb{R}^3} (x^j - \langle x^j\rangle)^2\,|\Psi(x)|^2\,\mathrm{d}^3x. \tag{16}$$

It is the second central moment of the closure density distribution in the j-th spatial direction. A state with small Δx_j is one whose closure density is tightly concentrated near the mean position $\langle x^j\rangle$ in that direction; a state with large Δx_j has closure density spread over a wide spatial region.

The momentum uncertainty $\Delta p_j = \Delta p_j$ is the root-mean-square spread of the momentum-space closure density $|\tilde{\Psi}(p)|^2$ in the j-th direction:

$$(\Delta p_j)^2 = \int_{\mathbb{R}^3} (p_j - \langle p_j \rangle)^2 \, |\tilde{\Psi}(p)|^2 \, \mathrm{d}^3 p, \tag{17}$$

where $\tilde{\Psi}(p)$ is the Fourier transform of Ψ at scale Φ_0, established in QM1 Proposition 7.1. A state with small Δp_j has momentum-space closure density concentrated near the mean momentum $\langle p_j \rangle$; a state with large Δp_j has momentum-space density spread over a wide range of transport momenta.

The Heisenberg relation Eq. (15) therefore states: for any admissible closure state $\Psi \in \mathcal{H}$, the product of the spatial concentration of the closure density and the momentum-space concentration of the momentum density is bounded below by $\Phi_0/2$. A closure state that is tightly concentrated in position (small Δx_j) must necessarily be broadly spread in momentum space (large Δp_j), and vice versa.

Remark 4.3. *This geometric interpretation differs fundamentally from the measurement-disturbance account. The measurement-disturbance account says: if you measure position precisely, the measurement disturbs the momentum, making a subsequent momentum measurement imprecise. The geometric account says: for any closure state Ψ, independently of any measurement, the closure density $|\Psi|^2$ and the momentum density $|\tilde{\Psi}|^2$ cannot both be sharply peaked simultaneously. The geometric account is the content of Theorem 4.1. The measurement-disturbance account is a separate phenomenon that involves the coherence-gated interaction structure of QB5 and QB6 and requires the full interaction theory for its derivation; it is not the content of the present theorem.*

In the NUVO framework, the distinction is especially clear because the transport closure state Ψ is a representational object that exists prior to and independently of any interaction event. The standard deviations Δx_j and Δp_j are properties of this representational object, not of any observable event or measurement outcome. The Heisenberg bound is therefore a statement about the geometry of the closure state space: no admissible element of $\mathcal{H}$ can simultaneously have arbitrarily small position spread and arbitrarily small momentum spread in the same spatial direction.

Remark 4.4. *The Heisenberg relation Eq.* (15) *can also be understood as a statement about the Fourier transform pair $(\Psi, \tilde{\Psi})$. A classical result of Fourier analysis—the Fourier uncertainty principle—states that a square-integrable function and its Fourier transform cannot both be sharply concentrated simultaneously, with the precise quantitative form being exactly $\Delta x_j \cdot \Delta p_j \geq \Phi_0/2$ where the scale factor Φ_0 enters through the convention $\tilde{\Psi}(p) = (2\pi\Phi_0)^{-3/2} \int \mathrm{e}^{-\mathrm{i}p\cdot x/\Phi_0} \Psi(x) \, \mathrm{d}^3 x$ [1]. In the NUVO framework, the Fourier transform at scale Φ_0 is not an independent mathematical choice but the natural transform associated with the momentum transport generators of QB2: the plane-wave generalized eigenstates of $\hat{p}_j$ are $\psi_p(x) = (2\pi\Phi_0)^{-3/2} \mathrm{e}^{\mathrm{i}p\cdot x/\Phi_0}$, as established in QM1 Definition 6.3. The Heisenberg relation is therefore simultaneously a theorem of the observable algebra (from the CCR via Robertson) and a classical theorem of Fourier analysis (the Fourier uncertainty principle), with the two approaches agreeing because the momentum transport generator is the infinitesimal generator of translations and the Fourier transform is the spectral decomposition of the translation group.*

4.3 The Full Three-Dimensional Uncertainty Structure

The canonical commutation relations of QM1 involve all pairs (j, k) of spatial indices. For $j = k$, the commutator is non-trivial and yields the Heisenberg bound; for $j \neq k$, the operators commute and the Robertson bound is trivial.

Corollary 4.5 (Full three-dimensional uncertainty structure)**.** *The canonical commutation relations* $[\hat{x}^j, \hat{p}_k] = \mathrm{i}\Phi_0\, \delta^j{}_k$ *of QM1 Proposition 5.4 yield the following complete structure of position-momentum uncertainty constraints:*

(i) Same-direction pairs *($j = k$): three independent uncertainty constraints,*

$$\Delta x_j \cdot \Delta p_j \geq \frac{\Phi_0}{2}, \qquad j = 1, 2, 3, \tag{18}$$

one for each spatial direction, each with the universal bound $\Phi_0/2$*.*

(ii) Cross-direction pairs *($j \neq k$): the operators* $\hat{x}^j$ *and* $\hat{p}_k$ *commute,* $[\hat{x}^j, \hat{p}_k] = 0$*, and the Robertson inequality yields only the trivial bound* $\Delta x_j \cdot \Delta p_k \geq 0$*. Position in one direction and momentum in a different direction are simultaneously resolvable without constraint from the canonical commutation structure.*

Proof. Part (i) is Theorem 4.1 with $j = k$. For part (ii): $[\hat{x}^j, \hat{p}_k] = \mathrm{i}\Phi_0\, \delta^j{}_k = 0$ for $j \neq k$ by QM1 Proposition 5.4. The Robertson inequality then gives $\Delta x_j \cdot \Delta p_k \geq \frac{1}{2}|0| = 0$, which is trivially satisfied. □

Remark 4.6. *The absence of a non-trivial uncertainty constraint for cross-direction pairs ($j \neq k$) is a direct consequence of the specific structure of the canonical commutation relations in three spatial dimensions. Position and momentum in orthogonal directions are compatible observables in the algebraic sense: they have a common eigenbasis (the simultaneous position-momentum eigenstates in orthogonal directions), and their standard deviations can in principle be simultaneously reduced to any desired values independently. This is consistent with the scalar–conformal transport picture: the closure density can be sharply concentrated in the* x*-direction (small* Δx_1*) while simultaneously having a broad spread in* p_2*-space (large* Δp_2*) or a sharp peak in* p_2 *(small* Δp_2*)—there is no geometric constraint relating these cross-direction quantities from the commutation structure. The angular momentum operators, which mix spatial directions, do generate non-trivial cross-component uncertainty constraints, as established in Sec. 7.*

4.4 Uncertainty and the Transport Phase Structure

The Heisenberg relation has a natural interpretation in terms of the transport phase structure of the closure state, connecting the abstract algebraic result to the geometric picture of the NUVO program.

The closure state $\Psi(x) = \sqrt{\rho(x)}\, \mathrm{e}^{\mathrm{i}\phi(x)/\Phi_0}$ encodes two fields: the closure density $\rho(x) = |\Psi(x)|^2$ and the transport phase $\phi(x) = \Phi_0 \arg \Psi(x)$. The mean momentum $\langle p_j \rangle$ is related to the mean phase gradient:

$$\langle p_j \rangle = -\mathrm{i}\Phi_0 \, \langle \Psi, \partial_j \Psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}^3} \rho(x)\, \partial_j \phi(x)\, \mathrm{d}^3 x, \tag{19}$$

which is the closure-density-weighted mean of the phase gradient $\partial_j \phi$. The momentum uncertainty Δp_j then measures the spread of the phase gradient field $\partial_j \phi(x)$ around its mean value, weighted by the closure density.

Remark 4.7. *Equation* (19) *reveals the physical content of the momentum uncertainty in the NUVO framework.* Δp_j *is large when the phase gradient* $\partial_j \phi(x)$ *varies significantly across the spatial support of the closure density* $\rho(x)$*: the transport closure configuration has different local transport momenta at different spatial positions.* Δp_j *is small when the phase gradient is approximately uniform across the support of* ρ*: the closure density is transported at nearly the same* *local momentum everywhere. The Heisenberg relation Eq.* (15) *then states: a closure state that is tightly spatially localized (small Δx_j) must have a rapidly varying phase gradient (large Δp_j) to accommodate the tight localization, and vice versa. This is the transport-phase interpretation of the uncertainty relation: spatial localization and phase gradient uniformity are geometrically incompatible beyond the threshold $\Phi_0/2$. The minimum-uncertainty Gaussian states of Sec. 6 are precisely those for which the phase gradient is linear in position—$\phi(x) \propto (x - \langle x \rangle)^2$—which is the configuration that optimally balances spatial concentration and phase gradient variation at the lower bound.*

5 The Energy-Time Uncertainty Relation

The Robertson inequality of Sec. 3 applies to pairs of self-adjoint operators on $\mathcal{H}$. The position-momentum pair of Sec. 4 is the canonical application, and the angular momentum pairs of Sec. 7 follow by the same route. The energy-time uncertainty relation occupies a different logical position within the QM-series: it cannot be derived by applying Robertson with $A = \hat{H}$ and $B = \hat{t}$ for a time operator $\hat{t}$, because no such operator exists in $\mathcal{H}$. The present section establishes why this is so, derives the energy-time relation by a distinct route through the rate-of-change identity for expectation values, and records its physical interpretation in terms of the transport phase coherence lifetime.

5.1 Time as a Parameter, Not an Operator

The absence of a self-adjoint time operator in $\mathcal{H}$ is not a limitation of the current framework but a structural theorem. It is established by an argument due to Pauli [2] and recorded here because the energy-time uncertainty relation is often incorrectly presented as a Robertson inequality applied to a "time operator," an error that is excluded from the outset by making the argument explicit.

Proposition 5.1 (No self-adjoint time operator in $\mathcal{H}$)**.** *There is no self-adjoint operator $\hat{t}$ on $\mathcal{H}$ satisfying*

$$[\hat{H}, \hat{t}] = -\mathrm{i}\Phi_0\, \hat{\mathbf{1}} \tag{20}$$

on a dense domain, if $\hat{H}$ is a self-adjoint operator that is bounded below: $\sigma(\hat{H}) \subseteq [E_{\min}, \infty)$ for some $E_{\min} \in \mathbb{R}$.

Proof. Suppose for contradiction that a self-adjoint $\hat{t}$ satisfying Eq. (20) exists on some dense domain $\mathcal{D}$. By the same argument as for position and momentum (the Weyl–Stone–von Neumann theorem for the Heisenberg commutation relations [3]), the pair $(\hat{H}/\Phi_0, \hat{t})$ would generate a unitary representation of the Weyl algebra, and the spectrum of $\hat{H}/\Phi_0$ would be forced to be all of $\mathbb{R}$. This implies $\sigma(\hat{H}) = \mathbb{R}$, contradicting the assumption that $\hat{H}$ is bounded below. Therefore no such $\hat{t}$ exists. □

Remark 5.2. *Proposition 5.1 applies to every physically admissible scalar-conformal Hamiltonian $\hat{H} = \hat{T} + \hat{V}$ of QM4, since all such Hamiltonians are bounded below: the kinetic operator $\hat{T} = -(\Phi_0^2/2m)\Delta$ has non-negative spectrum $[0, \infty)$, and the admissible potentials of Definition 3.2 of QM4 satisfy $V(x) \geq -C$ for some constant $C > 0$, making $\sigma(\hat{H})$ bounded below by $-C$. In the NUVO framework this is not a deficiency; it reflects the correct physical structure: time is the external parameter of the scalar-conformal transport evolution, not a dynamical observable encoded in the closure state. The Schrödinger evolution $\Psi(t) = U(t)\Psi_0$ treats t as a parameter of the unitary group $U(t)$, not as an observable with a spectrum. The energy-time uncertainty relation requires a different derivation that respects this structure.*

5.2 The Characteristic Evolution Time of an Observable

Since time cannot be treated as an operator, the energy-time uncertainty relation must involve a *time interval* derived from the dynamics of the closure state rather than from a time observable. The natural candidate is the characteristic time over which a physical observable changes appreciably under the Schrödinger evolution.

Definition 5.3 (Characteristic evolution time). *Let G be a self-adjoint observable on $\mathcal{H}$ and let $\Psi(t) = U(t)\Psi_0$ be a Schrödinger evolution with $\Psi_0 \in \mathcal{D}(\hat{H}) \cap \mathcal{D}(G)$. Suppose $\mathrm{d}\langle G\rangle(t)/\mathrm{d}t \neq 0$. The* characteristic evolution time of G *is*

$$\Delta t_G(t) := \frac{\Delta G(t)}{|\mathrm{d}\langle G\rangle(t)/\mathrm{d}t|}, \tag{21}$$

the time required for the expectation value $\langle G\rangle(t)$ to change by one standard deviation $\Delta G(t)$.

Remark 5.4. *Definition 5.3 defines Δt_G as a state-dependent, observable-dependent time scale extracted from the dynamics. It is not a universal "uncertainty in time" but the specific time interval over which the observable G undergoes a statistically significant change in its expectation value. Different observables G generally give different characteristic times Δt_G; the tightest energy-time bound at time t is obtained by choosing the observable G for which $\Delta t_G(t)$ is smallest, i.e., the observable whose expectation value is changing most rapidly relative to its standard deviation. The freedom to choose G is a feature of the energy-time relation that has no counterpart in the position-momentum relation, where the observables are fixed by the canonical commutation structure.*

5.3 Derivation from the Rate-of-Change Identity

The energy-time uncertainty relation is derived by applying the Robertson inequality to the pair $(\hat{H}, G)$ and using the Heisenberg equation of motion to express the commutator expectation value in terms of the time derivative of $\langle G\rangle$.

Theorem 5.5 (Energy-time uncertainty relation). *Let $\hat{H}$ be the scalar-conformal Hamiltonian of QM4 and let G be a self-adjoint observable on $\mathcal{H}$ with $\Psi(t) \in \mathcal{D}(\hat{H}) \cap \mathcal{D}(G)$ for all t. Then at each time t for which $\mathrm{d}\langle G\rangle(t)/\mathrm{d}t \neq 0$,*

$$\Delta E(t) \cdot \Delta t_G(t) \geq \frac{\Phi_0}{2}, \tag{22}$$

where $\Delta E(t) = \Delta H(t)$ is the energy standard deviation in the state $\Psi(t)$.

Proof. Apply the Robertson inequality Theorem 3.3 with $A = \hat{H}$ and $B = G$ at time t, using the state $\Psi = \Psi(t)$:

$$\Delta H(t) \cdot \Delta G(t) \geq \frac{1}{2}\left|\langle[\hat{H}, G]\rangle(t)\right|. \tag{23}$$

The commutator expectation value is related to the time derivative of $\langle G\rangle(t)$ by the Heisenberg equation of motion, established in QM4 Remark 7.1 (the rate-of-change identity for expectation values):

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle G\rangle(t) = \frac{\mathrm{i}}{\Phi_0}\langle[\hat{H}, G]\rangle(t). \tag{24}$$

Taking absolute values:

$$\left|\langle[\hat{H}, G]\rangle(t)\right| = \Phi_0\left|\frac{\mathrm{d}}{\mathrm{d}t}\langle G\rangle(t)\right|. \tag{25}$$

Substituting Eq. (25) into Eq. (23):

$$\Delta H(t) \cdot \Delta G(t) \geq \frac{\Phi_0}{2} \left| \frac{\mathrm{d}}{\mathrm{d}t} \langle G \rangle(t) \right|. \tag{26}$$

Divide both sides by $|\mathrm{d}\langle G \rangle(t)/\mathrm{d}t|$ (which is non-zero by assumption) and use the definition Eq. (21) of Δt_G:

$$\Delta H(t) \cdot \frac{\Delta G(t)}{|\mathrm{d}\langle G \rangle(t)/\mathrm{d}t|} = \Delta E(t) \cdot \Delta t_G(t) \geq \frac{\Phi_0}{2}.$$

□

Remark 5.6. *The derivation of Theorem 5.5 makes the logical structure of the energy-time relation transparent. It is not an independent uncertainty principle but a consequence of the Robertson inequality applied to* $(\hat{H}, G)$ *combined with the Heisenberg equation of motion from QM4. The Robertson inequality contributes the Cauchy–Schwarz bound; the Heisenberg equation converts the commutator into a time derivative; the characteristic time definition converts the time derivative into a time interval. The result is the energy-time relation Eq.* (22)*.*

This structure explains why the energy-time relation has a different character from the position-momentum relation. The Heisenberg relation Eq. (15) *involves a fixed, state-independent bound* $\Phi_0/2$ *because the commutator* $[\hat{x}^j, \hat{p}_j]$ *is a scalar multiple of the identity. The energy-time relation Eq.* (22) *involves a state-dependent and observable-dependent time interval* Δt_G *because the commutator* $[\hat{H}, G]$ *is in general a non-trivial operator whose expectation value depends on both the state and the observable. The bound* $\Phi_0/2$ *on the right-hand side is the same in both cases, but the quantities it constrains are structurally different.*

5.4 Interpretation as Transport Phase Coherence Lifetime

The energy-time uncertainty relation admits a natural interpretation in terms of the transport phase structure of the closure state, connecting it to the coherence properties of the transport system established in the Q-series.

A normalized closure state $\Psi \in \mathcal{H}$ with energy uncertainty ΔE is a superposition of energy eigenstates or a continuous superposition of scattering states with energy spread ΔE. Under the Schrödinger evolution, each spectral component Ψ_E acquires a phase factor $\mathrm{e}^{-\mathrm{i}Et/\Phi_0}$ at rate E/Φ_0. The relative phase between components of energy E and E' evolves as $\mathrm{e}^{-\mathrm{i}(E-E')t/\Phi_0}$, completing one cycle over a time $2\pi\Phi_0/(E - E')$. For a state with energy spread ΔE, the fastest-varying relative phases complete a cycle over a time of order $2\pi\Phi_0/\Delta E$. The transport phase coherence of the state—the degree to which the phase relationships among the spectral components remain organized—is therefore maintained over a characteristic time

$$\tau_c \sim \frac{\Phi_0}{\Delta E}, \tag{27}$$

beyond which the spectral dephasing randomizes the relative phases and the coherent structure of the state is lost. The energy-time relation $\Delta E \cdot \Delta t_G \geq \Phi_0/2$ states that the characteristic evolution time Δt_G of any observable is bounded below by $\Phi_0/(2\Delta E) \sim \tau_c/2$: the coherence time sets the scale below which no observable can undergo a statistically significant change in expectation value.

Remark 5.7. *In the NUVO framework, the coherence time* $\tau_c \sim \Phi_0/\Delta E$ *is the time over which the transport phase structure of the closure state remains coherent in the sense established in QB5: the relative phases between the spectral components of the closure state are well-defined and organized,* *supporting the interference structure analyzed in QM2. A state with small energy uncertainty ΔE (nearly definite energy) has a long coherence time: its spectral components dephase slowly and the interference pattern of QM2 is stable over long times. A state with large energy uncertainty ΔE (broad energy spread) has a short coherence time: the spectral components dephase rapidly and the interference pattern washes out over a short interval. The energy-time relation makes this qualitative picture quantitative: $\tau_c \geq \Phi_0/(2\Delta E)$, with the bound $\Phi_0/2$ fixed by the phase constant identified through the hydrogenic correspondence. This connection between energy uncertainty and coherence lifetime will be used in QM10 to interpret the linewidths of scattering resonances: a quasi-bound state with energy width ΔE has a coherence time $\tau_c \sim \Phi_0/\Delta E$, which is the lifetime of the resonance.*

Remark 5.8. *For a normalized energy eigenstate Ψ_n with $\hat{H}\Psi_n = E_n\Psi_n$, the energy uncertainty vanishes: $\Delta E = \Delta H = 0$ in the state Ψ_n. Theorem 5.5 does not apply in this case since the denominator of $\Delta t_G = \Delta G/|\mathrm{d}\langle G\rangle/\mathrm{d}t|$ may also vanish. Indeed, for a stationary state, $\langle G\rangle(t) = \langle \Psi_n, U(t)^* G\, U(t)\Psi_n\rangle_{\mathcal{H}} = \langle \Psi_n, G\Psi_n\rangle_{\mathcal{H}}$ is time-independent for any G that commutes with $\hat{H}$, giving $\mathrm{d}\langle G\rangle/\mathrm{d}t = 0$. The relation $\Delta E \cdot \Delta t_G \geq \Phi_0/2$ takes the indeterminate form $0 \cdot \infty$ and is consistent but vacuous for stationary states. This is the correct behavior: a state of definite energy has no finite characteristic evolution time, consistent with the fact that it evolves trivially (acquiring only an overall phase) under the Schrödinger dynamics.*

5.5 The Energy-Time Relation and the Double-Slit

The energy-time uncertainty relation connects directly to the which-path analysis of QM2, providing an energy-domain perspective on the coherence disruption established there.

In QM2 Theorem 6.1, which-path detection was shown to destroy interference by introducing an uncontrolled phase shift on one transport channel, with the phase shift distributed uniformly over $[0, 2\pi)$ across the interaction ensemble. This phase disturbance can be recast in energy-time language: the which-path interaction occurs over some finite time interval δt and introduces a phase uncertainty $\delta\phi \sim 2\pi$ (a full cycle, corresponding to complete decoherence). The corresponding energy uncertainty is, by the energy-time relation, $\Delta E \geq \Phi_0/(2\,\delta t)$.

Conversely, for the fringe pattern to survive, the which-path interaction must not introduce a phase uncertainty of order 2π within the coherence time $\tau_c \sim \Phi_0/\Delta E$ of the transport state. If the interaction time $\delta t \ll \tau_c$, the phase disturbance is small, partial coherence is maintained, and partial fringes survive—corresponding to the partial-interference regime of QM2 Corollary 7.3. This connection is recorded as a remark for program continuity rather than as a formal theorem, since its full development requires the interaction theory of QB5 and the scattering formalism of QM10.

Remark 5.9. *The complementarity relation $\mathcal{V}^2 + \mathcal{W}^2 \leq 1$ of QM2 Theorem 6.2 and the energy-time uncertainty relation $\Delta E \cdot \Delta t_G \geq \Phi_0/2$ of Theorem 5.5 are complementary perspectives on the same geometric constraint. The QM2 relation is formulated in terms of the coefficient structure of the two-path superposition state and bounds the trade-off between fringe visibility and path distinguishability. The present relation is formulated in terms of the energy spread of the closure state and bounds the coherence lifetime of the transport phase structure. Both bounds have the same right-hand side $\Phi_0/2$, fixed by the phase constant from the Q-series. Their relationship—the precise connection between the two-path coherence parameter μ_{AB} of QM2 and the energy uncertainty ΔE of the present section—involves the spectral decomposition of the two-path state and is a topic developed further in QM9 and QM10.*

6 Minimum-Uncertainty States

The Robertson inequality Eq. (11) and the Heisenberg relation Eq. (15) establish lower bounds on the product $\Delta x_j \cdot \Delta p_j$ for any admissible closure state. The present section identifies the states for which these bounds are achieved as equalities: the minimum-uncertainty states of the position-momentum pair. The identification proceeds by solving the saturation condition of Proposition 3.7 as a first-order differential equation in position space, yielding the Gaussian closure configurations. These Gaussian states are of broad programmatic significance within the QM-series: they are the single-particle precursors of the coherent states developed in QM6 and the natural initial states for semiclassical transport analysis throughout the series.

6.1 The Saturation Condition as a Differential Equation

Proposition 3.7 identified the saturation condition for the Robertson inequality as the operator equation $\widehat{\Delta B}\Psi = \mathrm{i}\mu\,\widehat{\Delta A}\Psi$ for some $\mu \in \mathbb{R}$. For the position-momentum pair $A = \hat{x}^j$ and $B = \hat{p}_j$ in one spatial dimension (suppressing the direction index for clarity), this operator equation becomes an explicit first-order ordinary differential equation for the closure state $\Psi(x)$.

Substituting $\widehat{\Delta A} = \hat{x} - \langle x \rangle\,\hat{\mathbf{1}}$ (multiplication by $x - \langle x \rangle$) and $\widehat{\Delta B} = \hat{p} - \langle p \rangle\,\hat{\mathbf{1}}$ (the operator $-\mathrm{i}\Phi_0\,\partial_x - \langle p \rangle$) into the saturation condition Eq. (13):

$$\big(-\mathrm{i}\Phi_0\,\partial_x - \langle p \rangle\big)\Psi(x) = \mathrm{i}\mu\,\big(x - \langle x \rangle\big)\Psi(x). \tag{28}$$

Rearranging:

$$\partial_x \Psi(x) = \left[\frac{\mathrm{i}\langle p \rangle}{\Phi_0} - \frac{\mu}{\Phi_0}\,(x - \langle x \rangle)\right]\Psi(x). \tag{29}$$

This is a linear first-order ODE for Ψ as a function of x. Its general solution is obtained by direct integration.

Lemma 6.1 (Solution of the saturation ODE)**.** *The general solution of Eq.* (29) *with* $\mu > 0$ *is*

$$\Psi(x) = N\,\exp\!\left(-\frac{\mu}{2\Phi_0}(x - \langle x \rangle)^2 + \frac{\mathrm{i}\langle p \rangle}{\Phi_0}\,x\right), \tag{30}$$

where $N \in \mathbb{C}$ *is a normalization constant. For* $\mu < 0$*, the solution is not square-integrable. For* $\mu = 0$*, the solution is a plane wave, which is not in* $\mathcal{H}$*. Therefore admissible (normalizable) solutions require* $\mu > 0$*.*

Proof. Equation (29) is separable: $\mathrm{d}\Psi/\Psi = [\mathrm{i}\langle p \rangle/\Phi_0 - \mu(x - \langle x \rangle)/\Phi_0]\,\mathrm{d}x$. Integrating both sides:

$$\ln \Psi(x) = \frac{\mathrm{i}\langle p \rangle}{\Phi_0}\,x - \frac{\mu}{2\Phi_0}(x - \langle x \rangle)^2 + \text{const},$$

giving Eq. (30) upon exponentiation. For $\mu > 0$, the Gaussian factor $\exp[-\mu(x - \langle x \rangle)^2/(2\Phi_0)]$ decays as $|x| \to \infty$, making the solution square-integrable; for $\mu < 0$, it grows and the solution is not in $\mathcal{H}$; for $\mu = 0$, the Gaussian factor is absent and the plane wave $\mathrm{e}^{\,\mathrm{i}\langle p \rangle x/\Phi_0}$ is not square-integrable. □

6.2 The Minimum-Uncertainty Gaussian Closure State

The normalization of the solution Eq. (30) fixes the constant N and identifies the Gaussian width parameter σ in terms of μ and Φ_0.

Theorem 6.2 (Minimum-uncertainty states are Gaussian closure states)**.** *For the position-momentum pair, equality holds in the Heisenberg uncertainty relation Eq.* (15) *if and only if the closure state* Ψ *is a Gaussian closure configuration of the form*

$$\Psi(x) = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\!\left(-\frac{(x-\langle x\rangle)^2}{4\sigma^2} + \frac{\mathrm{i}\langle p\rangle\, x}{\Phi_0}\right), \tag{31}$$

for parameters $\sigma > 0$*,* $\langle x\rangle \in \mathbb{R}$*, and* $\langle p\rangle \in \mathbb{R}$*, where* $\sigma = \sqrt{\Phi_0/(2\mu)}$ *relates the Gaussian width to the saturation parameter* μ*. The standard deviations of this state are* $\Delta x = \sigma$ *and* $\Delta p = \Phi_0/(2\sigma)$*, satisfying* $\Delta x \cdot \Delta p = \Phi_0/2$ *exactly.*

Proof. Normalization. Substituting Eq. (30) with $\mu > 0$ into the normalization condition $\int_{\mathbb{R}} |\Psi(x)|^2\,\mathrm{d}x = 1$:

$$|N|^2 \int_{\mathbb{R}} \exp\!\left[-\frac{\mu}{\Phi_0}(x-\langle x\rangle)^2\right] \mathrm{d}x = |N|^2 \sqrt{\frac{\pi\Phi_0}{\mu}} = 1,$$

giving $|N|^2 = \sqrt{\mu/(\pi\Phi_0)}$. Setting $\sigma^2 = \Phi_0/(2\mu)$, so that $\mu/\Phi_0 = 1/(2\sigma^2)$:

$$|N|^2 = \sqrt{\frac{1}{2\pi\sigma^2}}, \qquad N = (2\pi\sigma^2)^{-1/4}$$

(choosing N real and positive without loss of generality, since overall phase is unphysical). Substituting into Eq. (30) with $\mu = \Phi_0/(2\sigma^2)$:

$$\Psi(x) = (2\pi\sigma^2)^{-1/4} \exp\!\left(-\frac{(x-\langle x\rangle)^2}{4\sigma^2} + \frac{\mathrm{i}\langle p\rangle\, x}{\Phi_0}\right),$$

which is Eq. (31).

Standard deviations. The closure density is $|\Psi(x)|^2 = (2\pi\sigma^2)^{-1/2} \exp[-(x-\langle x\rangle)^2/(2\sigma^2)]$, a Gaussian with mean $\langle x\rangle$ and variance σ^2. Therefore $\Delta x = \Delta x = \sigma$. For the momentum standard deviation, the Fourier transform of $\Psi(x)$ at scale Φ_0 (QM1 Proposition 7.1) is

$$\tilde{\Psi}(p) = \frac{(2\pi\sigma^2)^{1/4}}{(\pi\Phi_0^2/(2\sigma^2))^{1/4}} \cdot (\textit{somethingGaussian} \text{ in } p \text{ with width } \Phi_0/(2\sigma)),$$

which gives $\Delta p = \Phi_0/(2\sigma)$. (The Fourier transform of a Gaussian of width σ at scale Φ_0 is a Gaussian of width $\Phi_0/(2\sigma)$; this is a classical result [1] and is not re-derived here.) Therefore $\Delta x \cdot \Delta p = \sigma \cdot \Phi_0/(2\sigma) = \Phi_0/2$, confirming saturation.

Converse. Any state achieving equality in Eq. (15) satisfies the saturation condition Eq. (13) with some $\mu > 0$ by Proposition 3.7, and hence satisfies the ODE Eq. (29). The unique normalizable solution of this ODE is Eq. (31) up to an overall phase, by Lemma 6.1. □

Remark 6.3. *The factor* $\mathrm{e}^{\mathrm{i}\langle p\rangle x/\Phi_0}$ *in Eq.* (31) *is a plane-wave phase carrier that encodes the mean transport momentum* $\langle p\rangle$ *of the closure state. In the polar decomposition* $\Psi(x) = \sqrt{\rho(x)}\,\mathrm{e}^{\mathrm{i}\phi(x)/\Phi_0}$*, the transport phase is*

$$\phi(x) = \langle p\rangle\, x - \mathrm{i}\Phi_0\, \frac{(x-\langle x\rangle)^2}{4\sigma^2},$$

where the imaginary contribution to the argument of the exponential comes from the Gaussian envelope. The phase gradient at the mean position is $\partial_x\phi|_{x=\langle x\rangle} = \langle p\rangle$*, consistent with the identification of the mean transport momentum as the mean phase gradient established in Eq.* (19)*. The phase gradient increases linearly away from the mean position, with slope* $\partial_x^2\phi = 0$ *at* $x = \langle x\rangle$ *(the phase is locally flat at the mean position) and the Gaussian spreading of the closure density providing the spatial localization.*

6.3 Properties of the Gaussian Closure State

The Gaussian closure states of Theorem 6.2 have several properties that are used throughout the remainder of the QM-series and are recorded here for reference.

Proposition 6.4 (Properties of Gaussian closure states)**.** *The Gaussian closure state Ψ_G of Theorem 6.2 with parameters $(\sigma, \langle x\rangle, \langle p\rangle)$ has the following properties.*

(i) Position-space closure density:

$$\rho(x) = |\Psi_G(x)|^2 = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - \langle x\rangle)^2}{2\sigma^2}\right),$$

a Gaussian with mean $\langle x\rangle$ and standard deviation σ.

(ii) Momentum-space closure density:

$$|\tilde{\Psi_G}(p)|^2 = \sqrt{\frac{2\sigma^2}{\pi\Phi_0^2}} \exp\left(-\frac{2\sigma^2(p - \langle p\rangle)^2}{\Phi_0^2}\right),$$

a Gaussian with mean $\langle p\rangle$ and standard deviation $\Phi_0/(2\sigma) = \Delta p$.

(iii) Simultaneous saturation: $\Delta x \cdot \Delta p = \sigma \cdot \Phi_0/(2\sigma) = \Phi_0/2$ *for all $\sigma > 0$: every member of the Gaussian family saturates the Heisenberg bound, regardless of the width parameter.*

(iv) Schrödinger improvement: *For the Gaussian state, the anti-commutator term of the Schrödinger inequality Eq. (12) vanishes: $\langle\{\widehat{\Delta x}, \widehat{\Delta p}\}\rangle = 0$. Consequently the Robertson bound and the Schrödinger bound coincide for Gaussian states.*

(v) Parametric family: *The set of all minimum-uncertainty states for the one-dimensional position-momentum pair is*

$$\mathcal{M} = \left\{ e^{i\vartheta}\Psi_{G(\sigma,\, \langle x\rangle,\, \langle p\rangle)} \;\middle|\; \sigma > 0,\ \langle x\rangle \in \mathbb{R},\ \langle p\rangle \in \mathbb{R},\ \vartheta \in [0, 2\pi) \right\},$$

parametrized by the width σ, the mean position $\langle x\rangle$, the mean momentum $\langle p\rangle$, and an overall phase ϑ (which is physically irrelevant).

Proof. Parts (i) and (ii) follow directly from the Gaussian form Eq. (31) and the Fourier transform at scale Φ_0 (QM1 Proposition 7.1; the transform of a Gaussian of width σ is a Gaussian of width $\Phi_0/(2\sigma)$, a classical result [1]). Part (iii) follows by direct computation from parts (i) and (ii). For part (iv): the anti-commutator expectation value is $\langle\{\widehat{\Delta x}, \widehat{\Delta p}\}\rangle = 2\,\mathrm{Re}(\left\langle \widehat{\Delta x}\Psi_G, \widehat{\Delta p}\Psi_G \right\rangle_{\mathcal{H}})$ by Eq. (9). Since the saturation condition Eq. (13) gives $\widehat{\Delta p}\Psi_G = i\mu\,\widehat{\Delta x}\Psi_G$, the cross inner product is $\left\langle \widehat{\Delta x}\Psi_G, \widehat{\Delta p}\Psi_G \right\rangle_{\mathcal{H}} = i\mu \left\| \widehat{\Delta x}\Psi_G \right\|_{\mathcal{H}}^2 = i\mu\sigma^2$, which is purely imaginary. Therefore $\mathrm{Re}(\left\langle \widehat{\Delta x}\Psi_G, \widehat{\Delta p}\Psi_G \right\rangle_{\mathcal{H}}) = 0$, confirming part (iv). Part (v) follows from Proposition 3.7 and Lemma 6.1: the minimum-uncertainty states are exactly the normalizable solutions of the saturation ODE, which form the Gaussian family parametrized as stated. □

6.4 The Three-Dimensional Gaussian State

The one-dimensional analysis extends directly to three spatial dimensions by taking the tensor product of three independent one-dimensional Gaussian states.

Corollary 6.5 (Three-dimensional minimum-uncertainty state)**.** *The minimum-uncertainty state for the three-dimensional position-momentum system, saturating all three uncertainty relations Eq.* (18) *simultaneously, is the product Gaussian*

$$\Psi(\mathbf{x}) = \prod_{j=1}^{3} \frac{1}{(2\pi\sigma_j^2)^{1/4}} \exp\left(-\frac{(x^j - \langle x^j \rangle)^2}{4\sigma_j^2} + \frac{\mathrm{i}\langle p_j \rangle\, x^j}{\Phi_0} \right), \tag{32}$$

with independent width parameters $\sigma_j > 0$, mean positions $\langle x^j \rangle \in \mathbb{R}$, and mean momenta $\langle p_j \rangle \in \mathbb{R}$ for each direction $j = 1, 2, 3$. Each direction independently saturates $\Delta x_j \cdot \Delta p_j = \Phi_0/2$, and there is no constraint coupling widths in different directions.

Proof. The three-dimensional position-momentum system has independent uncertainty constraints for each direction j (Corollary 4.5). Since the operators in orthogonal directions commute, the saturation conditions for different directions decouple, and the minimum-uncertainty state in three dimensions is the product of minimum-uncertainty states in each direction. The product form Eq. (32) follows from Theorem 6.2 applied independently in each direction. □

6.5 Programmatic Role: Precursors of Coherent States

The Gaussian minimum-uncertainty states of Theorem 6.2 are the structural foundation for the coherent states developed in QM6. The connection is recorded here as a forward reference that establishes the logical dependency between the two papers.

A *coherent state* of the harmonic oscillator is defined in QM6 as an eigenstate of the lowering operator $\hat{a}^- = (\hat{p} + \mathrm{i}m\omega\hat{x})/\sqrt{2m\omega\Phi_0}$: $\hat{a}^-\Psi_\alpha = \alpha\,\Psi_\alpha$ for $\alpha \in \mathbb{C}$. QM6 establishes that the position-space representation of Ψ_α is exactly the Gaussian Eq. (31) with width parameter $\sigma = \sqrt{\Phi_0/(2m\omega)}$ (the zero-point width of the harmonic oscillator ground state), mean position $\langle x \rangle = \sqrt{2\Phi_0/(m\omega)}\,\mathrm{Re}(\alpha)$, and mean momentum $\langle p \rangle = \sqrt{2m\omega\Phi_0}\,\mathrm{Im}(\alpha)$.

The connection between the present paper and QM6 is therefore:

- QM3 (present paper) identifies all Gaussian closure states as minimum-uncertainty states for the position-momentum pair, parametrized by $(\sigma, \langle x \rangle, \langle p \rangle)$. This is a purely algebraic characterization from the saturation condition of Proposition 3.7.
- QM6 identifies a specific one-parameter subfamily of Gaussian states—those with the harmonic oscillator zero-point width $\sigma = \sqrt{\Phi_0/(2m\omega)}$—as the coherent states, and shows that these states evolve under the harmonic oscillator dynamics without changing their Gaussian form: the width σ is preserved, only the mean position $\langle x \rangle(t)$ and mean momentum $\langle p \rangle(t)$ evolve according to the classical harmonic oscillator equations.

Remark 6.6. *The algebraic property (minimum uncertainty) and the dynamical property (shape preservation under harmonic oscillator evolution) together characterize coherent states in the NUVO framework. The algebraic property is established in the present paper from the saturation condition of the Robertson inequality; the dynamical property is established in QM6 from the Schrödinger equation of QM4 with a harmonic potential. The two characterizations are logically independent: not every minimum-uncertainty state is a coherent state (only those with the oscillator zero-point* *width are), and the shape-preservation property requires the dynamical framework of QM4 that is not available in the present paper. The present section establishes the necessary algebraic precondition; QM6 adds the dynamical content that completes the characterization.*

Remark 6.7. *The Gaussian minimum-uncertainty states of Theorem 6.2 are also the states for which the Ehrenfest theorem of QM4 is most naturally interpreted. For a general closure state, the Ehrenfest equations* $\mathrm{d}\langle x\rangle/\mathrm{d}t = \langle p\rangle/m$ *and* $\mathrm{d}\langle p\rangle/\mathrm{d}t = -\langle \partial V/\partial x\rangle$ *govern the centroid trajectory but do not constrain the width of the closure density distribution. For a Gaussian minimum-uncertainty state, the centroid trajectory is the classical trajectory and, for potentials up to second order in* x *(including the free particle and the harmonic oscillator), the Gaussian form is preserved under the dynamics: the state remains Gaussian with time-evolving centroid and constant or time-evolving width. For more general potentials, the Gaussian profile spreads as the state evolves, but the Ehrenfest theorem continues to govern the centroid motion. This spreading of the Gaussian profile under non-harmonic dynamics is the quantum feature that the classical Ehrenfest equations do not capture, and it is connected to the growth of the higher moments of the closure density beyond the uncertainty bound.*

7 Uncertainty Relations for Angular Momentum

The Robertson inequality of Sec. 3 is a universal result that applies to any pair of self-adjoint operators on $\mathcal{H}$ with a well-defined commutator on a common dense domain. The position-momentum application of Sec. 4 used the commutator $[\hat{x}^j, \hat{p}_k] = \mathrm{i}\Phi_0\, \delta^j{}_k$, whose expectation value is state-independent, yielding a universal constant bound. The angular momentum operators provide a structurally different application: their commutation algebra $[\hat{L}_j, \hat{L}_k] = \mathrm{i}\Phi_0\, \epsilon_{jkl}\hat{L}_l$ involves the operators themselves on the right-hand side, so the commutator expectation value is state-dependent, and the uncertainty bound depends on the closure state through $\langle \hat{L}_l\rangle$. This state-dependence is a structural feature of the angular momentum algebra that distinguishes it from the position-momentum algebra and will be prominent in the full angular momentum analysis of QM5.

The present section applies Robertson to the angular momentum commutation algebra, derives the angular momentum uncertainty relations as a proposition, discusses the state-dependence of the bound, and records the scope of the analysis relative to the full treatment deferred to QM5.

7.1 The Angular Momentum Commutation Relations

The angular momentum operators were introduced in QM4 as the generators of spatial rotations in the scalar–conformal transport system. Their definition and basic properties are recalled here in the form needed for the Robertson application.

The angular momentum operators are defined by

$$\hat{L}_j \coloneqq \epsilon_{jkl}\, \hat{x}^k \hat{p}_l = \epsilon_{jkl}\, x^k(-\mathrm{i}\Phi_0\, \partial_l), \qquad j = 1, 2, 3, \tag{33}$$

where ϵ_{jkl} is the Levi-Civita symbol and repeated indices are summed. These were introduced in QM4 Eq. (7.6) and shown in QM4 Propositions 7.2 and 7.3 to be self-adjoint on $\mathcal{S}(\mathbb{R}^3)$ and to commute with rotationally symmetric Hamiltonians. Their commutation algebra is established in QM5; for the purposes of the present section, it is recalled as an input.

Proposition 7.1 (Angular momentum commutation algebra, recalled from QM5)**.** *The angular momentum operators of Eq.* (33) *satisfy the commutation relations*

$$[\hat{L}_j, \hat{L}_k] = \mathrm{i}\Phi_0\, \epsilon_{jkl}\, \hat{L}_l \tag{34}$$

on the dense domain $\mathcal{S}(\mathbb{R}^3) \subset \mathcal{H}$*, where the sum over the repeated index* l *is understood. Equivalently, for the three cyclic pairs:*

$$[\hat{L}_1, \hat{L}_2] = \mathrm{i}\Phi_0\, \hat{L}_3, \tag{35}$$

$$[\hat{L}_2, \hat{L}_3] = \mathrm{i}\Phi_0\, \hat{L}_1, \tag{36}$$

$$[\hat{L}_3, \hat{L}_1] = \mathrm{i}\Phi_0\, \hat{L}_2. \tag{37}$$

Proof. The derivation of Eq. (34) from the definition Eq. (33) and the canonical commutation relations of QM1 is carried out in full in QM5 Sec. **??**. It is recalled here without re-derivation; the result is cited from QM5. □

Remark 7.2. *The commutation algebra Eq.* (34) *differs from the position-momentum CCR Eq.* (1) *in one fundamental structural respect: the right-hand side involves an angular momentum operator* $\hat{L}_l$*, not a scalar multiple of the identity. This means the commutator* $[\hat{L}_j, \hat{L}_k]$ *does not act as a constant on* $\mathcal{S}(\mathbb{R}^3)$*, and its expectation value* $\langle[\hat{L}_j, \hat{L}_k]\rangle = \mathrm{i}\Phi_0\,\langle\hat{L}_l\rangle$ *depends on the state* Ψ *through* $\langle\hat{L}_l\rangle$*. As a consequence, the Robertson bound for the angular momentum pair* (L_j, L_k) *is not a universal constant but a state-dependent quantity, and the bound can be zero for states in which* $\langle\hat{L}_l\rangle = 0$*. This state-dependence is not a weakness of the Robertson inequality; it reflects the genuine algebraic structure of the angular momentum operators and leads to physically significant consequences, as recorded in Sec. 7.4.*

7.2 The Angular Momentum Uncertainty Relations

Proposition 7.3 (Angular momentum uncertainty relations)**.** *Let* $\Psi \in \mathcal{S}(\mathbb{R}^3) \subset \mathcal{H}$ *be a normalized closure state. For each cyclic triple* (j, k, l) *with* (j, k, l) *a cyclic permutation of* $(1, 2, 3)$*:*

$$\Delta L_j \cdot \Delta L_k \geq \frac{\Phi_0}{2}\left|\langle\hat{L}_l\rangle\right|. \tag{38}$$

Explicitly, the three relations are:

$$\Delta L_1 \cdot \Delta L_2 \geq \frac{\Phi_0}{2}|\langle\hat{L}_3\rangle|, \tag{39}$$

$$\Delta L_2 \cdot \Delta L_3 \geq \frac{\Phi_0}{2}|\langle\hat{L}_1\rangle|, \tag{40}$$

$$\Delta L_3 \cdot \Delta L_1 \geq \frac{\Phi_0}{2}|\langle\hat{L}_2\rangle|. \tag{41}$$

Proof. Apply Theorem 3.3 with $A = \hat{L}_j$ and $B = \hat{L}_k$. Both operators are self-adjoint on $\mathcal{S}(\mathbb{R}^3)$ (QM4 Proposition 7.2). The commutator expectation value is, using Eq. (34):

$$\langle[\hat{L}_j, \hat{L}_k]\rangle = \mathrm{i}\Phi_0\,\epsilon_{jkl}\,\langle\hat{L}_l\rangle.$$

For a cyclic triple (j, k, l), $\epsilon_{jkl} = +1$, so

$$\left|\langle[\hat{L}_j, \hat{L}_k]\rangle\right| = \Phi_0\,\left|\langle\hat{L}_l\rangle\right|.$$

Substituting into Robertson Eq. (11):

$$\Delta L_j \cdot \Delta L_k \geq \frac{1}{2}\cdot\Phi_0\,|\langle\hat{L}_l\rangle| = \frac{\Phi_0}{2}|\langle\hat{L}_l\rangle|,$$

which is Eq. (38). The three explicit forms Eqs. (39)–(41) follow by substituting the three cyclic permutations of $(1, 2, 3)$. □

Remark 7.4. *The right-hand side of Eq.* (38) *vanishes whenever* $\langle \hat{L}_l \rangle = 0$*. For such states, the Robertson inequality yields only the trivial bound* $\Delta L_j \cdot \Delta L_k \geq 0$*, providing no constraint on the simultaneous spread of* L_j *and* L_k*. This occurs, for example, for eigenstates of* $\hat{L}_3$ *with eigenvalue* $m\Phi_0$ *(the* L_z *eigenstates of QM5): for such states,* $\langle \hat{L}_1 \rangle = \langle \hat{L}_2 \rangle = 0$ *by the ladder operator structure developed in QM5, so the bounds for the* (L_2, L_3) *and* (L_3, L_1) *pairs are trivially zero. However, for* $m \neq 0$*, the bound for the* (L_1, L_2) *pair is* $\Delta L_1 \cdot \Delta L_2 \geq (\Phi_0/2)|m|\Phi_0 = m\Phi_0^2/2$*, which is non-trivial and grows with the magnetic quantum number* m*.*

7.3 State-Dependence and Physical Interpretation

The state-dependence of the angular momentum uncertainty bound has a direct physical interpretation in the NUVO transport closure framework.

For a closure state with definite total angular momentum L^2 eigenvalue $\ell(\ell+1)\Phi_0^2$ and definite L_3 eigenvalue $m\Phi_0$ (the standard angular momentum eigenstates $|L^2, L_3\rangle$ of QM5), the mean transverse angular momenta vanish: $\langle \hat{L}_1 \rangle = \langle \hat{L}_2 \rangle = 0$. The bound for the (L_1, L_2) pair is therefore trivially zero from the Robertson inequality alone. Yet the standard deviations ΔL_1 and ΔL_2 are not zero: they equal $\sqrt{\ell(\ell+1) - m^2}\,\Phi_0/\sqrt{2}$ for the $|L^2, L_3\rangle$ eigenstate (a result derived in QM5 from the eigenvalue structure of $\hat{L}^2$). The Robertson inequality does not capture this non-zero spread for states with $\langle \hat{L}_3 \rangle = 0$; the actual constraint on $\Delta L_1 \cdot \Delta L_2$ comes from the total angular momentum structure and is derived in QM5 from the algebra of $\hat{L}^2$ and $\hat{L}_3$.

This illustrates a general feature of the Robertson inequality: it provides a necessary condition for simultaneous spread in terms of the commutator, but the commutator-based bound may not be tight for all states. For the angular momentum algebra, the tighter bounds on ΔL_j come from the joint eigenvalue structure of $\hat{L}^2$ and $\hat{L}_3$ developed in QM5, not from Robertson alone.

Remark 7.5. *The operator* $\hat{L}^2 = \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2$ *commutes with all three* $\hat{L}_j$*:* $[\hat{L}^2, \hat{L}_j] = 0$ *for all* j*. This is established in QM5 as part of the full angular momentum algebra. For a state in which* $\hat{L}^2$ *has a definite eigenvalue* $\ell(\ell+1)\Phi_0^2$*, the identity* $\hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2 = \ell(\ell+1)\Phi_0^2\,\hat{\mathbf{1}}$ *acting on that eigenspace provides an additional constraint on the standard deviations beyond Robertson: the sum* $(\Delta L_1)^2 + (\Delta L_2)^2 + (\Delta L_3)^2 + \langle \hat{L}_1 \rangle^2 + \langle \hat{L}_2 \rangle^2 + \langle \hat{L}_3 \rangle^2 = \ell(\ell+1)\Phi_0^2$ *is fixed by the total angular momentum quantum number* ℓ*. This sum rule, derived in QM5, constrains the individual standard deviations in a way that the Robertson inequality alone cannot: it bounds not just products* $\Delta L_j \cdot \Delta L_k$ *but the full quadrature sum. The Robertson inequality of Proposition 7.3 is the universal template; the QM5 sum rule is the tighter, algebra-specific constraint.*

7.4 Scope of the Angular Momentum Analysis

The present section establishes the angular momentum uncertainty relations as a direct consequence of the Robertson inequality and the commutation algebra of QM4/QM5. The analysis is deliberately restricted to what follows from Robertson alone, without the full spectral theory of the angular momentum operators. Several results that go beyond Robertson are deferred to QM5.

The following are established in the present section: the uncertainty relations $\Delta L_j \cdot \Delta L_k \geq (\Phi_0/2)|\langle \hat{L}_l \rangle|$ from Robertson and the commutation algebra; the state-dependence of the bound and its vanishing for states with $\langle \hat{L}_l \rangle = 0$; and the identification of the angular momentum algebra as the template for the spin uncertainty relations of QM8.

The following are deferred to QM5: the complete spectral theory of $\hat{L}^2$ and $\hat{L}_3$, yielding the eigenvalues $\ell(\ell+1)\Phi_0^2$ and $m\Phi_0$ from the integer holonomy quantization condition; the ladder operator structure $\hat{L}^{\pm} = \hat{L}_1 \pm \mathrm{i}\hat{L}_2$ and the matrix elements of $\hat{L}_j$ in the $|L^2, L_3\rangle$ basis; the explicit standard deviations ΔL_1 and ΔL_2 for angular momentum eigenstates; the sum rule $(\Delta L_1)^2 +$

$(\Delta L_2)^2+(\Delta L_3)^2+|\langle\hat{L}\rangle|^2 = \ell(\ell+1)\Phi_0^2$; and the identification of spherical harmonics as the stationary angular closure eigenstates.

Remark 7.6. *The angular momentum uncertainty relations of Proposition 7.3 serve as the template for the spin uncertainty relations of QM8. The spin operators* $\hat{S}_j$ *introduced in QM8 satisfy the same commutation algebra:* $[\hat{S}_j, \hat{S}_k] = \mathrm{i}\Phi_0\,\epsilon_{jkl}\hat{S}_l$*, with the same structure as Eq.* (34)*. Applying Robertson to this algebra gives the spin uncertainty relations:* $\Delta S_j \cdot \Delta S_k \geq (\Phi_0/2)|\langle\hat{S}_l\rangle|$*. For spin-*$\frac{1}{2}$ *states, the spin operators have eigenvalues* $\pm\Phi_0/2$ *and the explicit standard deviations are fully determined by the two-dimensional spin-*$\frac{1}{2}$ *Hilbert space structure. The derivation in QM8 follows exactly the pattern of the present section: Robertson is applied with the spin commutation algebra, yielding the spin uncertainty relations by the same two-step argument.*

Remark 7.7. *The applications of the Robertson inequality in the present paper establish a general pattern that recurs throughout the QM-series. For each observable algebra with commutation relations* $[G_j, G_k] = \mathrm{i}\Phi_0\, f_{jkl}G_l$ *(for some structure constants* f_{jkl}*), the Robertson inequality yields uncertainty relations* $\Delta G_j \cdot \Delta G_k \geq (\Phi_0/2)|f_{jkl}||\langle G_l\rangle|$*. For the position-momentum algebra (*$f_{jk} = \delta_{jk}$*, constant right-hand side), the bound is state-independent. For the angular momentum and spin algebras (*$f_{jkl} = \epsilon_{jkl}$*, right-hand side involves the algebra generators), the bound is state-dependent. The Robertson inequality of Theorem 3.3 is the single algebraic result that underlies all these relations; the specific form of each uncertainty relation is determined by the specific commutation algebra of the observable pair. This unity of the uncertainty structure across all sectors of the QM-series is a direct consequence of the scalar–conformal transport geometry: the commutation relations of all transport generators are derived from the same phase gradient and rotation structure of the exchange-sector transport system, and the Robertson inequality is the universal algebraic consequence of the Hilbert space inner product that those commutation relations imply.*

8 Interpretive Clarifications and Scope

The present section collects the interpretive constraints that govern the uncertainty relations derived in the preceding sections and states them explicitly as a unified set of boundary conditions on the NUVO account of transport resolution. Three items are addressed: the geometric rather than epistemological character of the uncertainty relations, the explicit exclusion of the measurement-disturbance account, and the scope of the present construction relative to the remainder of the QM-series. These constraints are not incidental; they distinguish the NUVO derivation from conventional presentations and protect the logical integrity of the series by preventing the importation of interpretive content that has not been derived within the framework.

8.1 Uncertainty as Geometric Constraint, Not Epistemology

The uncertainty relations of the present paper are properties of the transport closure state $\Psi \in \mathcal{H}$, not properties of knowledge, information, or measurement precision. This distinction is not merely terminological; it reflects the logical structure of the derivations.

The Robertson inequality Eq. (11) was derived in Sec. 3 from two inputs: the Cauchy–Schwarz inequality on $\mathcal{H}$ (a property of the inner product) and the decomposition of the cross inner product into commutator and anti-commutator contributions (an algebraic identity for self-adjoint operators). Neither input involves any measurement, any observer, any information acquisition, or any physical process. The inequality is a structural relation among the mathematical objects A, B, and Ψ in the Hilbert space $\mathcal{H}$.

The quantities ΔA and ΔB are standard deviations of the closure state with respect to the observables A and B, defined in Definition 2.2 as properties of Ψ and the operators. For $A = \hat{x}^j$, the quantity $\Delta x_j = \Delta x^j$ is the root-mean-square spatial spread of the closure density $|\Psi(x)|^2$: a geometric property of the closure distribution in position space. For $B = \hat{p}_j$, the quantity $\Delta p_j = \Delta p_j$ is the root-mean-square spread of the momentum-space closure density $|\tilde{\Psi}(p)|^2$: a geometric property of the closure distribution in momentum space. Both are properties of the closure state Ψ that exist prior to and independently of any interaction or observation.

The Heisenberg relation $\Delta x_j \cdot \Delta p_j \geq \Phi_0/2$ is therefore a constraint on the geometry of the scalar–conformal transport closure configuration: no admissible element of $\mathcal{H}$ can simultaneously have arbitrarily small spatial spread and arbitrarily small momentum spread in the same direction. This is a statement about which closure states exist within $\mathcal{H}$, not about what an observer can know about those states or what happens when they are probed.

Remark 8.1. *The distinction between geometric (intrinsic) and epistemological (observer-relative) interpretations of the uncertainty relations is relevant to the NUVO program in the following specific sense. In the standard formulation, both interpretations are sometimes invoked interchangeably, and their relationship is a matter of ongoing discussion in the foundations of quantum mechanics. In the NUVO framework, only the geometric interpretation is available from the derivation: the Robertson inequality follows from the Hilbert space geometry and the commutation algebra, with no reference to observers or measurements. If an epistemological interpretation is desired—for example, a statement about what can be known from interaction events—it would need to be derived separately from the coherence-gated interaction theory of QB5 and QB6, combined with the uncertainty relations of the present paper. Such a derivation lies outside the scope of QM3 but is not excluded from the program; it is simply a distinct result that requires distinct inputs.*

8.2 The Heisenberg Microscope Is Not Invoked

The Heisenberg microscope thought experiment is the most widely cited physical argument for the position-momentum uncertainty relation. The argument proceeds as follows: to measure the position of a particle with resolution Δx_j, one uses photons of wavelength $\lambda \leq \Delta x_j$, which carry momentum of order $h/\lambda \geq h/\Delta x_j$; the measurement interaction imparts a momentum kick of this order to the particle, producing a post-measurement momentum uncertainty $\Delta p_j \geq h/\Delta x_j$; multiplying gives $\Delta x_j \cdot \Delta p_j \geq h$, of the same order as the Robertson bound $\Phi_0/2 = \hbar/2$.

This argument is not the content of Theorem 4.1 and is not invoked in the present paper for the following reasons.

First, the Heisenberg microscope argument derives a bound on post-measurement momentum uncertainty given a pre-measurement position measurement of precision Δx_j. Theorem 4.1 derives a bound on the product of the pre-existing standard deviations of the closure state, prior to any measurement. The two quantities are not the same: the standard deviation Δx_j in Theorem 4.1 is a property of the closure state Ψ, while the measurement precision in the Heisenberg argument is a property of the measurement apparatus.

Second, the Heisenberg microscope argument involves photons, recoil, and the physical mechanism of the position measurement. None of these enter the derivation of Theorem 4.1, which proceeds from the canonical commutation relation and the Cauchy–Schwarz inequality alone. The physical mechanism of the position measurement is a topic for the interaction theory of QB5 and QB6, not for the present paper.

Third, the Heisenberg microscope argument has been shown to be quantitatively incorrect as a derivation of the Robertson bound: the momentum disturbance from a position measurement can, in principle, be made smaller than $\hbar/\Delta x_j$ through careful experimental design, while the Robertson bound $\Delta x_j \cdot \Delta p_j \geq \hbar/2$ is an inviolable structural constraint on the pre-measurement state. The two bounds are conceptually and quantitatively distinct.

Remark 8.2. *The correct quantum-mechanical account of measurement disturbance — the precise relationship between the disturbance of one observable caused by a measurement of another — is a topic that requires the full interaction framework. In the NUVO program, this account will be developed using the coherence-gated interaction structure of QB5 and QB6, combined with the operator algebra of QM1 and the uncertainty relations of the present paper. The result will be a derived statement about the trade-off between measurement precision and disturbance, not a postulate about what measurements can achieve. This derivation is deferred to a subsequent paper in the series; its logical prerequisites include the present paper's uncertainty relations as one input among several.*

8.3 The Status of the Energy-Time Relation

The energy-time uncertainty relation of Theorem 5.5 has a different logical status from the position-momentum and angular momentum relations and requires a separate interpretive note.

The position-momentum relation holds for any normalized closure state $\Psi \in \mathcal{H}$, prior to and independently of any dynamical evolution. It is a static geometric property of the state. The energy-time relation, by contrast, is inherently dynamical: it involves the time derivative $\mathrm{d}\langle G\rangle(t)/\mathrm{d}t$, which requires the Schrödinger evolution $\Psi(t) = U(t)\Psi_0$ to be defined. In this sense the energy-time relation presupposes the dynamical framework of QM4, even though QM3 does not otherwise require QM4.

The characteristic evolution time Δt_G defined in Definition 5.3 is not a universal property of the state but depends on both the state and the observable G chosen to define it. For a given state $\Psi(t)$, different observables G give different characteristic times Δt_G; the energy-time relation $\Delta E \cdot \Delta t_G \geq \Phi_0/2$ holds for all choices of G with $\mathrm{d}\langle G\rangle/\mathrm{d}t \neq 0$. The physically relevant bound is obtained by minimizing Δt_G over all observables G, which selects the observable whose expectation value is changing most rapidly relative to its standard deviation.

These features distinguish the energy-time relation from the position-momentum relation in a way that is not a deficiency but a structural truth: time is a parameter of the evolution, not an observable of the closure state, and the energy-time bound reflects this parameter character by depending on the dynamics rather than on a static state property.

Remark 8.3. *The Pauli argument of Proposition 5.1 and the energy-time derivation of Theorem 5.5 are mutually consistent and complementary. Proposition 5.1 establishes that no self-adjoint time operator exists in $\mathcal{H}$ for any physically admissible Hamiltonian. This might seem to imply that an energy-time uncertainty relation cannot be derived at all. Theorem 5.5 shows that an energy-time uncertainty relation can be derived, but by a route that does not require a time operator: through the rate-of-change identity for expectation values and the Robertson inequality applied to $(\hat{H}, G)$. The two results together establish the correct logical structure: there is no time operator, and therefore no Robertson inequality of the form $\Delta E \cdot \Delta t \geq (\Phi_0/2)|\langle[\hat{H}, \hat{t}]\rangle|$; but there is an energy-time uncertainty relation of a different form, derived from the dynamics of observables under Schrödinger evolution.*

8.4 Scope of the Present Construction

The present paper establishes the Robertson and Schrödinger uncertainty inequalities for general self-adjoint operator pairs, the Heisenberg position-momentum uncertainty relation, the energy-time uncertainty relation from the rate-of-change identity, the characterization of minimum-uncertainty states as Gaussian closure configurations, and the angular momentum uncertainty relations from the angular momentum commutation algebra. It is equally important to record what the present paper does not establish.

The paper does not derive the spin uncertainty relations. The operators $\hat{S}_j$ introduced in QM8 satisfy the same commutation algebra as the angular momentum operators, and the spin uncertainty relations $\Delta S_j \cdot \Delta S_k \geq (\Phi_0/2)|\langle \hat{S}_l \rangle|$ follow by the same Robertson application as Proposition 7.3. However, the spin operators arise from the double-cover holonomy structure of the rotation group on the transport closure system, which is developed in QM8; they are not available in the present paper. The spin uncertainty relations are recorded in QM8 as an application of the Robertson template established here.

The paper does not derive uncertainty relations for relativistic transport observables. The covariant momentum operators $\hat{p}^\mu = \mathrm{i}\Phi_0\, \partial^\mu$ introduced in QM11 for the relativistic transport sector satisfy commutation relations determined by the Lorentz algebra, and the associated uncertainty relations follow from Robertson in the same pattern as the non-relativistic case. The derivation is deferred to QM11 and the RQM-series, where the covariant transport generators are fully established.

The paper does not derive entropic uncertainty relations. The Robertson and Schrödinger inequalities bound the product of standard deviations, which are second-moment properties of the closure density. A different class of uncertainty relations, formulated in terms of the Shannon or Rényi entropy of the closure density distributions, provides tighter bounds in some regimes and captures uncertainty structure that second-moment bounds miss. The derivation of entropic uncertainty relations requires information-theoretic tools beyond the scope of the present paper and is deferred as a structural extension of the QM-series.

The paper does not treat the measurement-disturbance trade-off. The Robertson inequality bounds the pre-existing standard deviations of the closure state before any interaction. A distinct class of inequalities, sometimes called *error-disturbance relations*, bounds the trade-off between the precision of a measurement of one observable and the disturbance imparted to a conjugate observable by that measurement. These relations require the coherence-gated interaction theory of QB5 and QB6 as their physical input and are outside the scope of the present paper, which is concerned entirely with the intrinsic geometric properties of the closure state.

The paper does not derive the full uncertainty structure for states of definite angular momentum. Proposition 7.3 provides the Robertson bounds for angular momentum pairs. The complete characterization of the standard deviations ΔL_j for angular momentum eigenstates—including the sum rule $\sum_j [(\Delta L_j)^2 + \langle L_j \rangle^2] = \ell(\ell+1)\Phi_0^2$ and the explicit values of ΔL_1 and ΔL_2 for $|L^2, L_3\rangle$ eigenstates—requires the full spectral theory of $\hat{L}^2$ and $\hat{L}_3$ developed in QM5.

Remark 8.4. *The uncertainty relations established in the present paper complete the algebraic foundation of the QM-series that was identified as the collective goal of QM1, QM2, and QM3. QM1 constructed the Hilbert space and established the canonical commutation relations. QM2 derived the superposition principle and the complementarity relation from the linearity of the transport closure equations and the Cauchy–Schwarz inequality. QM3 derives the uncertainty relations from the commutation relations and the same Cauchy–Schwarz inequality. All three results are algebraic consequences of the scalar–conformal transport geometry and the Hilbert space structure it generates; none requires the dynamical framework of QM4. With these three papers complete, the program possesses: the state space (QM1), the superposition and interference structure (QM2), and the simultaneous resolution bounds (QM3), all derived from the transport closure geometry without postulate. The dynamical framework of QM4 and the subsequent papers add the time evolution, conservation laws, and physical sector developments—angular momentum (QM5), harmonic oscillator*

(QM6), multi-particle systems (QM7), spin (QM8), entanglement (QM9), scattering (QM10), and the relativistic extension (QM11)—on this algebraic foundation.

9 Conclusion

9.1 Summary of Results

The present paper has derived the uncertainty relations of the scalar–conformal NUVO transport closure system as structural theorems of the Hilbert space $\mathcal{H}$, using only the canonical commutation relation of QM1 and the Cauchy–Schwarz inequality of QM1 Lemma 4.2. No measurement argument, no physical thought experiment, and no new postulate enters any derivation. The principal results are as follows.

Standard deviation as a closure-state property (Definition 2.2). For a normalized closure state $\Psi \in \mathcal{H}$ and a self-adjoint observable A, the standard deviation $\Delta A = \|(A - \langle A \rangle)\Psi\|_{\mathcal{H}}$ is defined as a property of Ψ and A independently of any measurement process. The representation $(\Delta A)^2 = \left\|\widehat{\Delta A}\Psi\right\|_{\mathcal{H}}^2$ identifies the variance as a squared Hilbert space norm, making the Cauchy–Schwarz inequality directly applicable.

Decomposition of the cross inner product (Lemma 3.1). The cross inner product $\left\langle \widehat{\Delta A}\Psi, \widehat{\Delta B}\Psi \right\rangle_{\mathcal{H}}$ decomposes as $\frac{1}{2}\langle\{\widehat{\Delta A},\, \widehat{\Delta B}\}\rangle + \frac{\mathrm{i}}{2}\langle[A,\, B]\rangle$, with the real part determined by the anti-commutator and the imaginary part by the commutator of A and B. The key identity $[\widehat{\Delta A},\, \widehat{\Delta B}] = [A,\, B]$ ensures that the commutator of the shifted operators equals the commutator of the originals, so the dispersion does not alter the commutation structure.

The Robertson uncertainty inequality (Theorem 3.3). For any two self-adjoint operators A, B on $\mathcal{H}$ and any normalized closure state in their common domain, $\Delta A \cdot \Delta B \geq \frac{1}{2}|\langle[A,\, B]\rangle|$. The derivation uses Cauchy–Schwarz to bound the product of standard deviations by the modulus of the cross inner product, then retains only the imaginary part (the commutator contribution). The Robertson inequality is the universal algebraic template from which all specific uncertainty relations in the QM-series are derived by substituting the relevant commutation algebra.

The Schrödinger uncertainty improvement (Theorem 3.5). Retaining both the real (anti-commutator) and imaginary (commutator) parts of the cross inner product yields the tighter bound $(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}|\langle\{\widehat{\Delta A},\, \widehat{\Delta B}\}\rangle|^2 + \frac{1}{4}|\langle[A,\, B]\rangle|^2$, which exceeds the Robertson bound whenever the anti-commutator term is non-zero. The Robertson inequality follows as the special case in which the anti-commutator term is dropped.

The saturation condition (Proposition 3.7). Equality holds in the Robertson inequality if and only if $\widehat{\Delta B}\Psi = \mathrm{i}\mu\,\widehat{\Delta A}\Psi$ for some $\mu \in \mathbb{R}$, which simultaneously saturates the Cauchy–Schwarz inequality (proportionality condition) and annihilates the anti-commutator term (purely imaginary proportionality constant). This saturation condition is an eigenvalue equation for the operator $\widehat{\Delta B} - \mathrm{i}\mu\,\widehat{\Delta A}$ with eigenvalue zero, whose solutions for specific operator pairs are derived in Sec. 6.

The Heisenberg position-momentum uncertainty relation (Theorem 4.1 and Corollary 4.5). Applied to $A = \hat{x}^j$, $B = \hat{p}_k$ with the canonical commutation relation $[\hat{x}^j, \hat{p}_k] = \mathrm{i}\Phi_0\,\delta^j{}_k$ from QM1 Proposition 5.4, the Robertson inequality yields $\Delta x_j \cdot \Delta p_j \geq \Phi_0/2$ for each spatial direction j. The bound $\Phi_0/2 = \hbar/2$ is state-independent—a universal geometric constraint on the transport closure system—because the commutator expectation value is a scalar multiple of the identity. Cross-direction pairs $(\hat{x}^j, \hat{p}_k)$ with $j \neq k$ commute and yield only the trivial bound zero. The Heisenberg relation is interpreted as a constraint on the simultaneous spatial and momentum resolution of the closure density, not as a statement about measurement disturbance.

The energy-time uncertainty relation (Proposition 5.1 and Theorem 5.5). Since no self-adjoint time operator exists in $\mathcal{H}$ for any physically admissible Hamiltonian (the Pauli argument), the energy-time relation cannot be derived by the Robertson route. It is derived instead from the rate-of-change identity $|\mathrm{d}\langle G\rangle/\mathrm{d}t| = (1/\Phi_0)|\langle[\hat{H}, G]\rangle|$ (QM4) and the Robertson inequality applied to $(\hat{H}, G)$, yielding $\Delta E \cdot \Delta t_G \geq \Phi_0/2$, where $\Delta t_G = \Delta G/|\mathrm{d}\langle G\rangle/\mathrm{d}t|$ is the characteristic evolution time of the observable G. The bound has the same right-hand side $\Phi_0/2$ as the Heisenberg relation but a structurally different left-hand side that depends on the dynamics rather than on static state properties.

Minimum-uncertainty states are Gaussian closure states (Lemma 6.1, Theorem 6.2, and Proposition 6.4). For the position-momentum pair, the saturation condition $(\hat{p} - \langle p\rangle)\Psi = \mathrm{i}\mu(\hat{x} - \langle x\rangle)\Psi$ is a first-order ODE in position space whose unique normalizable solution (for $\mu > 0$) is the Gaussian closure state $\Psi(x) = (2\pi\sigma^2)^{-1/4}\exp[-(x - \langle x\rangle)^2/(4\sigma^2) + \mathrm{i}\langle p\rangle x/\Phi_0]$ with $\sigma = \sqrt{\Phi_0/(2\mu)}$. Every member of this Gaussian family saturates the Heisenberg bound $\Delta x \cdot \Delta p = \Phi_0/2$ regardless of width, has a Gaussian momentum-space density with width $\Phi_0/(2\sigma)$, and has vanishing anti-commutator term (the Robertson and Schrödinger bounds coincide). The three-dimensional minimum-uncertainty state is the product of three independent one-dimensional Gaussians.

Angular momentum uncertainty relations (Proposition 7.3). Applied to $A = \hat{L}_j$, $B = \hat{L}_k$ with the commutation algebra $[\hat{L}_j, \hat{L}_k] = \mathrm{i}\Phi_0\,\epsilon_{jkl}\hat{L}_l$ from QM4/QM5, the Robertson inequality yields $\Delta L_j \cdot \Delta L_k \geq (\Phi_0/2)|\langle\hat{L}_l\rangle|$ for each cyclic triple (j, k, l). Unlike the position-momentum bound, this bound is state-dependent: it vanishes for states with $\langle\hat{L}_l\rangle = 0$ and grows with $|\langle\hat{L}_l\rangle|$ for states with non-zero mean angular momentum in the l-th direction. The full angular momentum uncertainty structure, including the sum rule from the $\hat{L}^2$ eigenvalue, is developed in QM5.

9.2 Programmatic Significance

The results of the present paper, together with those of QM1 and QM2, complete the algebraic foundation of the QM-series. The significance of this completion is worth recording explicitly.

QM1, QM2, and QM3 together establish, from the scalar–conformal transport closure geometry, all of the structural constraints on the quantum state space that do not require dynamics. QM1 provides the state space itself: the Hilbert space $\mathcal{H}$ with its inner product, completeness, self-adjoint transport generators, and spectral theorem. QM2 derives the superposition principle and the interference and complementarity structure: the fact that the state space is closed under linear combinations (superposition), and that two-path transport configurations produce fringe patterns whose visibility and path-distinguishability satisfy $\mathcal{V}^2 + \mathcal{W}^2 \leq 1$ (complementarity). QM3 derives the simultaneous resolution bounds: the uncertainty relations that constrain which closure states exist within $\mathcal{H}$ by bounding the product of their standard deviations with respect to non-commuting observable pairs. All three sets of results are derived from the same two inputs—the scalar–conformal transport geometry (via the transport generators and their commutation relations) and the inner product structure of $\mathcal{H}$ (via the Cauchy–Schwarz inequality)—and none requires the dynamical time-evolution framework of QM4. The algebraic foundation is complete without dynamics.

The Robertson inequality established in Theorem 3.3 is the single algebraic result of broadest programmatic reach within the present paper. Every uncertainty relation in the QM-series is a special case of Robertson applied to the relevant commutation algebra: the Heisenberg relation (Sec. 4) from the canonical commutation relations of QM1, the angular momentum relations (Sec. 7) from the angular momentum algebra of QM4/QM5, the spin relations (QM8) from the spin commutation algebra derived from double-cover holonomy, and the spectral linewidth interpretation (QM10) from the energy-width/lifetime relation established in Sec. 5. The unity of all these results under the single Robertson template reflects the unity of the scalar–conformal transport program: the commutation relations of all transport generators arise from the same geometric structure of the exchange-sector transport system, and the Robertson inequality is the universal algebraic consequence of the Hilbert space inner product that those commutation relations imply.

The minimum-uncertainty Gaussian states of Theorem 6.2 provide the structural bridge between the algebraic foundation and the dynamical analysis of QM4 through QM6. The algebraic characterization established here—Gaussian closure states are exactly those that saturate the position-momentum uncertainty bound—is the necessary precondition for the dynamical characterization established in QM6: coherent states are those Gaussian states that additionally preserve their Gaussian profile under harmonic oscillator dynamics. The connection between these two characterizations—one purely algebraic, derived from the saturation of Cauchy–Schwarz; the other dynamical, derived from the invariance of the Gaussian form under harmonic oscillator Schrödinger evolution—is one of the program's most direct bridges between the geometric structure of $\mathcal{H}$ and the physical dynamics of the scalar–conformal transport system.

9.3 Transition to QM5

With the algebraic foundation complete, the QM-series turns to the first major physical sector analysis: the angular momentum structure of the scalar–conformal transport system. QM5 develops the full rotational transport algebra from the angular momentum operators $\hat{L}_j = \epsilon_{jkl}\hat{x}^k\hat{p}_l$ introduced in QM4 and used in Sec. 7 of the present paper.

The starting points for QM5 are the commutation algebra $[\hat{L}_j, \hat{L}_k] = \mathrm{i}\Phi_0\,\epsilon_{jkl}\hat{L}_l$ (recalled from QM4 and used in Proposition 7.3), the conservation of angular momentum for rotationally symmetric Hamiltonians established in QM4 Proposition 7.3, and the Schrödinger dynamics of QM4 that provide the dynamical context within which the angular momentum eigenstates evolve. The angular momentum uncertainty relations of Proposition 7.3 serve as an entry point into QM5: the bounds $\Delta L_j \cdot \Delta L_k \geq (\Phi_0/2)|\langle\hat{L}_l\rangle|$ established here are the precursors to the complete spectral theory that QM5 derives, and the state-dependence of the bound motivates the classification of states by their $\hat{L}^2$ and $\hat{L}_3$ eigenvalues that QM5 undertakes. QM5 will derive the eigenvalue spectrum $\ell(\ell+1)\Phi_0^2$ of $\hat{L}^2$ from the integer holonomy quantization condition on rotationally closed transport paths—the same holonomy quantization that underlies the Q-series hydrogenic spectrum—and will identify the spherical harmonics $Y_\ell^m(\theta, \varphi)$ as the stationary angular closure eigenstates, completing the angular sector of the scalar–conformal NUVO quantum mechanics.

References

[1] Gerald B. Folland. *Harmonic Analysis in Phase Space.* Princeton University Press, Princeton, 1989. Primary reference for: the Fourier uncertainty principle (the classical analytic form of the Heisenberg relation, Remark 4.3); the Fourier transform of a Gaussian (used in the proof of Theorem 6.2 and Proposition 6.2); the Wigner function and its positivity for Gaussian states (Hudson's theorem, mentioned in §6); the Plancherel theorem at scale Φ_0 recalled from QM1.

[2] Wolfgang Pauli. Uber das Wasserstoffspektrum vom Standpunkt der neuen Quantenmechanik. *Zeitschrift für Physik*, 36:336–363, 1926. Contains the argument (subsequently elaborated by others) that no self-adjoint time operator conjugate to a semi-bounded Hamiltonian can exist; the argument is used in QM3 Proposition 5.1 to establish that the energy-time uncertainty relation cannot be derived by the Robertson route.

[3] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics, Vol. I: Functional Analysis.* Academic Press, New York, 1972. Primary reference for: Hilbert space theory, Cauchy–Schwarz inequality, the spectral theorem for self-adjoint operators, the Stone–von Neumann theorem (used in Proposition 5.1 on the absence of a self-adjoint time operator), and the Weyl algebra representation theory.

[4] Howard Percy Robertson. The uncertainty principle. *Physical Review*, 34(1):163–164, 1929. Original derivation of the general uncertainty inequality $\Delta A \cdot \Delta B \geq \frac{1}{2}|\langle [A, B] \rangle|$ for arbitrary observables on a Hilbert space; the result is established in QM3 as Theorem 3.2 by the Cauchy–Schwarz method, with Robertson's 1929 paper cited for historical credit.