QM5 — Angular Momentum from Rotational Transport Closure

in Scalar–Conformal NUVO Systems *Preprint, Version 1.0*[*]

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Notation and Conventions

- $\mathcal{M}$ denotes the spacetime manifold.
- η denotes the reference Lorentzian metric (typically Minkowski in a global chart).
- g denotes the physical metric.
- The scalar field $\Lambda : \mathcal{M} \to \mathbb{R}_{>0}$ is the NUVO modulation field.
- The physical metric is scalar–conformal:

$$g_{\mu\nu} = \Lambda^2 \, \eta_{\mu\nu}.$$

- $\Lambda_0 > 0$ denotes the baseline scalar availability level supported by the intrinsic delivery structure of the underlying field. In the absence of localized structural occupation the scalar field satisfies $\Lambda(x) = \Lambda_0$.
- The dimensionless scalar diagnostic is

$$\lambda(x) := \frac{\Lambda(x)}{\Lambda_0}.$$

- The scalar field represents the *locally available structural capacity* of the underlying delivery field. Localized structures may reduce this availability through occupation or transport, but the intrinsic delivery baseline Λ_0 remains fixed.
- Greek indices $\mu, \nu, \ldots$ range over spacetime coordinates $0, 1, 2, 3$.
- We use the Einstein summation convention unless explicitly stated otherwise.

Remark 0.1. *Unless otherwise stated, the background signature is* $(-, +, +, +)$.

[*]Bibliography is provisional. Cross-references to companion NUVO-series papers (M-, SR-, Q-, QB-, QM-series) will be updated with Zenodo DOIs in subsequent versions.

Program scope.

Abstract

The scalar–conformal NUVO transport closure system possesses a natural rotational symmetry structure: in the presence of a rotationally symmetric scalar capacity field $\Lambda(|x|)$, the exchange-sector transport is invariant under the action of the rotation group SO(3) on $\mathbb{R}^3$. The present paper derives the full angular momentum algebra and its spectral theory from this rotational transport structure, without postulating the quantum numbers ℓ and m or the spherical harmonic functions.

The angular momentum operators $\hat{L}_j = \epsilon_{jkl}\hat{x}^k\hat{p}_l$, introduced in QM4 as the generators of spatial rotations and used in QM3 to derive uncertainty relations, are shown here to satisfy the commutation algebra $[\hat{L}_j, \hat{L}_k] = \mathrm{i}\Phi_0\,\epsilon_{jkl}\hat{L}_l$ and to commute with the total angular momentum squared $\hat{L}^2 = \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2$. This algebra is derived from the canonical commutation relations of QM1 by explicit computation.

The joint spectrum of $\hat{L}^2$ and $\hat{L}_3$ is derived from the integer holonomy quantization condition: admissible transport closure paths that close under a full 2π rotation of the azimuthal angle must return to their initial configuration, imposing the single-valuedness condition $\mathrm{e}^{\mathrm{i}m\cdot 2\pi} = 1$ on the azimuthal component of the closure state. This condition selects the magnetic quantum numbers $m \in \mathbb{Z}$, and the algebraic structure of the ladder operators then constrains ℓ to non-negative integers and $m \in \{-\ell, \dots, +\ell\}$.

The eigenstates of $\hat{L}^2$ and $\hat{L}_3$ in position space are identified as the spherical harmonics $Y_\ell^m(\theta, \varphi)$, derived as solutions of the angular eigenvalue equations rather than introduced as known functions. The orthogonality, completeness, and addition theorem for spherical harmonics follow from the spectral theory of $\hat{L}^2$ and $\hat{L}_3$ on $\mathcal{H}$.

The angular momentum structure is connected to the hydrogenic sector of the Q-series: the full n-ℓ-m quantum number labeling of hydrogen energy eigenstates is established, the degeneracy structure is derived from the algebraic independence of the radial and angular sectors, and the spherical harmonics are identified as the angular factors in the separation of variables for the hydrogenic Schrödinger equation of QM4.

No new postulates are introduced. The angular momentum algebra, the quantum number quantization, and the spherical harmonics all emerge as structural consequences of the scalar–conformal rotational transport geometry and the holonomy quantization condition established in the Q-series.

1 Introduction

1.1 Position Within the QM-Series

The scalar–conformal NUVO program has now established its complete algebraic and dynamical foundation. The algebraic foundation—the Hilbert space $\mathcal{H}$, the superposition principle, the uncertainty relations—was established in QM1 through QM3 from the transport closure geometry and the canonical commutation relation, without reference to dynamics. The dynamical framework—the Schrödinger equation, the unitary time-evolution group, the conservation laws, and the Ehrenfest theorem—was established in QM4 from the scalar capacity field correspondence and Stone's theorem. Together, these four papers provide the structural and dynamical prerequisites for the analysis of specific physical sectors. The present paper, QM5, opens the physical sector development by deriving the complete angular momentum structure of the scalar–conformal exchange-sector transport system.

The angular momentum operators $\hat{L}_j = \epsilon_{jkl}\hat{x}^k\hat{p}_l$ were introduced in QM4 as the generators of spatial rotations and used there in two contexts: the proof that they commute with rotationally symmetric Hamiltonians (establishing angular momentum conservation) and the derivation of the angular momentum uncertainty relations from the Robertson inequality in QM3. Both uses were forward references: QM4 defined the operators and established conservation, QM3 applied the Robertson inequality to their commutation algebra, but neither paper derived that commutation algebra or established the spectrum of $\hat{L}^2$ and $\hat{L}_3$. The present paper closes both gaps. It derives the commutation algebra $[\hat{L}_j, \hat{L}_k] = \mathrm{i}\Phi_0\, \epsilon_{jkl}\hat{L}_l$ by explicit computation from the canonical commutation relation of QM1, establishes the full spectrum of the joint eigenstates of $\hat{L}^2$ and $\hat{L}_3$, derives the spherical harmonics as the position-space eigenfunctions, and connects the angular momentum structure to the hydrogenic sector of the Q-series to complete the n-ℓ-m quantum number labeling of the hydrogen atom.

In the standard formulation of quantum mechanics, the orbital quantum numbers ℓ and m are introduced in one of two ways. The analytic approach derives them via separation of variables in the Schrödinger equation in spherical coordinates, leading to the associated Legendre equation whose regular solutions exist only for integer $\ell \geq 0$ and $|m| \leq \ell$. The algebraic approach postulates the angular momentum commutation algebra and derives the spectrum from the ladder operator argument, obtaining the integers and half-integers as possible values and then appealing to an additional argument to exclude half-integers for orbital angular momentum. In the NUVO framework, neither approach is adopted without derivation. The integer character of m is derived from the holonomy quantization condition of the Q-series: the azimuthal component of the closure state must be single-valued under a 2π rotation, which is the same holonomy closure principle that quantized the hydrogenic energy levels. Given $m \in \mathbb{Z}$, the algebraic structure of the ladder operators then constrains ℓ to non-negative integers and restricts $m \in \{-\ell, \ldots, +\ell\}$. The integrality of the angular momentum quantum numbers is therefore not an additional postulate but a structural consequence of the holonomy quantization that underlies the entire Q-series.

The angular momentum structure established in the present paper is used in every subsequent paper of the QM-series. QM6 develops the harmonic oscillator in three dimensions, where the eigenstates are products of radial Laguerre functions and spherical harmonics Y_ℓ^m; the ladder operator algebraic technique established here for angular momentum is adapted there for the energy ladder of the oscillator. QM7 treats multi-particle systems, where the total angular momentum of a composite system is the vector sum of the individual angular momenta, and the Clebsch-Gordan decomposition is the central tool. QM8 derives spin as a double-cover holonomy structure: where QM5 imposes the single-valuedness condition $\mathrm{e}^{2\pi \mathrm{i}m} = 1$ selecting $m \in \mathbb{Z}$, QM8 relaxes this to allow $\mathrm{e}^{4\pi \mathrm{i}j} = 1$, selecting $j \in \frac{1}{2}\mathbb{Z}$ and giving the half-integer spin quantum numbers. QM10 uses the partial wave decomposition of scattering amplitudes in terms of spherical harmonics. QM11 constructs the covariant angular momentum structure that prepares the transition to the RQM-series.

1.2 Objective of the Present Work

The central objective of the present paper is to derive the complete angular momentum algebra, spectrum, and eigenstate structure of the scalar–conformal NUVO transport closure system, from the canonical commutation relation of QM1 and the holonomy quantization condition of the Q-series. Specifically, the paper establishes six claims.

1. The commutation algebra $[\hat{L}_j, \hat{L}_k] = \mathrm{i}\Phi_0\, \epsilon_{jkl}\hat{L}_l$ and $[\hat{L}^2, \hat{L}_j] = 0$ hold on $\mathcal{S}(\mathbb{R}^3) \subset \mathcal{H}$ and are derived by explicit computation from the canonical commutation relations $[\hat{x}^j, \hat{p}_k] = \mathrm{i}\Phi_0\, \delta^j{}_k$, $[\hat{x}^j, \hat{x}^k] = 0$, and $[\hat{p}_j, \hat{p}_k] = 0$ of QM1 Proposition 5.4. The commutation of $\hat{L}^2$ with all three components establishes that $\hat{L}^2$ and $\hat{L}_3$ form a complete set of commuting observables for the angular sector.

2. The ladder operators $\hat{L}_+ = \hat{L}_1 + \mathrm{i}\hat{L}_2$ and $\hat{L}_- = \hat{L}_1 - \mathrm{i}\hat{L}_2$ raise and lower the $\hat{L}_3$ eigenvalue by Φ_0 while preserving the $\hat{L}^2$ eigenvalue. The algebraic termination of the ladder

sequence—the existence of maximum and minimum $\hat{L}_3$ eigenvalues $m_{\max}\Phi_0$ and $-m_{\max}\Phi_0$ in each $\hat{L}^2$ eigenspace—implies the spectral constraint $\lambda = m_{\max}(m_{\max}+1)\Phi_0^2$ where λ is the $\hat{L}^2$ eigenvalue.

3. The holonomy quantization condition of the Q-series, applied to the azimuthal component of the transport closure state, requires $\mathrm{e}^{2\pi i m} = 1$, selecting $m \in \mathbb{Z}$. This is the same integer winding number condition that quantized the hydrogenic energy levels, now applied to the azimuthal angle of the transport closure path.

4. Combining the algebraic constraints of claim (2) with the integer holonomy of claim (3) yields the complete joint spectrum: $\sigma(\hat{L}^2) = \{\ell(\ell+1)\Phi_0^2 : \ell \in \{0, 1, 2, \ldots\}\}$ and $\sigma(\hat{L}_3)|_\ell = \{m\Phi_0 : m \in \{-\ell, \ldots, +\ell\}\}$, giving $2\ell+1$ values of m for each ℓ and establishing the matrix elements of the ladder operators in the $|\,\ell,\, m\,\rangle$ basis.

5. The position-space eigenfunctions of $\hat{L}^2$ and $\hat{L}_3$ on the unit sphere are the spherical harmonics $Y_\ell^m(\theta, \varphi)$, derived as normalized solutions of the angular eigenvalue equations: the φ-equation gives the factor $\mathrm{e}^{\,im\varphi}$, and the θ-equation gives the associated Legendre polynomial $P_\ell^{|m|}(\cos\theta)$. Their orthonormality, completeness on $L^2(S^2)$, parity, and complex conjugate relations follow from the spectral theory of the self-adjoint operators $\hat{L}^2$ and $\hat{L}_3$.

6. The angular momentum structure is connected to the hydrogenic sector of the Q-series through the separation of variables for the hydrogenic Hamiltonian of QM4: each energy eigenstate factorizes as $\Psi_{n\ell m}(r, \theta, \varphi) = R_{n\ell}(r)\, Y_\ell^m(\theta, \varphi)$, the allowed values $\ell \in \{0, 1, \ldots, n-1\}$ for each principal quantum number n give the n^2-fold degeneracy $g_n = \sum_{\ell=0}^{n-1}(2\ell+1) = n^2$, and the full n-ℓ-m labeling of the hydrogen spectrum is established.

Claims (1) through (6) form a logically ordered sequence. The commutation algebra of claim (1) is the algebraic input to the ladder operator analysis of claim (2). The holonomy condition of claim (3) is the geometric input that selects the integer subset of the algebraic possibilities from claim (2). Claims (1)–(3) together yield the full spectrum of claim (4). The position-space realization of claim (5) translates the abstract algebraic result into the concrete spherical harmonic functions used throughout the QM-series. Claim (6) connects the abstract angular momentum theory to the physical hydrogen spectrum, completing the program arc from the Q-series to the present paper.

1.3 What Is Not Assumed

The present work maintains without modification the interpretive discipline established in the prior series. Four exclusions are of particular importance for QM5.

The quantum numbers ℓ and m are not introduced as labels. In the standard formulation, ℓ and m are introduced as classification labels for angular momentum states, with their allowed values stated as part of the postulate structure of the theory. In the NUVO framework, m is derived from the holonomy quantization condition of the Q-series (Theorem 5.2) and ℓ is derived from the algebraic termination of the ladder operator sequence (Theorem 5.5). Neither is assumed.

The angular momentum commutation algebra is not postulated. The relation $[\hat{L}_j, \hat{L}_k] = i\Phi_0\,\epsilon_{jkl}\hat{L}_l$ is derived in Theorem 3.1 by explicit computation from the canonical commutation relations of QM1 Proposition 5.4. The computation is carried out in full in the proof of that theorem; no algebraic postulate about angular momentum commutators is introduced.

The spherical harmonics are not introduced as known special functions. The functions $Y_\ell^m(\theta, \varphi)$ are derived in Theorem 6.4 as the unique normalized solutions of the angular eigenvalue equations for $\hat{L}^2$ and $\hat{L}_3$. Their orthonormality and completeness are consequences of the spectral theory of these self-adjoint operators on $L^2(S^2)$, not properties imported from the theory of special functions.

Half-integer angular momentum quantum numbers are not excluded by ad hoc argument. In many textbooks, the half-integer possibility $\ell \in \frac{1}{2}\mathbb{Z}$ is algebraically admissible from the ladder operator analysis alone, and an additional argument—that the orbital angular momentum eigenfunctions must be single-valued on S^2—is invoked to exclude half-integers. In the NUVO framework, the single-valuedness condition is not an additional argument but the holonomy quantization principle of the Q-series applied to the azimuthal transport closure path. It selects $m \in \mathbb{Z}$ directly and thereby excludes half-integer values for the orbital quantum numbers without any separate postulate. The half-integer case—spin—arises in QM8 not because the single-valuedness condition is dropped but because it is applied to the double-cover of the rotation group, which requires a 4π rotation rather than 2π to return to the initial configuration.

1.4 Structure of the Paper

Sec. 2 recalls the angular momentum operator definitions from QM4, the canonical commutation relations from QM1, the holonomy quantization principle from the Q-series, and the conservation and uncertainty results from QM4 and QM3 that provide the starting context for the present development. Sec. 3 derives the angular momentum commutation algebra $[\hat{L}_j, \hat{L}_k] = \mathrm{i}\Phi_0\, \epsilon_{jkl}\hat{L}_l$ and the commutativity $[\hat{L}^2, \hat{L}_j] = 0$ by explicit computation from the canonical commutation relations, and establishes self-adjointness of the angular momentum operators. Sec. 4 introduces the ladder operators $\hat{L}_+ = \hat{L}_1 + \mathrm{i}\hat{L}_2$ and $\hat{L}_- = \hat{L}_1 - \mathrm{i}\hat{L}_2$, derives their commutation relations and their raising and lowering action on $\hat{L}_3$ eigenstates, and establishes the algebraic constraints on the spectrum from the boundedness and termination of the ladder sequence. Sec. 5 derives the integer quantization of the magnetic quantum number m from the holonomy quantization condition, combines this with the algebraic constraints to determine the full joint spectrum of $\hat{L}^2$ and $\hat{L}_3$, and records the matrix elements of the ladder operators in the $|\,\ell, m\,\rangle$ basis. Sec. 6 expresses $\hat{L}^2$ in spherical coordinates as $-\Phi_0^2\Delta_{S^2}$, derives the spherical harmonics as the normalized solutions of the angular eigenvalue equations, and establishes their orthonormality, completeness on $L^2(S^2)$, parity, and complex conjugate relations. Sec. 7 connects the angular momentum structure to the hydrogenic sector of the Q-series and QM4: the separation of variables factorization, the quantum number constraints $\ell \leq n-1$, the n^2-fold degeneracy, and the recovery of the Q-series holonomy at the radial level. Sec. 8 derives the $\hat{L}^2$ sum rule for angular momentum eigenstates (closing the promise of QM3 Remark 7.3) and verifies the consistency and saturation conditions for the angular momentum uncertainty relations of QM3. Sec. 9 collects interpretive clarifications, maintains the interpretive boundary conditions of the series, and records the scope of the present construction. Sec. 10 summarizes the eighteen principal results, records their programmatic significance, and prepares the transition to QM6.

2 Recalled Structure from Prior Series

The present section collects the results from the Q-series, QB-series, QM1, QM3, and QM4 that are directly needed for the derivations of Secs. 3–8. Nothing in this section is new; the section makes the logical dependencies of the paper explicit and fixes notation before the main derivations begin.

2.1 The Angular Momentum Operators

The angular momentum operators were introduced in QM4 as the infinitesimal generators of spatial rotations in the scalar–conformal transport closure system. Their definition and basic symmetry properties are recalled here in the explicit component form needed for the algebraic computations of Sec. 3.

The three angular momentum operators are defined by

$$\hat{L}_j \coloneqq \epsilon_{jkl}\,\hat{x}^k\hat{p}_l = \epsilon_{jkl}\,x^k(-\mathrm{i}\Phi_0\,\partial_l), \qquad j = 1, 2, 3, \tag{1}$$

where ϵ_{jkl} is the Levi-Civita symbol, repeated indices are summed over $\{1, 2, 3\}$, $\hat{x}^k$ denotes multiplication by x^k, and $\hat{p}_l = -\mathrm{i}\Phi_0\,\partial_l$ is the l-th momentum transport generator of QB2 and QM1. In explicit component form:

$$\hat{L}_1 = \hat{x}^2\hat{p}_3 - \hat{x}^3\hat{p}_2 = -\mathrm{i}\Phi_0\,(x^2\partial_3 - x^3\partial_2), \tag{2}$$

$$\hat{L}_2 = \hat{x}^3\hat{p}_1 - \hat{x}^1\hat{p}_3 = -\mathrm{i}\Phi_0\,(x^3\partial_1 - x^1\partial_3), \tag{3}$$

$$\hat{L}_3 = \hat{x}^1\hat{p}_2 - \hat{x}^2\hat{p}_1 = -\mathrm{i}\Phi_0\,(x^1\partial_2 - x^2\partial_1). \tag{4}$$

The total angular momentum squared operator is

$$\hat{L}^2 \coloneqq \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2. \tag{5}$$

Remark 2.1. *The definition Eq.* (1) *identifies the angular momentum operators as the generators of the action of the rotation group* SO(3) *on the state space* $\mathcal{H}$*. In the scalar–conformal NUVO framework, spatial rotations are geometric symmetries of the delivery substrate whenever the scalar capacity field* Λ *is rotationally symmetric:* $\Lambda(x) = \Lambda(|x|)$*. The operators* $\hat{L}_j$ *are the infinitesimal generators of this symmetry at the level of the transport closure state, and QM4 Proposition 7.3 established that they commute with the Hamiltonian* $\hat{H}$ *for rotationally symmetric potentials, generating the conservation law* $\mathrm{d}\langle\hat{L}_j\rangle/\mathrm{d}t = 0$ *under the Schrödinger evolution. The present paper derives the full algebraic and spectral structure of these generators; the conservation law of QM4 is a structural consequence that will be re-derived here as a corollary of* $[\hat{L}^2, \hat{L}_j] = 0$ *and* $[\hat{H}, \hat{L}_j] = 0$.

The properties established in QM4 Proposition 7.2 and recalled here without re-derivation are:

- Each $\hat{L}_j$ is symmetric on $\mathcal{S}(\mathbb{R}^3)$ and essentially self-adjoint on $\mathcal{H}$.
- $\hat{L}^2$ is a non-negative self-adjoint operator on $\mathcal{H}$, since $\hat{L}^2 = \sum_j \hat{L}_j^2$ and $\left\langle \Psi, \hat{L}_j^2\Psi \right\rangle_{\mathcal{H}} = \left\| \hat{L}_j\Psi \right\|_{\mathcal{H}}^2 \geq 0$.
- For a rotationally symmetric potential $V = V(|x|)$, $[\hat{H}, \hat{L}_j] = 0$ on $\mathcal{S}(\mathbb{R}^3)$.

2.2 The Canonical Commutation Relations

The derivation of the angular momentum commutation algebra in Sec. 3 uses only three commutation relations, all established in QM1 Proposition 5.4 and recalled here in the form in which they will be applied.

On the dense domain $\mathcal{S}(\mathbb{R}^3) \subset \mathcal{H}$:

$$[\hat{x}^j, \hat{p}_k] = \mathrm{i}\Phi_0\,\delta^j{}_k, \tag{6}$$

$$[\hat{x}^j, \hat{x}^k] = 0, \tag{7}$$

$$[\hat{p}_j, \hat{p}_k] = 0. \tag{8}$$

Relation Eq. (6) is the canonical commutation relation derived in QB2 and promoted to $\mathcal{H}$ in QM1. Relations Eqs. (7) and (8) express the commutativity of position operators among themselves (multiplication operators with different coordinate functions commute pointwise) and of momentum operators among themselves (partial derivatives in different directions commute on smooth functions). All three are exact identities on $\mathcal{S}(\mathbb{R}^3)$.

Remark 2.2. *The three commutation relations Eqs.* (6)–(8) *are the complete algebraic input to the derivation of* $[\hat{L}_j, \hat{L}_k] = \mathrm{i}\Phi_0\, \epsilon_{jkl} \hat{L}_l$ *in Sec. 3. No other property of the position or momentum operators is needed. This means the angular momentum commutation algebra is a universal consequence of any system of operators satisfying the canonical commutation relations Eqs.* (6)–(8), *not a special property of the scalar–conformal transport system. The scalar–conformal specificity enters through the identification of these operators with transport generators in QB2 and QM1, and through the holonomy quantization that selects integer quantum numbers in Sec. 5.*

2.3 The Holonomy Quantization Principle

The Q-series established the holonomy quantization principle as the foundational quantization mechanism of the scalar–conformal NUVO program: on a closed transport path, the accumulated transport phase must be an integer multiple of $2\pi\Phi_0$, so that the complex state encoding Ψ returns to its initial value after traversing the closed path. This is the condition that makes the transport closure system self-consistent under path composition and underlies the quantization of the hydrogenic energy spectrum in Q4.

In the present paper, the holonomy quantization principle is applied to rotationally closed transport paths: paths generated by a full 2π rotation of the azimuthal angle φ about a fixed axis. The accumulated azimuthal phase after such a rotation is $\Delta\phi_\varphi = 2\pi\mu/\Phi_0$, where μ is the $\hat{L}_3$ eigenvalue of the closure state. The holonomy condition requires this phase to be a multiple of 2π:

$$\frac{\Delta\phi_\varphi}{\Phi_0} = \frac{2\pi\mu}{\Phi_0^2} \in 2\pi\mathbb{Z}, \tag{9}$$

which is equivalent to $\mu/\Phi_0 \in \mathbb{Z}$, i.e., $\mu = m\Phi_0$ for $m \in \mathbb{Z}$. The precise statement and proof of this condition are given in Theorem 5.2 of Sec. 5; the present subsection records it as a recalled input.

Remark 2.3. *The holonomy quantization of the azimuthal angle in the present paper and the holonomy quantization of the radial action in the Q-series are two manifestations of the same principle. In the Q-series, the closed transport path is the radial oscillation of the exchange-sector closure, and the integer winding number selects the discrete hydrogenic energy levels* E_n. *In QM5, the closed transport path is the azimuthal rotation of the closure state, and the integer winding number selects the discrete magnetic quantum numbers* m. *QM8 will show that if the transport structure is carried on the double cover of the rotation group (the group* SU(2) *rather than* SO(3)*), the holonomy condition selects half-integer values, yielding the spin quantum numbers. The holonomy quantization principle is thus the single geometric mechanism underlying all quantization in the NUVO framework.*

2.4 Angular Momentum Conservation and Uncertainty from QM4 and QM3

Two results from QM4 and QM3 provide the starting context for the present development and are recalled here to close the forward references opened in those papers.

Angular momentum conservation (QM4 Proposition 7.3). For a rotationally symmetric Hamiltonian $\hat{H} = \hat{T} + \hat{V}$ with $V = V(|x|)$, the commutation $[\hat{H}, \hat{L}_j] = 0$ holds on $\mathcal{S}(\mathbb{R}^3)$, and the expectation value $\langle \hat{L}_j \rangle(t)$ is conserved under Schrödinger evolution. The proof in QM4 Proposition 7.3 established this result using the commutators $[\hat{T}, \hat{L}_j] = 0$ and $[V(|x|), \hat{L}_j] = 0$. The first commutator $[\hat{T}, \hat{L}_j] = 0$ was stated there as a consequence of $\hat{T} = \hat{p}^2/(2m)$ being rotationally symmetric; the present paper derives it as a corollary of $[\hat{L}^2, \hat{L}_j] = 0$ in Theorem 3.3.

Angular momentum uncertainty relations (QM3 Proposition 7.1). For any normalized $\Psi \in \mathcal{S}(\mathbb{R}^3)$:

$$\Delta L_j \cdot \Delta L_k \geq \frac{\Phi_0}{2} |\langle \hat{L}_l \rangle| \tag{10}$$

for each cyclic triple (j, k, l). This inequality was derived in QM3 from the Robertson uncertainty theorem applied to the commutation algebra $[\hat{L}_j, \hat{L}_k] = \mathrm{i}\Phi_0\, \epsilon_{jkl} \hat{L}_l$, which was cited there from QM4 as a forward reference. The present paper derives that commutation algebra in Theorem 3.1, retroactively completing the derivation chain for QM3 Proposition 7.1. Additionally, QM3 Remark 7.3 stated the sum rule $\sum_j [(\Delta L_j)^2 + \langle \hat{L}_j \rangle^2] = \ell(\ell+1)\Phi_0^2$ as a QM5 result; this sum rule is established in Sec. 8 of the present paper.

Remark 2.4. *The use of the angular momentum commutation algebra in QM3 Proposition 7.1 before its derivation in the present paper created a forward-reference dependency that is now resolved. QM3 cited the commutation algebra $[\hat{L}_j, \hat{L}_k] = \mathrm{i}\Phi_0\, \epsilon_{jkl} \hat{L}_l$ from QM4 Eq. (7.6), where it appeared as a forward reference to QM5. The derivation of this algebra in Theorem 3.1 of the present paper closes the logical chain: the CCR of QM1 $\rightarrow$ the angular momentum commutation algebra of QM5 $\rightarrow$ the angular momentum uncertainty relations of QM3. The result is now fully grounded without circularity.*

3 The Angular Momentum Commutation Algebra

The present section derives the two foundational algebraic results of the angular momentum theory: the commutation relations $[\hat{L}_j, \hat{L}_k] = \mathrm{i}\Phi_0\, \epsilon_{jkl} \hat{L}_l$ among the three components, and the commutativity $[\hat{L}^2, \hat{L}_j] = 0$ of the total angular momentum squared with each component. Both derivations proceed by explicit computation from the canonical commutation relations Eqs. (6)–(8) and the definition Eq. (1). No new algebraic postulate is introduced; every step follows from the CCR of QM1 and the Leibniz rule for commutators.

The key algebraic tool used throughout is the *commutator Leibniz rule*:

$$[A, BC] = [A, B]\, C + B\, [A, C], \tag{11}$$

which holds for any operators A, B, C on a common domain.

3.1 The Fundamental Commutation Relations

The derivation of the full algebra proceeds by computing one representative commutator $[\hat{L}_1, \hat{L}_2]$ explicitly and then invoking cyclic symmetry to obtain the remaining two.

Theorem 3.1 (Angular momentum commutation algebra)**.** *On the dense domain $\mathcal{S}(\mathbb{R}^3) \subset \mathcal{H}$, the angular momentum operators satisfy*

$$[\hat{L}_j, \hat{L}_k] = \mathrm{i}\Phi_0\, \epsilon_{jkl}\, \hat{L}_l \tag{12}$$

for all $j, k \in \{1, 2, 3\}$, where the repeated index l is summed. Equivalently, the three cyclic relations are:

$$[\hat{L}_1, \hat{L}_2] = \mathrm{i}\Phi_0\, \hat{L}_3, \tag{13}$$
$$[\hat{L}_2, \hat{L}_3] = \mathrm{i}\Phi_0\, \hat{L}_1, \tag{14}$$
$$[\hat{L}_3, \hat{L}_1] = \mathrm{i}\Phi_0\, \hat{L}_2. \tag{15}$$

Proof. It suffices to derive Eq. (13); the remaining two follow by cyclic permutation of the index labels $\{1, 2, 3\}$.

Substituting Eqs. (2) and (3):

$$\begin{aligned}[\hat{L}_1, \hat{L}_2] &= \left[\hat{x}^2\hat{p}_3 - \hat{x}^3\hat{p}_2,\ \hat{x}^3\hat{p}_1 - \hat{x}^1\hat{p}_3\right] \\ &= [\hat{x}^2\hat{p}_3,\ \hat{x}^3\hat{p}_1] - [\hat{x}^2\hat{p}_3,\ \hat{x}^1\hat{p}_3] - [\hat{x}^3\hat{p}_2,\ \hat{x}^3\hat{p}_1] + [\hat{x}^3\hat{p}_2,\ \hat{x}^1\hat{p}_3].\end{aligned}$$

Each of the four brackets is evaluated using the Leibniz rule Eq. (11) and the CCR Eqs. (6)–(8).

First bracket: $[\hat{x}^2\hat{p}_3, \hat{x}^3\hat{p}_1]$.

$$\begin{aligned}[\hat{x}^2\hat{p}_3, \hat{x}^3\hat{p}_1] &= \hat{x}^2[\hat{p}_3, \hat{x}^3]\hat{p}_1 + \hat{x}^2\hat{x}^3[\hat{p}_3, \hat{p}_1] + [\hat{x}^2, \hat{x}^3]\hat{p}_3\hat{p}_1 + \hat{x}^3[\hat{x}^2, \hat{p}_1]\hat{p}_3 \\ &= \hat{x}^2(-\mathrm{i}\Phi_0)\hat{p}_1 + 0 + 0 + 0 = -\mathrm{i}\Phi_0\, \hat{x}^2\hat{p}_1,\end{aligned}$$

where we used $[\hat{p}_3, \hat{x}^3] = -[\hat{x}^3, \hat{p}_3] = -\mathrm{i}\Phi_0$, $[\hat{p}_3, \hat{p}_1] = 0$, $[\hat{x}^2, \hat{x}^3] = 0$, and $[\hat{x}^2, \hat{p}_1] = 0$ (since $j \neq k$ in Eq. (6) gives zero).

Second bracket: $[\hat{x}^2\hat{p}_3, \hat{x}^1\hat{p}_3]$.

$$\begin{aligned}[\hat{x}^2\hat{p}_3, \hat{x}^1\hat{p}_3] &= \hat{x}^2[\hat{p}_3, \hat{x}^1]\hat{p}_3 + \hat{x}^2\hat{x}^1[\hat{p}_3, \hat{p}_3] + [\hat{x}^2, \hat{x}^1]\hat{p}_3\hat{p}_3 + \hat{x}^1[\hat{x}^2, \hat{p}_3]\hat{p}_3 \\ &= 0 + 0 + 0 + 0 = 0,\end{aligned}$$

since $[\hat{p}_3, \hat{x}^1] = 0$ (different indices), $[\hat{p}_3, \hat{p}_3] = 0$, $[\hat{x}^2, \hat{x}^1] = 0$, and $[\hat{x}^2, \hat{p}_3] = 0$ (different indices).

Third bracket: $[\hat{x}^3\hat{p}_2, \hat{x}^3\hat{p}_1]$.

$$\begin{aligned}[\hat{x}^3\hat{p}_2, \hat{x}^3\hat{p}_1] &= \hat{x}^3[\hat{p}_2, \hat{x}^3]\hat{p}_1 + \hat{x}^3\hat{x}^3[\hat{p}_2, \hat{p}_1] + [\hat{x}^3, \hat{x}^3]\hat{p}_2\hat{p}_1 + \hat{x}^3[\hat{x}^3, \hat{p}_1]\hat{p}_2 \\ &= 0 + 0 + 0 + 0 = 0,\end{aligned}$$

since $[\hat{p}_2, \hat{x}^3] = 0$, $[\hat{p}_2, \hat{p}_1] = 0$, $[\hat{x}^3, \hat{x}^3] = 0$, and $[\hat{x}^3, \hat{p}_1] = 0$.

Fourth bracket: $[\hat{x}^3\hat{p}_2, \hat{x}^1\hat{p}_3]$.

$$\begin{aligned}[\hat{x}^3\hat{p}_2, \hat{x}^1\hat{p}_3] &= \hat{x}^3[\hat{p}_2, \hat{x}^1]\hat{p}_3 + \hat{x}^3\hat{x}^1[\hat{p}_2, \hat{p}_3] + [\hat{x}^3, \hat{x}^1]\hat{p}_2\hat{p}_3 + \hat{x}^1[\hat{x}^3, \hat{p}_3]\hat{p}_2 \\ &= 0 + 0 + 0 + \hat{x}^1(\mathrm{i}\Phi_0)\hat{p}_2 = \mathrm{i}\Phi_0\, \hat{x}^1\hat{p}_2,\end{aligned}$$

where we used $[\hat{p}_2, \hat{x}^1] = 0$, $[\hat{p}_2, \hat{p}_3] = 0$, $[\hat{x}^3, \hat{x}^1] = 0$, and $[\hat{x}^3, \hat{p}_3] = \mathrm{i}\Phi_0$.

Combining the four brackets:

$$[\hat{L}_1, \hat{L}_2] = (-\mathrm{i}\Phi_0\, \hat{x}^2\hat{p}_1) - 0 - 0 + (\mathrm{i}\Phi_0\, \hat{x}^1\hat{p}_2) = \mathrm{i}\Phi_0\, (\hat{x}^1\hat{p}_2 - \hat{x}^2\hat{p}_1) = \mathrm{i}\Phi_0\, \hat{L}_3,$$

which is Eq. (13). The relations Eqs. (14) and (15) follow by cyclic permutation: relabelling $(1, 2, 3) \to (2, 3, 1)$ in the above computation yields $[\hat{L}_2, \hat{L}_3] = \mathrm{i}\Phi_0\, \hat{L}_1$, and relabelling $(1, 2, 3) \to (3, 1, 2)$ yields $[\hat{L}_3, \hat{L}_1] = \mathrm{i}\Phi_0\, \hat{L}_2$. The compact form Eq. (12) encodes all three relations using the Levi-Civita symbol. □

Remark 3.2. *The computation in the proof of Theorem 3.1 uses only the three CCR identities Eqs.* (6)–(8) *and the Leibniz rule Eq.* (11)*. Of the sixteen terms generated by expanding the four brackets (four terms per bracket via the Leibniz rule), twelve vanish by the commutativity of position operators, of momentum operators, and of position and momentum operators in different spatial directions. The two surviving terms come from the non-trivial commutators* $[\hat{p}_3, \hat{x}^3] = -\mathrm{i}\Phi_0$ *in the first bracket and* $[\hat{x}^3, \hat{p}_3] = \mathrm{i}\Phi_0$ *in the fourth. This pattern—two non-trivial contributions from the CCR, twelve trivial contributions from commutativity—is a consequence of the antisymmetric structure of the angular momentum definition* $\hat{L}_j = \epsilon_{jkl}\hat{x}^k\hat{p}_l$ *and will recur in all subsequent commutator computations in the present paper.*

3.2 Commutativity of $\hat{L}^2$ with Each Component

The commutativity of $\hat{L}^2$ with each $\hat{L}_j$ is the key structural result that makes $\hat{L}^2$ and $\hat{L}_3$ jointly diagonalizable. The proof uses Theorem 3.1 rather than the CCR directly.

Theorem 3.3 ($\hat{L}^2$ commutes with each angular momentum component)**.** *On* $\mathcal{S}(\mathbb{R}^3)$*:*

$$[\hat{L}^2, \hat{L}_j] = 0 \tag{16}$$

for all $j \in \{1, 2, 3\}$*.*

Proof. It suffices to prove $[\hat{L}^2, \hat{L}_3] = 0$; the other two cases follow by cyclic symmetry. Expand using $\hat{L}^2 = \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2$:

$$[\hat{L}^2, \hat{L}_3] = [\hat{L}_1^2, \hat{L}_3] + [\hat{L}_2^2, \hat{L}_3] + [\hat{L}_3^2, \hat{L}_3]. \tag{17}$$

The third term vanishes trivially: $[\hat{L}_3^2, \hat{L}_3] = \hat{L}_3[\hat{L}_3, \hat{L}_3] + [\hat{L}_3, \hat{L}_3]\hat{L}_3 = 0$.

For the first term, apply the Leibniz rule $[\hat{L}_1^2, \hat{L}_3] = \hat{L}_1[\hat{L}_1, \hat{L}_3] + [\hat{L}_1, \hat{L}_3]\hat{L}_1$. From Eq. (15): $[\hat{L}_3, \hat{L}_1] = \mathrm{i}\Phi_0\,\hat{L}_2$, so $[\hat{L}_1, \hat{L}_3] = -\mathrm{i}\Phi_0\,\hat{L}_2$. Therefore:

$$[\hat{L}_1^2, \hat{L}_3] = \hat{L}_1(-\mathrm{i}\Phi_0\,\hat{L}_2) + (-\mathrm{i}\Phi_0\,\hat{L}_2)\hat{L}_1 = -\mathrm{i}\Phi_0\,(\hat{L}_1\hat{L}_2 + \hat{L}_2\hat{L}_1). \tag{18}$$

For the second term, apply the Leibniz rule $[\hat{L}_2^2, \hat{L}_3] = \hat{L}_2[\hat{L}_2, \hat{L}_3] + [\hat{L}_2, \hat{L}_3]\hat{L}_2$. From Eq. (14): $[\hat{L}_2, \hat{L}_3] = \mathrm{i}\Phi_0\,\hat{L}_1$. Therefore:

$$[\hat{L}_2^2, \hat{L}_3] = \hat{L}_2(\mathrm{i}\Phi_0\,\hat{L}_1) + (\mathrm{i}\Phi_0\,\hat{L}_1)\hat{L}_2 = \mathrm{i}\Phi_0\,(\hat{L}_2\hat{L}_1 + \hat{L}_1\hat{L}_2). \tag{19}$$

Adding Eqs. (18) and (19):

$$[\hat{L}_1^2, \hat{L}_3] + [\hat{L}_2^2, \hat{L}_3] = -\mathrm{i}\Phi_0\,(\hat{L}_1\hat{L}_2 + \hat{L}_2\hat{L}_1) + \mathrm{i}\Phi_0\,(\hat{L}_2\hat{L}_1 + \hat{L}_1\hat{L}_2) = 0.$$

Substituting into Eq. (17): $[\hat{L}^2, \hat{L}_3] = 0 + 0 + 0 = 0$. By cyclic symmetry, $[\hat{L}^2, \hat{L}_1] = 0$ and $[\hat{L}^2, \hat{L}_2] = 0$ follow by relabelling indices. □

Remark 3.4. *The cancellation in the proof of Theorem 3.3 has a transparent algebraic structure. The terms* $[\hat{L}_1^2, \hat{L}_3]$ *and* $[\hat{L}_2^2, \hat{L}_3]$ *each produce an anti-commutator* $\{L_j, L_k\} = L_jL_k + L_kL_j$ *with the opposite sign from the commutation algebra of Theorem 3.1. Specifically,* $[\hat{L}_1^2, \hat{L}_3] = -\mathrm{i}\Phi_0\,\{\hat{L}_1, \hat{L}_2\}/2 \cdot 2$ *and* $[\hat{L}_2^2, \hat{L}_3] = +\mathrm{i}\Phi_0\,\{\hat{L}_2, \hat{L}_1\}/2 \cdot 2$*, and since* $\{\hat{L}_1, \hat{L}_2\} = \{\hat{L}_2, \hat{L}_1\}$ *(the anti-commutator is symmetric), the two terms cancel exactly. This cancellation is the algebraic expression of the rotational isotropy of* $\hat{L}^2$*: the squared length of a rotation generator is invariant under the rotation it generates.*

Remark 3.5. *Theorems 3.1 and 3.3 together establish the complete commutation structure of the angular momentum operators on* $\mathcal{S}(\mathbb{R}^3)$*:* $[\hat{L}^2, \hat{L}_3] = 0$ *(simultaneous diagonalizability) while* $[\hat{L}_1, \hat{L}_2] \neq 0$ *(no third simultaneously diagonalizable component). The pair* $(\hat{L}^2, \hat{L}_3)$ *therefore constitutes a complete set of commuting observables (CSCO) for the angular sector: the joint spectrum of* $\hat{L}^2$ *and* $\hat{L}_3$ *labels the angular eigenstates without redundancy. The choice of* $\hat{L}_3$ *as the distinguished component is conventional;* $\hat{L}_1$ *or* $\hat{L}_2$ *could equally serve, and the spectral results of Secs. 4–5 are independent of this choice. In the NUVO framework, the natural choice of* $\hat{L}_3$ *is reinforced by the azimuthal holonomy condition of Sec. 5: the holonomy condition is formulated in terms of the azimuthal angle* φ*, and the generator of* φ*-rotations is* $\hat{L}_3 = -\mathrm{i}\Phi_0\,\partial_\varphi$ *in spherical coordinates, making* $\hat{L}_3$ *the natural partner of the azimuthal holonomy.*

3.3 Self-Adjointness and Non-Negativity

The self-adjointness of $\hat{L}_j$ on $\mathcal{H}$ was established in QM4 Proposition 7.2. The present subsection records the consequence for $\hat{L}^2$ and derives the non-negativity that will be used in the ladder operator analysis of Sec. 4.

Proposition 3.6 (Self-adjointness and non-negativity of $\hat{L}^2$)**.** *The operators* $\hat{L}_j$ *are essentially self-adjoint on* $\mathcal{S}(\mathbb{R}^3)$ *with self-adjoint closures on* $\mathcal{H}$ *(QM4 Proposition 7.2). The operator* $\hat{L}^2 = \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2$ *is self-adjoint and non-negative on* $\mathcal{H}$*:*

$$\left\langle \Psi, \hat{L}^2\,\Psi \right\rangle_{\mathcal{H}} = \left\| \hat{L}_1\Psi \right\|_{\mathcal{H}}^2 + \left\| \hat{L}_2\Psi \right\|_{\mathcal{H}}^2 + \left\| \hat{L}_3\Psi \right\|_{\mathcal{H}}^2 \geq 0 \tag{20}$$

for all $\Psi \in \mathcal{D}(\hat{L}^2)$*.*

Proof. Self-adjointness of each $\hat{L}_j$ is QM4 Proposition 7.2. Self-adjointness of $\hat{L}^2$: a finite sum of self-adjoint operators is self-adjoint on the intersection of their domains, which is dense in $\mathcal{H}$. Non-negativity: for each j, since $\hat{L}_j$ is self-adjoint, $\left\langle \Psi, \hat{L}_j^2\Psi \right\rangle_{\mathcal{H}} = \left\langle \hat{L}_j\Psi, \hat{L}_j\Psi \right\rangle_{\mathcal{H}} = \left\| \hat{L}_j\Psi \right\|_{\mathcal{H}}^2 \geq 0$. Summing over j gives Eq. (20). □

Remark 3.7. *The non-negativity Eq.* (20) *has two immediate consequences used in Sec. 4. First, the* $\hat{L}^2$ *eigenvalue* $\lambda = \ell(\ell+1)\Phi_0^2$ *is non-negative:* $\lambda \geq 0$*. Second, for a joint eigenstate with* $\hat{L}^2$ *eigenvalue* λ *and* $\hat{L}_3$ *eigenvalue* μ*:*

$$0 \leq \left\| \hat{L}_1\Psi \right\|_{\mathcal{H}}^2 + \left\| \hat{L}_2\Psi \right\|_{\mathcal{H}}^2 = \left\langle \Psi, (\hat{L}^2 - \hat{L}_3^2)\Psi \right\rangle_{\mathcal{H}} = (\lambda - \mu^2)\,\|\Psi\|_{\mathcal{H}}^2\,,$$

giving the bound $\mu^2 \leq \lambda$*, i.e.,* $|\mu| \leq \sqrt{\lambda}$*. This is the key inequality establishing the boundedness of the* $\hat{L}_3$ *spectrum in each* $\hat{L}^2$ *eigenspace, which forces the ladder sequence to terminate.*

4 Ladder Operators and the Spectrum

With the angular momentum commutation algebra established as a theorem in Sec. 3, the present section deploys the standard algebraic technique of ladder operators to extract the spectrum of $\hat{L}^2$ and $\hat{L}_3$ from the algebra alone, prior to the holonomy quantization input of Sec. 5. The ladder operators raise and lower the $\hat{L}_3$ eigenvalue by Φ_0 while preserving the $\hat{L}^2$ eigenvalue; since the $\hat{L}_3$ eigenvalue is bounded (by the non-negativity of $\hat{L}^2$, Proposition 3.6), the ladder sequence must terminate at a maximum and minimum value, and these terminal conditions impose algebraic constraints on the spectrum. The output of the present section is the constraint that the $\hat{L}^2$ eigenvalue takes the form $m_{\max}(m_{\max}+1)\Phi_0^2$ for some non-negative $m_{\max}$; the holonomy quantization in Sec. 5 then restricts $m_{\max}$ to the non-negative integers, completing the spectrum.

4.1 Definition and Commutation Relations of the Ladder Operators

Definition 4.1 (Raising and lowering operators)**.** *The* raising operator *and* lowering operator *are defined by*

$$\hat{L}_+ := \hat{L}_1 + \mathrm{i}\hat{L}_2, \qquad \hat{L}_- := \hat{L}_1 - \mathrm{i}\hat{L}_2, \tag{21}$$

on the domain $\mathcal{S}(\mathbb{R}^3) \subset \mathcal{H}$*.*

Remark 4.2. *The ladder operators are adjoint to each other:* $(\hat{L}_+)^\dagger = (\hat{L}_1 + \mathrm{i}\hat{L}_2)^\dagger = \hat{L}_1^\dagger - \mathrm{i}\hat{L}_2^\dagger = \hat{L}_1 - \mathrm{i}\hat{L}_2 = \hat{L}_-$*, using the self-adjointness* $\hat{L}_j^\dagger = \hat{L}_j$ *established in Proposition 3.6. Neither* $\hat{L}_+$ *nor* $\hat{L}_-$ *is self-adjoint. The non-self-adjoint character of the ladder operators is essential to their raising and lowering action: a self-adjoint operator preserves the eigenspace of any commuting self-adjoint operator, whereas the non-self-adjoint* $\hat{L}_+$ *moves between eigenspaces.*

The commutation relations of the ladder operators with $\hat{L}_3$ and $\hat{L}^2$, and with each other, follow from Theorem 3.1 by direct substitution.

Lemma 4.3 (Ladder operator commutation relations)**.** *On* $\mathcal{S}(\mathbb{R}^3)$*:*

$$[\hat{L}_3, \hat{L}_+] = \Phi_0\, \hat{L}_+, \tag{22}$$

$$[\hat{L}_3, \hat{L}_-] = -\Phi_0\, \hat{L}_-, \tag{23}$$

$$[\hat{L}_+, \hat{L}_-] = 2\Phi_0\, \hat{L}_3, \tag{24}$$

$$[\hat{L}^2, \hat{L}_+] = 0, \tag{25}$$

$$[\hat{L}^2, \hat{L}_-] = 0. \tag{26}$$

Proof. Equation (22)*:* Using Theorem 3.1:

$$[\hat{L}_3, \hat{L}_+] = [\hat{L}_3, \hat{L}_1 + \mathrm{i}\hat{L}_2] = [\hat{L}_3, \hat{L}_1] + \mathrm{i}[\hat{L}_3, \hat{L}_2].$$

From Eq. (15): $[\hat{L}_3, \hat{L}_1] = \mathrm{i}\Phi_0\, \hat{L}_2$. From Eq. (14): $[\hat{L}_2, \hat{L}_3] = \mathrm{i}\Phi_0\, \hat{L}_1$, so $[\hat{L}_3, \hat{L}_2] = -\mathrm{i}\Phi_0\, \hat{L}_1$. Therefore:

$$[\hat{L}_3, \hat{L}_+] = \mathrm{i}\Phi_0\, \hat{L}_2 + \mathrm{i}(-\mathrm{i}\Phi_0\, \hat{L}_1) = \mathrm{i}\Phi_0\, \hat{L}_2 + \Phi_0\, \hat{L}_1 = \Phi_0(\hat{L}_1 + \mathrm{i}\hat{L}_2) = \Phi_0\, \hat{L}_+.$$

Equation (23)*:* Take the adjoint of Eq. (22) using $[\hat{L}_3, \hat{L}_+]^\dagger = (\hat{L}_+)^\dagger \hat{L}_3^\dagger - \hat{L}_3^\dagger (\hat{L}_+)^\dagger = \hat{L}_- \hat{L}_3 - \hat{L}_3 \hat{L}_- = -[\hat{L}_3, \hat{L}_-]$, which must equal $(\Phi_0\, \hat{L}_+)^\dagger = \Phi_0\, \hat{L}_-$. Therefore $-[\hat{L}_3, \hat{L}_-] = \Phi_0\, \hat{L}_-$, giving $[\hat{L}_3, \hat{L}_-] = -\Phi_0\, \hat{L}_-$.

Equation (24)*:*

$$\begin{aligned}
[\hat{L}_+, \hat{L}_-] &= [\hat{L}_1 + \mathrm{i}\hat{L}_2,\ \hat{L}_1 - \mathrm{i}\hat{L}_2] \\
&= [\hat{L}_1, \hat{L}_1] - \mathrm{i}[\hat{L}_1, \hat{L}_2] + \mathrm{i}[\hat{L}_2, \hat{L}_1] - \mathrm{i}^2[\hat{L}_2, \hat{L}_2] \\
&= 0 - \mathrm{i}(\mathrm{i}\Phi_0\, \hat{L}_3) + \mathrm{i}(-\mathrm{i}\Phi_0\, \hat{L}_3) + 0 \\
&= \Phi_0\, \hat{L}_3 + \Phi_0\, \hat{L}_3 = 2\Phi_0\, \hat{L}_3,
\end{aligned}$$

using $[\hat{L}_1, \hat{L}_2] = \mathrm{i}\Phi_0\, \hat{L}_3$ from Eq. (13) and $[\hat{L}_2, \hat{L}_1] = -\mathrm{i}\Phi_0\, \hat{L}_3$.

Equations (25) *and* (26)*:* From Theorem 3.3, $[\hat{L}^2, \hat{L}_1] = [\hat{L}^2, \hat{L}_2] = 0$. Therefore $[\hat{L}^2, \hat{L}_+] = [\hat{L}^2, \hat{L}_1] + \mathrm{i}[\hat{L}^2, \hat{L}_2] = 0$, and similarly for $\hat{L}_-$. □

4.2 The Raising and Lowering Action on Eigenstates

The commutation relations of Lemma 4.3 imply that $\hat{L}_+$ and $\hat{L}_-$ act on joint eigenstates of $\hat{L}^2$ and $\hat{L}_3$ by shifting the $\hat{L}_3$ eigenvalue while preserving the $\hat{L}^2$ eigenvalue.

Proposition 4.4 (Raising and lowering action on eigenstates)**.** *Let $\Psi \in \mathcal{S}(\mathbb{R}^3)$ satisfy $\hat{L}^2\Psi = \lambda\,\Psi$ and $\hat{L}_3\Psi = \mu\,\Psi$ for $\lambda, \mu \in \mathbb{R}$. Then, if $\hat{L}_+\Psi \neq 0$, the state $\hat{L}_+\Psi$ satisfies*

$$\hat{L}^2(\hat{L}_+\Psi) = \lambda\,(\hat{L}_+\Psi), \qquad \hat{L}_3(\hat{L}_+\Psi) = (\mu + \Phi_0)(\hat{L}_+\Psi); \tag{27}$$

and if $\hat{L}_-\Psi \neq 0$, the state $\hat{L}_-\Psi$ satisfies

$$\hat{L}^2(\hat{L}_-\Psi) = \lambda\,(\hat{L}_-\Psi), \qquad \hat{L}_3(\hat{L}_-\Psi) = (\mu - \Phi_0)(\hat{L}_-\Psi). \tag{28}$$

Proof. For the $\hat{L}^2$ action on $\hat{L}_+\Psi$: using Eq. (25) and the $\hat{L}^2$ eigenvalue equation,

$$\hat{L}^2(\hat{L}_+\Psi) = (\hat{L}^2\hat{L}_+)\Psi = (\hat{L}_+\hat{L}^2 + [\hat{L}^2, \hat{L}_+])\Psi = \hat{L}_+(\lambda\Psi) + 0 = \lambda(\hat{L}_+\Psi).$$

For the $\hat{L}_3$ action on $\hat{L}_+\Psi$: using Eq. (22) and the $\hat{L}_3$ eigenvalue equation,

$$\hat{L}_3(\hat{L}_+\Psi) = (\hat{L}_3\hat{L}_+)\Psi = (\hat{L}_+\hat{L}_3 + [\hat{L}_3, \hat{L}_+])\Psi = \hat{L}_+(\mu\Psi) + \Phi_0\hat{L}_+\Psi = (\mu + \Phi_0)(\hat{L}_+\Psi).$$

The argument for $\hat{L}_-\Psi$ is identical using Eqs. (26) and (23), with the sign of the Φ_0 contribution reversed. □

Remark 4.5. *Proposition 4.4 gives the raising and lowering operators their names. Starting from a joint eigenstate with $\hat{L}_3$ eigenvalue μ and $\hat{L}^2$ eigenvalue λ, repeated application of $\hat{L}_+$ generates a sequence of eigenstates $\Psi, \hat{L}_+\Psi, \hat{L}_+^2\Psi, \ldots$ with $\hat{L}_3$ eigenvalues $\mu, \mu + \Phi_0, \mu + 2\Phi_0, \ldots$ and the same $\hat{L}^2$ eigenvalue λ throughout. Similarly, repeated application of $\hat{L}_-$ generates $\Psi, \hat{L}_-\Psi, \hat{L}_-^2\Psi, \ldots$ with eigenvalues $\mu, \mu - \Phi_0, \mu - 2\Phi_0, \ldots$. In the NUVO transport closure framework, this raising and lowering of the $\hat{L}_3$ eigenvalue corresponds to changing the azimuthal winding number of the transport closure configuration by one unit at a time. The ladder operators are not physical operations on the transport system but algebraic tools that generate the complete family of angular closure eigenstates from any single member.*

4.3 Termination of the Ladder and the Spectral Constraint

Since the $\hat{L}_3$ eigenvalue in the λ-eigenspace is bounded by $|\mu| \leq \sqrt{\lambda}$ (Remark 3.7), the ladder sequences generated by $\hat{L}_+$ and $\hat{L}_-$ cannot continue indefinitely. Each must terminate: there exist states $\Psi_{\max}$ and $\Psi_{\min}$ in the λ-eigenspace with $\hat{L}_3$ eigenvalues $\mu_{\max}$ and $\mu_{\min}$ respectively such that $\hat{L}_+\Psi_{\max} = 0$ and $\hat{L}_-\Psi_{\min} = 0$. The termination conditions yield the algebraic constraints on λ.

Theorem 4.6 (Algebraic constraints on the spectrum from ladder termination)**.** *Let $\lambda > 0$ be an eigenvalue of $\hat{L}^2$ with at least one joint eigenstate of $\hat{L}^2$ and $\hat{L}_3$. Let $\mu_{\max}$ and $\mu_{\min}$ denote the maximum and minimum $\hat{L}_3$ eigenvalues in the λ-eigenspace. Then:*

(i) $\mu_{\min} = -\mu_{\max}$.

(ii) $\lambda = \mu_{\max}(\mu_{\max} + \Phi_0)$.

(iii) *The $\hat{L}_3$ eigenvalues in the λ-eigenspace form the arithmetic progression $\{\mu_{\min}, \mu_{\min}+\Phi_0, \ldots, \mu_{\max}-\Phi_0, \mu_{\max}\}$, a sequence of $N + 1$ values where $N = (\mu_{\max} - \mu_{\min})/\Phi_0 \in \mathbb{Z}_{\geq 0}$.*

Proof. Termination at $\mu_{\max}$: Since $\hat{L}_+\Psi_{\max} = 0$, compute $\hat{L}_-\hat{L}_+\Psi_{\max} = 0$. The key identity is

$$\hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_3^2 - \Phi_0\,\hat{L}_3, \tag{29}$$

derived as follows: from Eq. (24), $[\hat{L}_+, \hat{L}_-] = 2\Phi_0\,\hat{L}_3$, so $\hat{L}_-\hat{L}_+ = \hat{L}_+\hat{L}_- - 2\Phi_0\,\hat{L}_3$. Also, $\hat{L}_+\hat{L}_- = (\hat{L}_1 + \mathrm{i}\hat{L}_2)(\hat{L}_1 - \mathrm{i}\hat{L}_2) = \hat{L}_1^2 + \hat{L}_2^2 - \mathrm{i}[\hat{L}_1, \hat{L}_2] + \mathrm{i}[\hat{L}_2, \hat{L}_1] + \hat{L}_2\hat{L}_1 - \hat{L}_1\hat{L}_2$. More directly: $\hat{L}_+\hat{L}_- = \hat{L}^2 - \hat{L}_3^2 + \Phi_0\,\hat{L}_3$ (a standard identity derived by expanding $(\hat{L}_1 \pm \mathrm{i}\hat{L}_2)^2$ and using the commutation algebra). Therefore $\hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_3^2 + \Phi_0\,\hat{L}_3 - 2\Phi_0\,\hat{L}_3 = \hat{L}^2 - \hat{L}_3^2 - \Phi_0\,\hat{L}_3$, which is Eq. (29).

Applying Eq. (29) to $\Psi_{\max}$:

$$0 = \hat{L}_-\hat{L}_+\Psi_{\max} = (\hat{L}^2 - \hat{L}_3^2 - \Phi_0\,\hat{L}_3)\Psi_{\max} = (\lambda - \mu_{\max}^2 - \Phi_0\,\mu_{\max})\Psi_{\max}.$$

Since $\Psi_{\max} \neq 0$:

$$\lambda = \mu_{\max}(\mu_{\max} + \Phi_0), \tag{30}$$

which is part (ii).

Termination at $\mu_{\min}$: By the analogous argument with $\hat{L}_+\hat{L}_- = \hat{L}^2 - \hat{L}_3^2 + \Phi_0\,\hat{L}_3$:

$$0 = \hat{L}_+\hat{L}_-\Psi_{\min} = (\lambda - \mu_{\min}^2 + \Phi_0\,\mu_{\min})\Psi_{\min},$$

giving $\lambda = \mu_{\min}(\mu_{\min} - \Phi_0)$. Comparing with Eq. (30):

$$\mu_{\max}^2 + \Phi_0\,\mu_{\max} = \mu_{\min}^2 - \Phi_0\,\mu_{\min},$$

i.e., $(\mu_{\max}^2 - \mu_{\min}^2) + \Phi_0(\mu_{\max} + \mu_{\min}) = 0$, i.e., $(\mu_{\max} + \mu_{\min})(\mu_{\max} - \mu_{\min} + \Phi_0) = 0$. Since the ladder from $\mu_{\min}$ to $\mu_{\max}$ takes at least one step (if $\mu_{\min} \neq \mu_{\max}$) and the step size is $\Phi_0 > 0$, we have $\mu_{\max} - \mu_{\min} \geq 0$, so $\mu_{\max} - \mu_{\min} + \Phi_0 > 0$. Therefore the first factor must vanish: $\mu_{\max} + \mu_{\min} = 0$, giving part (i).

The arithmetic progression, part (iii): Starting from $\Psi_{\max}$ and applying $\hat{L}_-$ repeatedly, Proposition 4.4 gives a sequence of eigenstates with $\hat{L}_3$ eigenvalues $\mu_{\max}, \mu_{\max} - \Phi_0, \mu_{\max} - 2\Phi_0, \ldots$. This sequence terminates at $\mu_{\min} = -\mu_{\max}$, having taken $N = (\mu_{\max} - \mu_{\min})/\Phi_0 = 2\mu_{\max}/\Phi_0$ steps, so N is a positive rational. Since the sequence takes integer steps and terminates, N must be a non-negative integer: $\mu_{\max} = N\Phi_0/2$ for $N \in \mathbb{Z}_{\geq 0}$. The $\hat{L}_3$ eigenvalues are therefore $\{-N\Phi_0/2, (-N/2+1)\Phi_0, \ldots, N\Phi_0/2\}$, an arithmetic progression of $N+1$ values with spacing Φ_0. □

Remark 4.7. *Theorem 4.6 establishes that $\mu_{\max} = N\Phi_0/2$ for some $N \in \mathbb{Z}_{\geq 0}$, which allows both integer and half-integer values: if N is even, $\mu_{\max} = \ell\Phi_0$ for integer $\ell = N/2$; if N is odd, $\mu_{\max} = j\Phi_0$ for half-integer $j = N/2$. The algebraic analysis alone cannot distinguish these two cases. The holonomy quantization of Sec. 5 selects $m \in \mathbb{Z}$ for the orbital angular momentum of the scalar–conformal transport closure state, excluding half-integers. In QM8, the half-integer case is recovered by applying the holonomy condition to the double cover* SU(2) *of the rotation group rather than to* SO(3)*: the 4π single-valuedness condition $\mathrm{e}^{4\pi\mathrm{i}j} = 1$ selects $j \in \frac{1}{2}\mathbb{Z}$. Thus the two cases are not competing alternatives but distinct physical situations distinguished by the topology of the relevant holonomy path.*

4.4 The Identity $\hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_3^2 - \Phi_0\,\hat{L}_3$

The identity Eq. (29) used in the proof of Theorem 4.6 is sufficiently important to be recorded as a standalone result, as it and its companion identity are used again in Sec. 5 and Sec. 8.

Lemma 4.8 (Ladder product identities)**.** *On $\mathcal{S}(\mathbb{R}^3)$:*

$$\hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_3^2 - \Phi_0\,\hat{L}_3, \tag{31}$$

$$\hat{L}_+\hat{L}_- = \hat{L}^2 - \hat{L}_3^2 + \Phi_0\,\hat{L}_3. \tag{32}$$

Proof. Expand $\hat{L}_-\hat{L}_+ = (\hat{L}_1 - \mathrm{i}\hat{L}_2)(\hat{L}_1 + \mathrm{i}\hat{L}_2) = \hat{L}_1^2 + \mathrm{i}\hat{L}_1\hat{L}_2 - \mathrm{i}\hat{L}_2\hat{L}_1 + \hat{L}_2^2 = \hat{L}_1^2 + \hat{L}_2^2 + \mathrm{i}[\hat{L}_1, \hat{L}_2]$. Using Eq. (13): $\mathrm{i}[\hat{L}_1, \hat{L}_2] = \mathrm{i}(\mathrm{i}\Phi_0\,\hat{L}_3) = -\Phi_0\,\hat{L}_3$. Since $\hat{L}^2 = \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2$, we have $\hat{L}_1^2 + \hat{L}_2^2 = \hat{L}^2 - \hat{L}_3^2$. Combining: $\hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_3^2 - \Phi_0\,\hat{L}_3$, which is Eq. (31). The companion identity Eq. (32) follows by the same expansion with signs reversed, or equivalently by adding $[\hat{L}_+, \hat{L}_-] = 2\Phi_0\,\hat{L}_3$ to both sides of Eq. (31): $\hat{L}_+\hat{L}_- = \hat{L}_-\hat{L}_+ + 2\Phi_0\,\hat{L}_3 = \hat{L}^2 - \hat{L}_3^2 + \Phi_0\,\hat{L}_3$. □

Remark 4.9. *The two identities of Lemma 4.8 can be combined to express $\hat{L}^2$ entirely in terms of $\hat{L}_+$, $\hat{L}_-$, and $\hat{L}_3$:*

$$\hat{L}^2 = \hat{L}_-\hat{L}_+ + \hat{L}_3^2 + \Phi_0\,\hat{L}_3 = \hat{L}_+\hat{L}_- + \hat{L}_3^2 - \Phi_0\,\hat{L}_3. \tag{33}$$

These representations of $\hat{L}^2$ in terms of the ladder operators and $\hat{L}_3$ are useful in computing matrix elements and in establishing the sum rules of Sec. 8. In particular, Eq. (33) *shows that the ground state of the ladder (the state annihilated by $\hat{L}_-$) satisfies $\hat{L}^2\Psi_{\min} = \hat{L}_3^2\Psi_{\min} - \Phi_0\,\hat{L}_3\Psi_{\min} = (\mu_{\min}^2 - \Phi_0\,\mu_{\min})\Psi_{\min}$, consistent with $\lambda = \mu_{\min}(\mu_{\min} - \Phi_0)$ from the proof of Theorem 4.6.*

5 Integer Holonomy Quantization and the Quantum Numbers

The ladder operator analysis of Sec. 4 established that the $\hat{L}_3$ eigenvalues in any $\hat{L}^2$ eigenspace form an arithmetic progression with spacing Φ_0, terminating at $\pm\mu_{\max}$ where $\mu_{\max} = N\Phi_0/2$ for some $N \in \mathbb{Z}_{\geq 0}$. This allows both integer and half-integer multiples of Φ_0 as candidate values of $\mu_{\max}$. The present section applies the holonomy quantization principle of the Q-series to the azimuthal transport closure path, deriving that the $\hat{L}_3$ eigenvalues must be integer multiples of Φ_0. This selects $\mu_{\max} = \ell\Phi_0$ for integer $\ell \geq 0$, completing the determination of the spectrum. The section closes by recording the complete joint spectrum of $\hat{L}^2$ and $\hat{L}_3$ and the matrix elements of the ladder operators in the $|\,\ell, m\,\rangle$ basis.

5.1 The Azimuthal Holonomy Condition

In spherical coordinates (r, θ, φ), the operator $\hat{L}_3$ takes a particularly simple form that makes the holonomy condition transparent.

Lemma 5.1 ($\hat{L}_3$ in spherical coordinates)**.** *In spherical coordinates (r, θ, φ) related to Cartesian coordinates by $x^1 = r\sin\theta\cos\varphi$, $x^2 = r\sin\theta\sin\varphi$, $x^3 = r\cos\theta$, the operator $\hat{L}_3 = -\mathrm{i}\Phi_0\,(x^1\partial_2 - x^2\partial_1)$ takes the form*

$$\hat{L}_3 = -\mathrm{i}\Phi_0\,\frac{\partial}{\partial\varphi}. \tag{34}$$

Proof. By the chain rule, $\partial/\partial\varphi = (\partial x^1/\partial\varphi)\partial_1 + (\partial x^2/\partial\varphi)\partial_2 + (\partial x^3/\partial\varphi)\partial_3$. Computing the partial derivatives: $\partial x^1/\partial\varphi = -r\sin\theta\sin\varphi = -x^2$, $\partial x^2/\partial\varphi = r\sin\theta\cos\varphi = x^1$, $\partial x^3/\partial\varphi = 0$. Therefore $\partial/\partial\varphi = -x^2\partial_1 + x^1\partial_2 = x^1\partial_2 - x^2\partial_1$, giving $-\mathrm{i}\Phi_0\,\partial/\partial\varphi = -\mathrm{i}\Phi_0\,(x^1\partial_2 - x^2\partial_1) = \hat{L}_3$. □

With $\hat{L}_3 = -\mathrm{i}\Phi_0\,\partial_\varphi$, a $\hat{L}_3$-eigenstate with eigenvalue μ has azimuthal dependence determined by the eigenvalue equation $-\mathrm{i}\Phi_0\,\partial_\varphi\Psi = \mu\,\Psi$, whose solution is $\Psi \propto \mathrm{e}^{\mathrm{i}\mu\varphi/\Phi_0}$. The holonomy condition selects the admissible values of μ.

Theorem 5.2 (Integer quantization of the magnetic quantum number)**.** *The eigenvalues of $\hat{L}_3$ for admissible transport closure states on $\mathcal{H}$ are*

$$\mu = m\Phi_0, \qquad m \in \mathbb{Z}. \tag{35}$$

These are selected by the holonomy quantization condition: an admissible closure state must be single-valued under the full 2π rotation of the azimuthal angle,

$$\Psi(r, \theta, \varphi + 2\pi) = \Psi(r, \theta, \varphi) \tag{36}$$

for all (r, θ, φ). The condition Eq. (36) requires $\mathrm{e}^{2\pi \mathrm{i}\mu/\Phi_0} = 1$, which holds if and only if $\mu/\Phi_0 \in \mathbb{Z}$.

Proof. A $\hat{L}_3$-eigenstate with eigenvalue μ has azimuthal dependence $\Psi \propto \mathrm{e}^{\mathrm{i}\mu\varphi/\Phi_0}$ by the eigenvalue equation $-\mathrm{i}\Phi_0\,\partial_\varphi\Psi = \mu\Psi$. Applying the single-valuedness condition Eq. (36):

$$\mathrm{e}^{\mathrm{i}\mu(\varphi+2\pi)/\Phi_0} = \mathrm{e}^{\mathrm{i}\mu\varphi/\Phi_0},$$

which requires $\mathrm{e}^{2\pi \mathrm{i}\mu/\Phi_0} = 1$. This holds if and only if $2\pi\mu/\Phi_0 = 2\pi m$ for some $m \in \mathbb{Z}$, i.e., $\mu = m\Phi_0$. □

Remark 5.3. *The single-valuedness condition Eq. (36) is the NUVO holonomy quantization principle of the Q-series applied to the azimuthal transport closure path. In the Q-series, the holonomy condition required the transport phase accumulated along a complete radial closure cycle to be an integer multiple of $2\pi\Phi_0$. The condition Eq. (36) is the azimuthal analogue: after the closure state traverses a full 2π rotation in φ, it must return to its initial value. The accumulated azimuthal phase is $2\pi\mu/\Phi_0$, and the integer winding number condition $\mathrm{e}^{2\pi \mathrm{i}\mu/\Phi_0} = 1$ selects $\mu \in \Phi_0\mathbb{Z}$. This is the same holonomy principle; the two applications differ only in the geometric nature of the closed path (radial cycle versus azimuthal rotation).*

Remark 5.4. *Theorem 5.2 resolves the half-integer question raised in Remark 4.7 without any additional postulate. The algebraic analysis of Sec. 4 admitted $\mu_{\max} = N\Phi_0/2$ for any $N \in \mathbb{Z}_{\geq 0}$. The holonomy condition now restricts $\mu/\Phi_0 \in \mathbb{Z}$, which requires $\mu_{\max}/\Phi_0 = N/2 \in \mathbb{Z}$, so N must be even. Writing $N = 2\ell$ for $\ell \in \mathbb{Z}_{\geq 0}$, the maximum $\hat{L}_3$ eigenvalue is $\mu_{\max} = \ell\Phi_0$ and the $\hat{L}^2$ eigenvalue is $\lambda = \ell(\ell+1)\Phi_0^2$. The half-integer possibility $N = 2j + 1$ (odd) is excluded by the holonomy condition, not by a separate postulate. In QM8, the double-cover holonomy applied to the transport closure structure on $\mathrm{SU}(2)$ relaxes the condition to $\mathrm{e}^{4\pi \mathrm{i}\mu/\Phi_0} = 1$, allowing $\mu/\Phi_0 \in \frac{1}{2}\mathbb{Z}$ and thereby permitting the half-integer case.*

5.2 The Complete Joint Spectrum

Combining Theorem 4.6 (algebraic constraints from ladder termination) with Theorem 5.2 (integer holonomy quantization) yields the complete joint spectrum of $\hat{L}^2$ and $\hat{L}_3$.

Theorem 5.5 (Complete joint spectrum of $\hat{L}^2$ and $\hat{L}_3$)**.** *The joint spectrum of $\hat{L}^2$ and $\hat{L}_3$ on $\mathcal{H}$ is:*

$$\sigma(\hat{L}^2) = \big\{\ell(\ell+1)\Phi_0^2 \;\big|\; \ell \in \{0, 1, 2, \ldots\}\big\}, \tag{37}$$

$$\sigma(\hat{L}_3)\big|_\ell = \big\{m\Phi_0 \;\big|\; m \in \{-\ell, -\ell+1, \ldots, \ell-1, \ell\}\big\}, \tag{38}$$

giving $2\ell+1$ values of m for each value of ℓ. Each $\hat{L}^2$ eigenvalue $\ell(\ell+1)\Phi_0^2$ is $(2\ell+1)$-fold degenerate with respect to $\hat{L}_3$. The joint eigenstates $|\,\ell,\, m\,\rangle$ satisfy

$$\hat{L}^2\,|\,\ell,\, m\,\rangle = \ell(\ell+1)\Phi_0^2\,|\,\ell,\, m\,\rangle, \qquad \hat{L}_3\,|\,\ell,\, m\,\rangle = m\Phi_0\,|\,\ell,\, m\,\rangle, \tag{39}$$

with $\langle \ell', m' | \ell, m, =\rangle\, \delta_{\ell'\ell}\delta_{m'm}$.

Proof. From Theorem 4.6 (iii), the $\hat{L}_3$ eigenvalues in the λ-eigenspace are an arithmetic progression with spacing Φ_0 and maximum value $\mu_{\max} = N\Phi_0/2$. From Theorem 5.2, each $\hat{L}_3$ eigenvalue must be an integer multiple of Φ_0, so $\mu_{\max}/\Phi_0 = N/2 \in \mathbb{Z}_{\geq 0}$, requiring $N = 2\ell$ for $\ell \in \mathbb{Z}_{\geq 0}$. Therefore $\mu_{\max} = \ell\Phi_0$ and $\mu_{\min} = -\ell\Phi_0$ (Theorem 4.6 (i)). The $\hat{L}^2$ eigenvalue is $\lambda = \mu_{\max}(\mu_{\max} + \Phi_0) = \ell\Phi_0(\ell\Phi_0 + \Phi_0) = \ell(\ell+1)\Phi_0^2$ (Theorem 4.6 (ii)), giving Eq. (37). The $\hat{L}_3$ eigenvalues range from $-\ell\Phi_0$ to $+\ell\Phi_0$ in integer steps of Φ_0, giving the values in Eq. (38) with $2\ell + 1$ elements. Orthonormality of the joint eigenstates follows from the self-adjointness of $\hat{L}^2$ and $\hat{L}_3$ and the distinct eigenvalues in each case. □

Remark 5.6. *In the NUVO framework, the integer ℓ is the maximum azimuthal winding number accessible to the transport closure configuration with $\hat{L}^2$ eigenvalue $\ell(\ell+1)\Phi_0^2$, and the integer m is the actual azimuthal winding number of the specific eigenstate $|\,\ell, m\,\rangle$. The relation $|m| \leq \ell$ reflects the geometric constraint that the z-component of the angular momentum cannot exceed the total angular momentum: the azimuthal winding number cannot exceed the maximum winding number of the multiplet. The degeneracy $2\ell+1$ is the number of distinct azimuthal winding numbers accessible within the ℓ-multiplet, ranging from $-\ell$ (maximum clockwise winding) through 0 (no net azimuthal winding) to $+\ell$ (maximum counterclockwise winding).*

5.3 Matrix Elements of the Ladder Operators

With the complete spectrum established, the matrix elements of the ladder operators in the $|\,\ell, m\,\rangle$ basis are determined by normalization.

Proposition 5.7 (Matrix elements of the ladder operators)**.** *In the $|\,\ell, m\,\rangle$ basis:*

$$\hat{L}_+ \,|\,\ell, m\,\rangle = \Phi_0 \sqrt{\ell(\ell+1) - m(m+1)}\,|\,\ell, m+1\,\rangle, \tag{40}$$

$$\hat{L}_- \,|\,\ell, m\,\rangle = \Phi_0 \sqrt{\ell(\ell+1) - m(m-1)}\,|\,\ell, m-1\,\rangle. \tag{41}$$

The square-root factors vanish at the termination points: for $m = +\ell$, $\hat{L}_+\,|\,\ell, \ell\,\rangle = 0$; for $m = -\ell$, $\hat{L}_-\,|\,\ell, -\ell\,\rangle = 0$.

Proof. By Proposition 4.4, $\hat{L}_+\,|\,\ell, m\,\rangle$ is proportional to $|\,\ell, m+1\,\rangle$ (for $m < \ell$). Write $\hat{L}_+\,|\,\ell, m\,\rangle = c^+_{\ell,m}\,|\,\ell, m+1\,\rangle$ for some constant $c^+_{\ell,m}$. Compute $|c^+_{\ell,m}|^2 = \left\|\hat{L}_+\,|\,\ell, m\,\rangle\right\|^2_{\mathcal{H}}$:

$$\begin{aligned}
\left\|\hat{L}_+\,|\,\ell, m\,\rangle\right\|^2_{\mathcal{H}} &= \Big\langle |\,\ell, m\,\rangle, \hat{L}_-\hat{L}_+\,|\,\ell, m\,\rangle \Big\rangle_{\mathcal{H}} \\
&= \Big\langle |\,\ell, m\,\rangle, (\hat{L}^2 - \hat{L}_3^2 - \Phi_0\,\hat{L}_3)|\,\ell, m\,\rangle \Big\rangle_{\mathcal{H}} \\
&= \ell(\ell+1)\Phi_0^2 - m^2\Phi_0^2 - m\Phi_0^2 \\
&= \Phi_0^2\big[\ell(\ell+1) - m(m+1)\big],
\end{aligned}$$

where in the second step we used the identity $\hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_3^2 - \Phi_0\,\hat{L}_3$ from Lemma 4.8, and in the third step we substituted the eigenvalues from Eq. (39). Choosing the positive real square root: $c^+_{\ell,m} = \Phi_0\sqrt{\ell(\ell+1) - m(m+1)}$, which gives Eq. (40). The result for $\hat{L}_-$ follows by the same argument using the identity $\hat{L}_+\hat{L}_- = \hat{L}^2 - \hat{L}_3^2 + \Phi_0\,\hat{L}_3$ from Lemma 4.8.

For the termination: at $m = \ell$, the factor $\ell(\ell+1) - \ell(\ell+1) = 0$ makes $c^+_{\ell,\ell} = 0$, so $\hat{L}_+\,|\,\ell, \ell\,\rangle = 0$. Similarly at $m = -\ell$, $\hat{L}_-\,|\,\ell, -\ell\,\rangle = 0$. □

Remark 5.8. *The square-root factors in Eqs.* (40) *and* (41) *provide a useful consistency check on the spectral result. At the top of the ladder ($m = \ell$), the factor for $\hat{L}_+$ is $\sqrt{\ell(\ell+1) - \ell(\ell+1)} = 0$, as required. At one step below the top ($m = \ell - 1$), the factor is $\sqrt{\ell(\ell+1) - (\ell-1)\ell} = \sqrt{2\ell}$, which grows with ℓ as expected: higher multiplets have larger ladder matrix elements. At the center of the multiplet ($m = 0$), the factor is $\sqrt{\ell(\ell+1)}$, which is the maximum value within the multiplet. These features are structural consequences of the spectral geometry and will be used in the hydrogen atom matrix element computations of Sec. 7.*

Remark 5.9. *Theorem 5.5 together with Proposition 5.7 establishes that the family $\{|\,\ell,\, m\,\rangle : \ell \in \mathbb{Z}_{\geq 0},\, m \in \{-\ell, \dots, \ell\}\}$ is a complete orthonormal family of joint eigenstates of $\hat{L}^2$ and $\hat{L}_3$ in the angular sector Hilbert space $\mathcal{H}_{\text{ang}} = L^2(S^2, \mathbb{C})$. This completeness follows from the spectral theorem of QM1 applied to the commuting self-adjoint pair $(\hat{L}^2, \hat{L}_3)$: the joint spectral decomposition of any state in $\mathcal{H}_{\text{ang}}$ into the $|\,\ell,\, m\,\rangle$ basis is guaranteed by the spectral theorem, and the resolution of the identity on $\mathcal{H}_{\text{ang}}$ is*

$$\mathbf{1}_{\mathcal{H}_{\text{ang}}} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} |\ell, m\rangle\langle \ell, m|. \tag{42}$$

The position-space realization of the states $|\,\ell,\, m\,\rangle$ as spherical harmonics $Y_\ell^m(\theta, \varphi)$ is the subject of Sec. 6.

6 Spherical Harmonics as Angular Closure Eigenstates

The abstract joint eigenstates $|\,\ell,\, m\,\rangle$ of $\hat{L}^2$ and $\hat{L}_3$ established in Sec. 5 are elements of the angular sector Hilbert space $\mathcal{H}_{\text{ang}} = L^2(S^2, \mathbb{C})$. The present section derives their position-space representation: the functions on S^2 that realize the abstract eigenstates as square-integrable functions of (θ, φ). The derivation proceeds by solving the angular eigenvalue equations directly in spherical coordinates. The azimuthal equation, already solved by the holonomy condition of Sec. 5, gives the factor $\mathrm{e}^{\,im\varphi}$. The polar equation, obtained by substituting the azimuthal solution into the $\hat{L}^2$ eigenvalue equation, yields the associated Legendre equation whose regular solutions are the associated Legendre polynomials $P_\ell^{|m|}(\cos\theta)$. The normalized product of these two factors is the spherical harmonic $Y_\ell^m(\theta, \varphi)$.

6.1 $\hat{L}^2$ in Spherical Coordinates

Before solving the angular eigenvalue equations, it is necessary to express $\hat{L}^2$ in spherical coordinates. The result shows that $\hat{L}^2$ acts only on the angular variables (θ, φ) and not on the radial coordinate r, confirming that the angular eigenvalue problem is well-posed on the unit sphere S^2.

Proposition 6.1 ($\hat{L}^2$ as the Laplace-Beltrami operator on S^2)**.** *In spherical coordinates (r, θ, φ), the operator $\hat{L}^2$ takes the form*

$$\hat{L}^2 = -\Phi_0^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right] =: -\Phi_0^2\, \Delta_{S^2}, \tag{43}$$

where Δ_{S^2} is the Laplace–Beltrami operator on the unit sphere S^2. In particular, $\hat{L}^2$ acts only on the angular variables (θ, φ) and commutes with any function of r alone.

Proof. The Laplacian in spherical coordinates is

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \Delta_{S^2},$$

where Δ_{S^2} is as defined in Eq. (43). The operator $\hat{L}^2$ is related to the full Laplacian by the identity

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\hat{L}^2}{\Phi_0^2 r^2}, \tag{44}$$

which is established by direct coordinate transformation from the Cartesian expression $\hat{L}^2 = -\Phi_0^2\sum_{j,k}(\delta_{jk}x^m x^m - x^j x^k)\partial_j\partial_k + \ldots$ (using the explicit Cartesian components Eqs. (2)–(4) and the standard coordinate transformation formulas; the computation is standard and cited from [1]). Solving Eq. (44) for $\hat{L}^2$ and substituting the spherical form of Δ_{S^2} gives Eq. (43). □

Remark 6.2. *Proposition 6.1 confirms that $\hat{L}^2$ is a purely angular operator. Combined with Eq. (44), this gives the decomposition of the kinetic operator:*

$$\hat{T} = -\frac{\Phi_0^2}{2m}\Delta = -\frac{\Phi_0^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\hat{L}^2}{2mr^2}, \tag{45}$$

which separates the kinetic energy into a radial part and an angular (centrifugal) part $\hat{L}^2/(2mr^2)$. This decomposition is used in Sec. 7 to separate the hydrogenic Schrödinger equation into independent radial and angular equations.

6.2 The Angular Eigenvalue Equations

The joint eigenvalue equations $\hat{L}^2 f = \ell(\ell+1)\Phi_0^2 f$ and $\hat{L}_3 f = m\Phi_0 f$ on $L^2(S^2)$ are now solved explicitly. The ansatz of separation of variables $f(\theta,\varphi) = \Theta(\theta)\Phi(\varphi)$ decouples the two equations.

The azimuthal equation. The $\hat{L}_3$ eigenvalue equation with $\hat{L}_3 = -\mathrm{i}\Phi_0\,\partial_\varphi$ (Lemma 5.1) gives

$$-\mathrm{i}\Phi_0\,\Phi'(\varphi) = m\Phi_0\,\Phi(\varphi),$$

with solution $\Phi(\varphi) = \mathrm{e}^{\mathrm{i}m\varphi}$ normalized on $[0,2\pi)$ as $\Phi(\varphi) = (2\pi)^{-1/2}\mathrm{e}^{\mathrm{i}m\varphi}$. By Theorem 5.2, the single-valuedness condition requires $m \in \mathbb{Z}$, already established.

The polar equation. Substituting $f = \Theta(\theta)\mathrm{e}^{\mathrm{i}m\varphi}$ and $\hat{L}_3^2 f = m^2\Phi_0^2 f$ into the $\hat{L}^2$ eigenvalue equation Eq. (43) with eigenvalue $\ell(\ell+1)\Phi_0^2$:

$$-\Phi_0^2\left[\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta\right] = \ell(\ell+1)\Phi_0^2\,\Theta. \tag{46}$$

Dividing by $-\Phi_0^2$ and substituting $u = \cos\theta$, $\Theta(\theta) = P(u)$:

$$\frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)\frac{\mathrm{d}P}{\mathrm{d}u}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-u^2}\right]P = 0, \tag{47}$$

which is the *associated Legendre equation* with parameters ℓ and m.

Lemma 6.3 (Regular solutions of the associated Legendre equation)**.** *For $\ell \in \{0,1,2,\ldots\}$ and $|m| \le \ell$, the unique solution of Eq. (47) regular at $u = \pm 1$ (i.e., at $\theta = 0$ and $\theta = \pi$) is the* associated Legendre polynomial

$$P_\ell^{|m|}(u) = \frac{(-1)^{|m|}}{2^\ell \ell!}(1-u^2)^{|m|/2}\frac{\mathrm{d}^{\ell+|m|}}{\mathrm{d}u^{\ell+|m|}}(u^2-1)^\ell, \tag{48}$$

the Rodrigues formula for the associated Legendre function. For $|m| > \ell$, no regular solution exists.

Proof. This is a classical result of the theory of ordinary differential equations applied to the Legendre equation [1, 2]. The regularity condition at $u = \pm 1$ (the poles of the sphere) requires ℓ to be a non-negative integer and $|m| \le \ell$ for the power series solution to terminate and remain bounded. The explicit Rodrigues formula Eq. (48) is the standard form of the regular solution; its derivation is cited for orientation. □

6.3 The Spherical Harmonics

The normalized product of the azimuthal and polar solutions defines the spherical harmonics.

Theorem 6.4 (Spherical harmonics as angular closure eigenstates)**.** *For $\ell \in \{0, 1, 2, \ldots\}$ and $m \in \{-\ell, \ldots, +\ell\}$, the function*

$$Y_\ell^m(\theta, \varphi) := \sqrt{\frac{(2\ell+1)}{4\pi} \cdot \frac{(\ell-|m|)!}{(\ell+|m|)!}}\, P_\ell^{|m|}(\cos\theta)\, \mathrm{e}^{\,\mathrm{i}m\varphi} \tag{49}$$

is the position-space realization of the abstract eigenstate $|\,\ell,\, m\,\rangle$:

$$\hat{L}^2\, Y_\ell^m = \ell(\ell+1)\Phi_0^2\, Y_\ell^m, \qquad \hat{L}_3\, Y_\ell^m = m\Phi_0\, Y_\ell^m, \tag{50}$$

and is normalized on the unit sphere: $\int_{S^2} |Y_\ell^m|^2\, \mathrm{d}\Omega = 1$, where $\mathrm{d}\Omega = \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi$.

Proof. The eigenvalue equations Eq. (50) follow directly: the $\hat{L}_3$ equation holds by construction from the factor $\mathrm{e}^{\,\mathrm{i}m\varphi}$ (Lemma 5.1); the $\hat{L}^2$ equation holds because $P_\ell^{|m|}(\cos\theta)$ is the regular solution of Eq. (47) with parameters (ℓ, m) (Lemma 6.3).

For normalization, verify $\int_{S^2} |Y_\ell^m|^2\, \mathrm{d}\Omega = 1$:

$$\begin{aligned}\int_{S^2} |Y_\ell^m|^2\, \mathrm{d}\Omega &= \frac{(2\ell+1)}{4\pi} \cdot \frac{(\ell-|m|)!}{(\ell+|m|)!} \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} |P_\ell^{|m|}(\cos\theta)|^2 \sin\theta\, \mathrm{d}\theta \\ &= \frac{(2\ell+1)}{4\pi} \cdot \frac{(\ell-|m|)!}{(\ell+|m|)!} \cdot 2\pi \cdot \frac{2}{2\ell+1} \cdot \frac{(\ell+|m|)!}{(\ell-|m|)!} = 1,\end{aligned}$$

using the orthogonality integral for associated Legendre polynomials: $\int_{-1}^{1} [P_\ell^{|m|}(u)]^2\, \mathrm{d}u = \frac{2}{2\ell+1} \frac{(\ell+|m|)!}{(\ell-|m|)!}$ (cited from [1]). The normalization constant in Eq. (49) is precisely that required for unit norm. □

Remark 6.5. *The phase convention in Eq.* (49) *follows the Condon-Shortley convention, in which the associated Legendre functions carry the factor $(-1)^{|m|}$ for $m > 0$ (absorbed into the Rodrigues formula Eq.* (48)*) and the spherical harmonics satisfy the complex conjugate relation $\overline{Y_\ell^m} = (-1)^m Y_\ell^{-m}$ recorded in Proposition 6.7 below. This convention is adopted here for consistency with the standard literature and with the subsequent papers QM6 through QM11. Different phase conventions are used in different references; all give the same orthonormality and completeness properties but differ by sign in individual matrix elements.*

The lowest-degree spherical harmonics are recorded explicitly for reference.

Remark 6.6. *The spherical harmonics for $\ell = 0$ and $\ell = 1$ are:*

$$Y_0^0(\theta, \varphi) = \frac{1}{\sqrt{4\pi}},$$

$$Y_1^0(\theta, \varphi) = \sqrt{\frac{3}{4\pi}} \cos\theta,$$

$$Y_1^{\pm 1}(\theta, \varphi) = \mp\sqrt{\frac{3}{8\pi}} \sin\theta\, \mathrm{e}^{\,\pm\mathrm{i}\varphi}.$$

These are verified by direct substitution into Eq. (49) *with $P_0^0(u) = 1$, $P_1^0(u) = u$, and $P_1^1(u) = -(1-u^2)^{1/2}$ from the Rodrigues formula. The Y_0^0 state represents a spherically symmetric angular closure configuration (constant on S^2, no angular structure). The Y_1^0 state has a $\cos\theta$ polar modulation (aligned with the z-axis) and no azimuthal variation. The $Y_1^{\pm 1}$ states have $\sin\theta$ polar modulation (equatorial concentration) and azimuthal winding number ± 1, representing closure configurations with one unit of azimuthal angular momentum.*

6.4 Orthonormality, Completeness, and Key Properties

Proposition 6.7 (Properties of spherical harmonics). *The spherical harmonics Y_ℓ^m of Theorem 6.4 satisfy the following properties.*

(i) Orthonormality on S^2:

$$\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \, \sin\theta \, \overline{Y_\ell^m(\theta,\varphi)} \, Y_{\ell'}^{m'}(\theta,\varphi) = \delta_{\ell\ell'}\delta_{mm'}. \tag{51}$$

(ii) Completeness on $L^2(S^2)$:

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^m(\theta,\varphi) \, \overline{Y_\ell^m(\theta',\varphi')} = \delta_{S^2}\big((\theta,\varphi),(\theta',\varphi')\big), \tag{52}$$

where δ_{S^2} is the Dirac delta on the unit sphere. Every $f \in L^2(S^2)$ expands as $f = \sum_{\ell,m} c_{\ell m} Y_\ell^m$ with $c_{\ell m} = \int_{S^2} \overline{Y_\ell^m} f \, \mathrm{d}\Omega$.

(iii) Parity: *Under the inversion $\mathbf{x} \to -\mathbf{x}$, equivalent to $(\theta,\varphi) \to (\pi-\theta, \varphi+\pi)$:*

$$Y_\ell^m(\pi-\theta,\varphi+\pi) = (-1)^\ell \, Y_\ell^m(\theta,\varphi). \tag{53}$$

(iv) Complex conjugate:

$$\overline{Y_\ell^m(\theta,\varphi)} = (-1)^m \, Y_\ell^{-m}(\theta,\varphi). \tag{54}$$

Proof. Part (i): Orthogonality for $(\ell,m) \neq (\ell',m')$ follows from the self-adjointness of $\hat{L}^2$ and $\hat{L}_3$: eigenstates with different eigenvalues are orthogonal. If $m \neq m'$, orthogonality follows from the azimuthal integral $\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(m'-m)\varphi} \, \mathrm{d}\varphi = 2\pi\delta_{mm'}$. If $m = m'$ but $\ell \neq \ell'$, orthogonality follows from the associated Legendre orthogonality: $\int_0^{\pi} P_\ell^{|m|}(\cos\theta) P_{\ell'}^{|m|}(\cos\theta) \sin\theta \, \mathrm{d}\theta = 0$ for $\ell \neq \ell'$ [1]. Unit normalization was verified in the proof of Theorem 6.4.

Part (ii): Completeness follows from the spectral theorem of QM1 applied to the commuting pair $(\hat{L}^2, \hat{L}_3)$ on $\mathcal{H}_{\mathrm{ang}} = L^2(S^2)$: since both operators are self-adjoint and their joint spectrum is discrete (Theorem 5.5), the complete orthonormal family of joint eigenstates forms a basis for $\mathcal{H}_{\mathrm{ang}}$ by the spectral theorem [3]. The completeness relation Eq. (52) is the position-space expression of the resolution of the identity Eq. (42).

Part (iii): Under $(\theta,\varphi) \to (\pi-\theta,\varphi+\pi)$: $\cos\theta \to -\cos\theta$, so $P_\ell^{|m|}(\cos\theta) \to P_\ell^{|m|}(-\cos\theta) = (-1)^{\ell+|m|} P_\ell^{|m|}(\cos\theta)$ by the parity of associated Legendre functions [1]; and $\mathrm{e}^{\mathrm{i}m\varphi} \to \mathrm{e}^{\mathrm{i}m(\varphi+\pi)} = (-1)^m \mathrm{e}^{\mathrm{i}m\varphi}$. The combined factor is $(-1)^{\ell+|m|}(-1)^m = (-1)^{\ell+|m|+m}$. For $m \geq 0$: $|m| = m$, so the exponent is $\ell + 2m \equiv \ell \pmod 2$. For $m < 0$: $|m| = -m$, so the exponent is $\ell + (-m) + m = \ell$. In both cases the parity factor is $(-1)^\ell$, giving Eq. (53).

Part (iv): Taking the complex conjugate of Eq. (49): $\overline{Y_\ell^m} \propto \overline{P_\ell^{|m|}(\cos\theta)} \, \mathrm{e}^{-\mathrm{i}m\varphi}$. Since $P_\ell^{|m|}$ is real-valued, $\overline{P_\ell^{|m|}} = P_\ell^{|m|} = P_\ell^{|-m|}$. The relation between the normalization constants for m and $-m$ is $\sqrt{(\ell-|m|)!/(\ell+|m|)!} = \sqrt{(\ell-|-m|)!/(\ell+|-m|)!}$, so the normalization factors are identical. The Condon-Shortley phase gives the extra factor $(-1)^m$ [1], yielding Eq. (54). □

Remark 6.8. *In the NUVO transport closure framework, the spherical harmonics $Y_\ell^m(\theta,\varphi)$ are the angular components of closure states with definite total angular momentum $\ell(\ell+1)\Phi_0^2$ and definite azimuthal winding number m. The parity property (iii) reflects the behavior of the closure state* *under spatial inversion $\mathbf{x} \to -\mathbf{x}$: states with even ℓ are parity-even (symmetric under inversion) and states with odd ℓ are parity-odd (antisymmetric). This parity structure is inherited by the full hydrogenic eigenstates $\Psi_{n\ell m} = R_{n\ell}(r)Y_\ell^m(\theta,\varphi)$ of Sec. 7 (since $R_{n\ell}(r)$ is parity-even as a function of $r > 0$), giving the selection rule for electric dipole transitions: transitions between states of the same parity are forbidden. The completeness property (ii) means that any square-integrable function on S^2—equivalently, the angular dependence of any closure state in $\mathcal{H}$—can be expanded in spherical harmonics. This expansion is the generalization of the Fourier series to functions on the sphere and is the tool used throughout the QM-series wherever the angular structure of physical states is analyzed.*

7 Connection to the Hydrogenic Sector

The angular momentum structure derived in Secs. 3–6 is now connected to the hydrogenic sector of the Q-series and QM4. The hydrogenic Hamiltonian $\hat{H}_\mathrm{H} = \hat{T} + \hat{V}_\mathrm{H}$ commutes with all three angular momentum operators (QM4 Proposition 7.3), so its eigenstates can be chosen to be simultaneous eigenstates of $\hat{L}^2$ and $\hat{L}_3$. The present section shows explicitly how this is achieved through separation of variables, derives the constraint $\ell \leq n-1$ on the orbital quantum number from the radial eigenvalue problem, establishes the n^2-fold degeneracy of each energy level, and records how the two holonomy quantization conditions—azimuthal (the present paper) and radial (the Q-series)—combine to give the full n-ℓ-m labeling of the hydrogen spectrum.

7.1 Separation of Variables for the Hydrogenic Hamiltonian

The hydrogenic Hamiltonian from QM4 is

$$\hat{H}_\mathrm{H} = \hat{T} + \hat{V}_\mathrm{H} = -\frac{\Phi_0^2}{2m}\Delta - \frac{e^2}{4\pi\epsilon_0|x|}. \tag{55}$$

Using the decomposition Eq. (44) from Proposition 6.1:

$$\hat{H}_\mathrm{H} = -\frac{\Phi_0^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\hat{L}^2}{2mr^2} - \frac{e^2}{4\pi\epsilon_0 r}. \tag{56}$$

Since $\hat{L}^2$ acts only on the angular variables (Proposition 6.1) and the Coulomb potential depends only on r, the eigenvalue equation $\hat{H}_\mathrm{H}\Psi = E\Psi$ separates under the ansatz $\Psi_{n\ell m}(r,\theta,\varphi) = R_{n\ell}(r)Y_\ell^m(\theta,\varphi)$.

Proposition 7.1 (Separation of variables for hydrogenic eigenstates)**.** *The eigenstate ansatz*

$$\Psi_{n\ell m}(r,\theta,\varphi) = R_{n\ell}(r)\,Y_\ell^m(\theta,\varphi) \tag{57}$$

reduces the hydrogenic eigenvalue equation $\hat{H}_\mathrm{H}\Psi = E\Psi$ to a purely radial ordinary differential equation for $R_{n\ell}(r)$:

$$\left[-\frac{\Phi_0^2}{2m}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\right) + \frac{\ell(\ell+1)\Phi_0^2}{2mr^2} - \frac{e^2}{4\pi\epsilon_0 r}\right] R_{n\ell}(r) = E\,R_{n\ell}(r), \tag{58}$$

in which the angular momentum quantum number ℓ enters as a parameter through the centrifugal term $\ell(\ell+1)\Phi_0^2/(2mr^2)$.

Proof. Substitute the ansatz Eq. (57) into $\hat{H}_{\mathrm{H}}\Psi = E\Psi$ using the form Eq. (56). Since $\hat{L}^2 Y_\ell^m = \ell(\ell+1)\Phi_0^2 Y_\ell^m$ (Theorem 6.4), the operator $\hat{L}^2$ acting on the product $R_{n\ell}(r)Y_\ell^m(\theta,\varphi)$ gives $\ell(\ell+1)\Phi_0^2 R_{n\ell}(r)Y_\ell^m(\theta,\varphi)$, replacing $\hat{L}^2$ by the scalar $\ell(\ell+1)\Phi_0^2$. The radial differential operator and the Coulomb term act only on $R_{n\ell}(r)$. Dividing through by $Y_\ell^m(\theta,\varphi) \neq 0$ yields the radial equation Eq. (58), which depends only on r and the quantum number ℓ, not on m. □

Remark 7.2. *The radial equation Eq. (58) is independent of the magnetic quantum number m. This independence is the algebraic origin of the $2\ell+1$ degeneracy in m for each ℓ: all $2\ell+1$ states $\Psi_{n\ell m}$ for $m \in \{-\ell,\dots,+\ell\}$ share the same radial wave function $R_{n\ell}(r)$ and the same energy eigenvalue E_n. This is a direct consequence of the rotational symmetry of the Coulomb potential: since the Hamiltonian commutes with all rotations, no physical distinction can be made between states related by rotation, and all states in the (ℓ,m)-multiplet are degenerate.*

7.2 The Radial Equation and the Constraint on ℓ

The radial equation Eq. (58) is the hydrogen atom radial Schrödinger equation. Its bound state solutions exist only for discrete negative energies $E_n < 0$ and impose a constraint on the relationship between the principal quantum number n and the orbital quantum number ℓ.

Proposition 7.3 (Radial bound state solutions and the constraint $\ell \le n-1$)**.** *For the radial equation Eq. (58) with $\ell \in \{0,1,2,\dots\}$ fixed, the normalizable (square-integrable) solutions exist only for discrete energies*

$$E_n = -\frac{me^4}{2\Phi_0^2 n^2}, \qquad n \in \{1,2,3,\dots\}, \tag{59}$$

which are the energy levels established in the Q-series hydrogenic correspondence. For a given ℓ, the principal quantum number n satisfies

$$n \ge \ell+1, \qquad \textit{equivalently}, \qquad \ell \le n-1. \tag{60}$$

Proof. The radial equation Eq. (58) is solved by the substitution $R_{n\ell}(r) = r^{-1}u(r)$, which transforms it into the standard form of the hydrogen radial equation in terms of $u(r)$. The bound state solutions ($E < 0$) are found by the power series method: the series terminates to give a polynomial times an exponential (ensuring square-integrability) if and only if the series truncation condition is satisfied. This condition requires the principal quantum number n, defined as $n = n_r + \ell + 1$ where $n_r \in \{0,1,2,\dots\}$ is the radial quantum number (the degree of the associated Laguerre polynomial in the solution), to be a positive integer [2]. Since $n_r \ge 0$, the constraint $n = n_r + \ell + 1 \ge \ell + 1$ follows, giving Eq. (60). The energy eigenvalues Eq. (59) are recovered from the series truncation condition and agree with the Q-series result; their derivation is cited from [2] as a classical result of the radial analysis. □

Remark 7.4. *The energy quantization Eq. (59) is the Q-series holonomy quantization applied to the radial transport closure path. In the Q-series, the integer winding number n of the radial closure cycle was shown to select the discrete energy levels $E_n = -me^4/(2\Phi_0^2 n^2)$. The present paper has established the azimuthal holonomy quantization (selecting integer m) and the algebraic structure of the ladder operators (constraining ℓ to non-negative integers via Theorem 5.5). The radial constraint $\ell \le n-1$ from Proposition 7.3 completes the quantum number structure, relating the azimuthal and radial holonomy integers through the inequality $\ell \le n-1$. All three quantization conditions—radial holonomy (Q-series), azimuthal holonomy (QM5 Sec. 5), and algebraic ladder termination (QM5 Sec. 4)—are manifestations of the single holonomy quantization principle.*

7.3 The Complete Quantum Number Structure and Degeneracy

Theorem 7.5 (Hydrogenic quantum number structure and energy degeneracy)**.** *The complete set of quantum numbers for the hydrogenic system is* (n, ℓ, m) *with*

$$\begin{aligned} &n \in \{1, 2, 3, \dots\}, \\ &\ell \in \{0, 1, \dots, n-1\}, \\ &m \in \{-\ell, -\ell+1, \dots, \ell-1, \ell\}. \end{aligned}$$

Each energy level E_n *is* n^2*-fold degenerate:*

$$g_n = \sum_{\ell=0}^{n-1} (2\ell + 1) = n^2. \tag{61}$$

The complete set of hydrogenic energy eigenstates is

$$\left\{\Psi_{n\ell m} = R_{n\ell}(r)\, Y_\ell^m(\theta, \varphi) \;\middle|\; n \geq 1,\ 0 \leq \ell \leq n-1,\ |m| \leq \ell\right\}, \tag{62}$$

and these form a complete orthonormal system in $\mathcal{H}$ *for the bound-state sector of the hydrogenic spectrum.*

Proof. The allowed values of (n, ℓ, m) follow from combining: $n \geq 1$ (positive integer, from the radial holonomy quantization of the Q-series), $\ell \leq n - 1$ (from Proposition 7.3), $\ell \geq 0$ (from Theorem 5.5), and $|m| \leq \ell$ (from Theorem 5.5). For the degeneracy: the number of states with principal quantum number n is

$$g_n = \sum_{\ell=0}^{n-1} (2\ell + 1) = 2\sum_{\ell=0}^{n-1} \ell + n = 2 \cdot \frac{(n-1)n}{2} + n = n(n-1) + n = n^2,$$

using the identity $\sum_{\ell=0}^{n-1} \ell = n(n-1)/2$. Completeness of the eigenstate family in the bound-state sector follows from the spectral theorem applied to $\hat{H}_\mathrm{H}$ on $\mathcal{H}$ (QM1 Theorem 6.1 applied to the self-adjoint operator $\hat{H}_\mathrm{H}$ of QM4 Theorem 4.2): the discrete spectral family corresponding to $\sigma_\mathrm{disc}(\hat{H}_\mathrm{H}) = \{E_n\}_{n\geq 1}$ forms a complete basis for the discrete-spectrum subspace of $\mathcal{H}$. □

Remark 7.6. *The* n^2*-fold degeneracy of the hydrogenic energy level* E_n *has a two-part origin in the NUVO framework. The* $(2\ell + 1)$*-fold degeneracy in* m *for fixed* ℓ *is the azimuthal rotational degeneracy: all* $2\ell + 1$ *states* $\Psi_{n\ell m}$ *with* $m \in \{-\ell, \dots, +\ell\}$ *are related by rotations about the z-axis and are degenerate because the Coulomb potential is rotationally symmetric. The additional* n*-fold degeneracy in* ℓ *(the fact that states with different* ℓ *but the same* n *are also degenerate) is the accidental or dynamical degeneracy of the Coulomb potential, which has a higher symmetry than* SO(3)*: it is invariant under the full* SO(4) *symmetry group of the Kepler problem. The* SO(4) *symmetry of the hydrogen atom is the generator of the Runge-Lenz vector conservation, which in the quantum-mechanical setting commutes with* $\hat{H}_\mathrm{H}$ *and generates transitions between different* ℓ *values at the same energy. A full treatment of the* SO(4) *symmetry is beyond the scope of the present paper and is deferred as a structural extension of the QM-series.*

7.4 The Hydrogenic Eigenstates and the Q-Series

The quantum number structure established in Theorem 7.5 completes the arc from the Q-series to the present paper.

Remark 7.7. *The Q-series derived the hydrogenic energy spectrum $E_n = -me^4/(2\Phi_0^2 n^2)$ from the radial holonomy quantization condition, identifying n as the integer winding number of the radial transport closure cycle and fixing $\Phi_0 = \hbar$ through the correspondence limit. At that stage, the angular structure of the hydrogenic states was not analyzed: the Q-series established the energy eigenvalues but not the full wave functions. The present paper completes this analysis. The angular structure of each energy eigenstate is encoded in the spherical harmonic $Y_\ell^m(\theta, \varphi)$, whose quantum numbers (ℓ, m) are determined by the azimuthal holonomy and the angular momentum algebra. The full hydrogenic wave function $\Psi_{n\ell m}(r, \theta, \varphi) = R_{n\ell}(r)Y_\ell^m(\theta, \varphi)$ is thus the product of two separately derived structures: the radial factor $R_{n\ell}(r)$ determined by the Q-series radial holonomy and the radial equation, and the angular factor $Y_\ell^m(\theta, \varphi)$ determined by the azimuthal holonomy and the angular momentum algebra of the present paper. The hydrogenic sector of the NUVO program is thereby complete: energy levels from the Q-series, angular structure from QM5, dynamics from QM4, and the full quantum number structure from all three.*

Proposition 7.8 (Orthonormality and completeness of hydrogenic eigenstates)**.** *The hydrogenic eigenstates Eq.* (62) *satisfy:*

(i) Orthonormality: $\langle \Psi_{n'\ell' m'}, \Psi_{n\ell m} \rangle_{\mathcal{H}} = \delta_{nn'}\delta_{\ell\ell'}\delta_{mm'}$.

(ii) Completeness in the discrete sector: $\sum_{n=1}^{\infty} \sum_{\ell=0}^{n-1} \sum_{m=-\ell}^{\ell} |\Psi_{n\ell m}\rangle\langle\Psi_{n\ell m}| = \hat{P}_{\mathrm{disc}}$*, where* $\hat{P}_{\mathrm{disc}}$ *is the projection onto the discrete-spectrum subspace of* $\mathcal{H}$*.*

Proof. Part (i): For the angular factor: orthonormality of the spherical harmonics (Proposition 6.7 (i)) gives $\int_{S^2} \overline{Y_{\ell'}^{m'}} Y_\ell^m \, \mathrm{d}\Omega = \delta_{\ell\ell'}\delta_{mm'}$. For the radial factor: the radial functions $R_{n\ell}(r)$ are orthogonal with respect to the radial inner product $\int_0^\infty r^2 |R_{n'\ell}(r)|^2 |R_{n\ell}(r)|^2 \, \mathrm{d}r$; this is established by self-adjointness of the radial Hamiltonian operator for each fixed ℓ [2]. Combining the angular and radial orthonormality: $\langle \Psi_{n'\ell' m'}, \Psi_{n\ell m} \rangle_{\mathcal{H}} = \delta_{nn'}\delta_{\ell\ell'}\delta_{mm'}$.

Part (ii): Follows from the spectral theorem applied to the discrete part of $\sigma(\hat{H}_{\mathrm{H}})$ (QM1 Theorem 6.1 and QM4 Proposition 4.4): the discrete eigenstates form a complete basis for the discrete-spectrum subspace. □

8 Angular Momentum Sum Rules and Uncertainty Structure

The present section derives the $\hat{L}^2$ sum rule for angular momentum eigenstates, a result promised in QM3 Remark 7.3 and now established with the full spectral theory of Secs. 4 and 5 available. The section then verifies the consistency of the Robertson uncertainty bounds of QM3 Proposition 7.1 against the explicit standard deviations of the angular momentum eigenstates, identifies the conditions under which the Robertson bound is saturated, and records the full uncertainty structure that the sum rule imposes on the family of $|\, \ell, \, m \,\rangle$ states.

8.1 Expectation Values in Angular Momentum Eigenstates

The explicit eigenvalue equations Eq. (39) immediately yield the expectation values of $\hat{L}^2$, $\hat{L}_3$, and their squares. The expectation values of the transverse components $\hat{L}_1$ and $\hat{L}_2$ require the ladder operator structure.

Lemma 8.1 (Expectation values in $|\,\ell,\,m\,\rangle$)**.** *For the normalized eigenstate* $|\,\ell,\,m\,\rangle$*:*

$$\langle \hat{L}^2 \rangle = \ell(\ell+1)\Phi_0^2, \tag{63}$$

$$\langle \hat{L}_3 \rangle = m\Phi_0, \tag{64}$$

$$\langle \hat{L}_3^2 \rangle = m^2\Phi_0^2, \tag{65}$$

$$\langle \hat{L}_1 \rangle = 0, \tag{66}$$

$$\langle \hat{L}_2 \rangle = 0. \tag{67}$$

Proof. Equations (63)–(65) follow directly from the eigenvalue equations $\hat{L}^2|\,\ell,\,m\,\rangle = \ell(\ell+1)\Phi_0^2|\,\ell,\,m\,\rangle$ and $\hat{L}_3|\,\ell,\,m\,\rangle = m\Phi_0|\,\ell,\,m\,\rangle$ by taking the inner product with $|\,\ell,\,m\,\rangle$.

For Eq. (66): write $\hat{L}_1 = (\hat{L}_+ + \hat{L}_-)/2$. Then

$$\langle \hat{L}_1 \rangle = \frac{1}{2}\left(\left\langle \ell, m, \hat{L}_+|\ell, m\right\rangle + \left\langle \ell, m, \hat{L}_-|\ell, m\right\rangle\right).$$

By Proposition 5.7, $\hat{L}_+|\,\ell,\,m\,\rangle \propto |\,\ell,\,m+1\,\rangle$ (for $m < \ell$) or zero (for $m = \ell$), and $\hat{L}_-|\,\ell,\,m\,\rangle \propto |\,\ell,\,m-1\,\rangle$ (for $m > -\ell$) or zero (for $m = -\ell$). In all cases, the resulting states are orthogonal to $|\,\ell,\,m\,\rangle$ by the orthonormality of the basis (Theorem 5.5), so both inner products vanish and $\langle \hat{L}_1 \rangle = 0$. The argument for $\langle \hat{L}_2 \rangle = 0$ is identical using $\hat{L}_2 = (\hat{L}_+ - \hat{L}_-)/(2\mathrm{i})$. □

Remark 8.2. *The vanishing of $\langle \hat{L}_1 \rangle$ and $\langle \hat{L}_2 \rangle$ in any $\hat{L}_3$-eigenstate reflects the rotational symmetry of the angular momentum algebra: a state with a definite z-component of angular momentum has no preferred direction in the x-y plane, so the mean transverse angular momentum must vanish. This is the angular momentum analogue of the statement that a plane wave (a definite-momentum state) has zero mean position: the state is maximally spread in the conjugate variable. Notably, the vanishing of $\langle \hat{L}_1 \rangle$ and $\langle \hat{L}_2 \rangle$ does not mean the transverse angular momentum is absent — the standard deviations ΔL_1 and ΔL_2 are in general non-zero, as established in Theorem 8.3 below.*

8.2 The $\hat{L}^2$ Sum Rule

With the expectation values of Lemma 8.1 in hand, the standard deviations of all three angular momentum components in the eigenstate $|\,\ell,\,m\,\rangle$ are determined.

Theorem 8.3 ($\hat{L}^2$ sum rule for angular momentum eigenstates)**.** *For the normalized eigenstate $|\,\ell,\,m\,\rangle$ of $\hat{L}^2$ and $\hat{L}_3$, the standard deviations of the three angular momentum components satisfy:*

$$(\Delta L_3)^2 = \langle \hat{L}_3^2 \rangle - \langle \hat{L}_3 \rangle^2 = 0, \tag{68}$$

$$(\Delta L_1)^2 = (\Delta L_2)^2 = \frac{\ell(\ell+1) - m^2}{2}\,\Phi_0^2, \tag{69}$$

$$\sum_{j=1}^{3}\left[(\Delta L_j)^2 + \langle \hat{L}_j \rangle^2\right] = \ell(\ell+1)\Phi_0^2, \tag{70}$$

where Eq. (70) *is the* $\hat{L}^2$ sum rule.

Proof. Equation (68)*:* Since $|\,\ell,\,m\,\rangle$ is an eigenstate of $\hat{L}_3$ with eigenvalue $m\Phi_0$: $(\Delta L_3)^2 = \langle \hat{L}_3^2 \rangle - \langle \hat{L}_3 \rangle^2 = m^2\Phi_0^2 - m^2\Phi_0^2 = 0$.

Equation (69)*:* From $\hat{L}^2 = \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2$ and the eigenvalue of $\hat{L}^2$:

$$\langle \hat{L}_1^2 \rangle + \langle \hat{L}_2^2 \rangle = \langle \hat{L}^2 \rangle - \langle \hat{L}_3^2 \rangle = \ell(\ell+1)\Phi_0^2 - m^2\Phi_0^2 = [\ell(\ell+1) - m^2]\Phi_0^2. \tag{71}$$

To show $\langle \hat{L}_1^2 \rangle = \langle \hat{L}_2^2 \rangle$, note that the eigenstate $|\,\ell,\, m\,\rangle$ is invariant in expectation values under the discrete rotation $\hat{x}^1 \to \hat{x}^2$, $\hat{x}^2 \to -\hat{x}^1$ (rotation by $\pi/2$ about the z-axis), which maps $\hat{L}_1 \to \hat{L}_2$ and $\hat{L}_2 \to -\hat{L}_1$. Since $\hat{L}_3$-eigenstates have azimuthal symmetry under rotation about the z-axis (the azimuthal factor $\mathrm{e}^{\,im\varphi}$ acquires only a phase under such rotation, which cancels in expectation values of Hermitian operators), the expectation values of $\hat{L}_1^2$ and $\hat{L}_2^2$ are equal. Therefore $\langle \hat{L}_1^2 \rangle = \langle \hat{L}_2^2 \rangle = [\ell(\ell+1) - m^2]\Phi_0^2/2$. Since $\langle \hat{L}_1 \rangle = \langle \hat{L}_2 \rangle = 0$ (Lemma 8.1): $(\Delta L_1)^2 = \langle \hat{L}_1^2 \rangle - \langle \hat{L}_1 \rangle^2 = \langle \hat{L}_1^2 \rangle = [\ell(\ell+1) - m^2]\Phi_0^2/2$, and similarly for $(\Delta L_2)^2$.

The sum rule Eq. (70)*:*

$$\begin{aligned}\sum_{j=1}^{3}[(\Delta L_j)^2 + \langle \hat{L}_j \rangle^2] &= [(\Delta L_1)^2 + 0] + [(\Delta L_2)^2 + 0] + [0 + m^2\Phi_0^2] \\ &= \frac{\ell(\ell+1) - m^2}{2}\Phi_0^2 + \frac{\ell(\ell+1) - m^2}{2}\Phi_0^2 + m^2\Phi_0^2 \\ &= [\ell(\ell+1) - m^2]\Phi_0^2 + m^2\Phi_0^2 = \ell(\ell+1)\Phi_0^2,\end{aligned}$$

using Eqs. (66)–(67) and (68)–(69). □

Remark 8.4. *The sum rule Eq.* (70) *expresses the conservation of total angular momentum in a geometric form. The quantity* $\sum_j[(\Delta L_j)^2 + \langle \hat{L}_j \rangle^2]$ *is the sum of the squared standard deviations and squared means of all three angular momentum components, which equals the expected value of* $\sum_j \hat{L}_j^2 = \hat{L}^2$*. In the* $|\,\ell,\, m\,\rangle$ *eigenstate, this sum is exactly* $\ell(\ell+1)\Phi_0^2$*, independent of* m*. The* m*-dependence of the individual contributions—smaller transverse spread* $\Delta L_1 = \Delta L_2$ *for larger* $|m|$*, exactly compensated by larger* z*-mean* $\langle \hat{L}_3 \rangle = m\Phi_0$*—reflects the geometric constraint that increasing the* z*-component of the angular momentum reduces the available transverse spread within the total angular momentum budget* $\ell(\ell+1)\Phi_0^2$*.*

8.3 Consistency with the Robertson Uncertainty Bounds

The explicit standard deviations from Theorem 8.3 can now be checked against the Robertson uncertainty bounds established in QM3 Proposition 7.1. This verification closes the forward reference in QM3 and establishes when the Robertson bounds are tight.

Proposition 8.5 (Consistency and saturation of Robertson bounds for angular momentum eigenstates)**.** *For the eigenstate* $|\,\ell,\, m\,\rangle$*, the Robertson uncertainty relations of QM3 Proposition 7.1 are satisfied. The* (L_1, L_2)*-bound with the* L_3 *right-hand side is:*

$$\Delta L_1 \cdot \Delta L_2 = \frac{\ell(\ell+1) - m^2}{2}\,\Phi_0^2 \geq \frac{|m|}{2}\,\Phi_0^2 = \frac{\Phi_0}{2}\big|\langle \hat{L}_3 \rangle\big|, \tag{72}$$

with equality if and only if $|m| = \ell$ *(the maximally polarized* stretched states $|\,\ell,\, \pm\ell\,\rangle$*). By cyclic symmetry, the* (L_2, L_3) *and* (L_3, L_1) *bounds are:*

$$\Delta L_2 \cdot \Delta L_3 = 0 \geq 0 = \frac{\Phi_0}{2}\big|\langle \hat{L}_1 \rangle\big|, \tag{73}$$

$$\Delta L_3 \cdot \Delta L_1 = 0 \geq 0 = \frac{\Phi_0}{2}\big|\langle \hat{L}_2 \rangle\big|, \tag{74}$$

both trivially satisfied since $\Delta L_3 = 0$ *and* $\langle \hat{L}_1 \rangle = \langle \hat{L}_2 \rangle = 0$*.*

Proof. Equation (72)*:* From Theorem 8.3, $\Delta L_1 = \Delta L_2 = \Phi_0\sqrt{[\ell(\ell+1) - m^2]/2}$. Therefore $\Delta L_1 \cdot \Delta L_2 = [\ell(\ell+1) - m^2]\Phi_0^2/2$. The Robertson bound is $(\Phi_0/2)|\langle \hat{L}_3\rangle| = |m|\Phi_0^2/2$. The inequality $[\ell(\ell+1) - m^2]/2 \geq |m|/2$ is equivalent to $\ell(\ell+1) \geq m^2 + |m| = |m|(|m|+1)$. Since $|m| \leq \ell$ and the function $t(t+1)$ is increasing for $t \geq 0$, we have $|m|(|m|+1) \leq \ell(\ell+1)$, confirming the inequality. Equality holds iff $|m|(|m|+1) = \ell(\ell+1)$, which (for non-negative integers) holds iff $|m| = \ell$.

Equations (73) *and* (74)*:* $\Delta L_3 = 0$ (Theorem 8.3), and $\langle \hat{L}_1\rangle = \langle \hat{L}_2\rangle = 0$ (Lemma 8.1), so both sides of each inequality are zero; both are trivially satisfied. □

Remark 8.6. *The saturation condition $|m| = \ell$ identifies the* stretched states $|\,\ell,\,\ell\,\rangle$ *and* $|\,\ell,\,-\ell\,\rangle$ *as the angular momentum eigenstates for which the Robertson bound Eq.* (72) *is saturated. For the stretched state $|\,\ell,\,\ell\,\rangle$: $\Delta L_1 = \Delta L_2 = \Phi_0\sqrt{\ell}/\sqrt{2}$ (from Theorem 8.3 with $m = \ell$, giving $[\ell(\ell+1) - \ell^2]/2 = \ell/2$), and the Robertson bound is $(\Phi_0/2)|\langle \hat{L}_3\rangle| = \ell\Phi_0^2/2$. The product is $\Delta L_1 \cdot \Delta L_2 = (\Phi_0^2/2)\cdot\ell = \ell\Phi_0^2/2$, exactly equal to the bound. Geometrically, the stretched states are those in which the angular momentum vector is maximally aligned with the z-axis, minimizing the transverse spread consistent with the total angular momentum budget.*

Remark 8.7. *The Robertson inequality is non-saturated for all eigenstates $|\,\ell,\,m\,\rangle$ with $|m| < \ell$ (the non-stretched states). For example, the state $|\,\ell,\,0\,\rangle$ (maximum total angular momentum, zero z-component) has $\Delta L_1 \cdot \Delta L_2 = \ell(\ell+1)\Phi_0^2/2$, while the Robertson bound is zero (since $\langle \hat{L}_3\rangle = 0$). The gap between the actual product and the Robertson bound grows as $|m|$ decreases from ℓ to* 0*. This illustrates the general feature noted in QM3 Sec. 9: the Robertson inequality provides a necessary condition on the uncertainty product in terms of the commutator expectation value, but the commutator-based bound may be far from tight for specific states. The sum rule Eq.* (70) *provides the additional constraint beyond Robertson that fully determines the individual standard deviations ΔL_1 and ΔL_2 for angular momentum eigenstates.*

8.4 The Uncertainty Structure Across the Multiplet

The sum rule and the Robertson consistency check together determine the complete uncertainty structure across the ℓ-multiplet: the family of $2\ell+1$ states $\{|\,\ell,\,m\,\rangle : m = -\ell, \ldots, +\ell\}$.

Corollary 8.8 (Uncertainty structure of the ℓ-multiplet)**.** *For the ℓ-multiplet $\{|\,\ell,\,m\,\rangle\}$:*

(i) $\Delta L_3 = 0$ *for all states in the multiplet: each state has definite z-component of angular momentum.*

(ii) $\Delta L_1 = \Delta L_2 = \Phi_0\sqrt{[\ell(\ell+1) - m^2]/2}$ *for each m. The transverse spread is maximum for $m = 0$ (the equatorial state) and minimum (but non-zero for $\ell > 0$) for $|m| = \ell$ (the stretched states).*

(iii) *The Robertson bound Eq.* (72) *is saturated only for the two stretched states $|\,\ell,\,\pm\ell\,\rangle$.*

(iv) *The sum rule Eq.* (70) *is constant across the multiplet: every state $|\,\ell,\,m\,\rangle$, for any m, satisfies $\sum_j[(\Delta L_j)^2 + \langle \hat{L}_j\rangle^2] = \ell(\ell+1)\Phi_0^2$.*

Proof. All four parts follow directly from Theorem 8.3 and Proposition 8.5. Part (i) is Eq. (68). Part (ii) is Eq. (69) with the explicit values at $m = 0$ (giving $\ell(\ell+1)/2$ per component) and $|m| = \ell$ (giving $\ell/2$ per component, since $\ell(\ell+1) - \ell^2 = \ell$). Part (iii) is the saturation condition of Proposition 8.5. Part (iv) is the sum rule Eq. (70). □

Remark 8.9. *Corollary 8.8 completes the program of QM3 Remark 7.3, which stated the sum rule* $\sum_j[(\Delta L_j)^2 + \langle \hat{L}_j\rangle^2] = \ell(\ell+1)\Phi_0^2$ *as a result to be established in QM5. The derivation given here requires, in order: the angular momentum commutation algebra of Theorem 3.1 (for the vanishing of transverse means via ladder operators), the spectral theorem of QM1 applied to* $\hat{L}^2$ *and* $\hat{L}_3$ *(for the eigenvalue equations), and the azimuthal symmetry of the* $\hat{L}_3$*-eigenstates (for the equality* $\Delta L_1 = \Delta L_2$*). None of these inputs was available in QM3; the forward reference was therefore necessary and is now closed. The sum rule, together with the Robertson uncertainty relations of QM3, provides a complete characterization of the angular momentum uncertainty structure for eigenstates: the Robertson relations give lower bounds on products of standard deviations, while the sum rule gives the sum of all squared standard deviations and squared means. Together these two constraints uniquely determine* ΔL_1*,* ΔL_2*, and* ΔL_3 *for every* $|\,\ell, m\,\rangle$ *state.*

9 Interpretive Clarifications and Scope

The present section collects the interpretive constraints that govern the angular momentum analysis of the preceding sections and states them as a unified set of boundary conditions on the NUVO account of rotational transport structure. Three items are addressed: the interpretation of the quantum numbers ℓ and m as holonomy invariants rather than classification labels, the status of the spherical harmonics as geometric transport eigenstates rather than imported special functions, and the scope of the present construction relative to the remainder of the QM-series. These constraints protect the logical integrity of the series by preventing the importation of interpretive content that has not been derived within the scalar–conformal NUVO framework.

9.1 Quantum Numbers as Holonomy Invariants

In the standard formulation of quantum mechanics, the quantum numbers ℓ and m are introduced as integer labels that classify angular momentum states. Their integer character is established either by the regularity of the solutions to the associated Legendre equation (in the analytic approach) or by an ad hoc argument excluding half-integers after the algebraic ladder analysis (in the algebraic approach). In neither approach is the physical origin of the integrality made explicit.

In the NUVO framework, the quantum numbers have a precise geometric origin. The magnetic quantum number m is the winding number of the azimuthal transport closure path: the integer that counts how many times the transport phase completes a full cycle $2\pi\Phi_0$ as the azimuthal angle φ advances through one period $[0, 2\pi)$. This is the holonomy quantization condition of the Q-series applied to the azimuthal transport structure, established in Theorem 5.2. The condition is not an additional postulate but the same holonomy principle that quantized the hydrogenic energy levels in the Q-series, now applied to a different closed path.

The orbital quantum number ℓ is the maximum azimuthal winding number accessible within a given total angular momentum multiplet. Its integer character and non-negativity follow from combining the holonomy condition (which requires $m \in \mathbb{Z}$) with the algebraic termination of the ladder sequence (which requires $\mu_{\max} = \ell\Phi_0$ for integer $\ell \geq 0$), as established in Theorem 5.5. The quantum number ℓ is not introduced as a label; it is derived as the algebraic consequence of the holonomy selection acting on the ladder structure.

Remark 9.1. *The three quantum numbers of the hydrogenic system have the following holonomy origins within the NUVO program:*

Quantum number	*Physical origin*	*NUVO derivation*
n (principal)	*Radial closure winding number*	*Q-series holonomy quantization*
ℓ (orbital)	*Maximum azimuthal winding number*	*Ladder algebra + azimuthal holonomy*
m (magnetic)	*Azimuthal closure winding number*	*QM5 holonomy (Theorem 5.2)*

All three quantum numbers arise from the same geometric mechanism — the holonomy quantization of transport closure paths — applied at different levels of the transport structure. The radial, azimuthal, and algebraic aspects of the quantization are not independent postulates but manifestations of a single principle: on a closed transport path, the accumulated phase must be an integer multiple of $2\pi\Phi_0$.

9.2 Spherical Harmonics as Geometric Transport Eigenstates

The spherical harmonics $Y_\ell^m(\theta, \varphi)$ established in Theorem 6.4 are, in the standard mathematical literature, a well-known family of special functions defined as solutions of the Laplace-Beltrami eigenvalue problem on S^2. Their properties — orthonormality, completeness, parity, the addition theorem, and the connection to associated Legendre polynomials — are classical results whose derivation predates quantum mechanics.

In the NUVO framework, the spherical harmonics are not imported as known functions. They are derived as the position-space realization of the abstract angular momentum eigenstates $|\,\ell,\, m\,\rangle$, whose existence and quantum number labeling are established algebraically and holonomically in Secs. 3–5 prior to and independently of any coordinate representation. The derivation in Sec. 6 then identifies these abstract eigenstates with specific functions on S^2 by solving the angular eigenvalue equations in spherical coordinates. The spherical harmonics emerge from this identification; they are not assumed.

Several properties of the spherical harmonics that are classically derived from their analytic definition are, in the NUVO framework, consequences of prior algebraic or spectral results.

Orthonormality (Proposition 6.7 (i)): follows from the self-adjointness of $\hat{L}^2$ and $\hat{L}_3$ and the distinctness of their joint eigenvalues — not from the orthogonality theory of the associated Legendre functions as such.

Completeness (Proposition 6.7 (ii)): follows from the spectral theorem of QM1 applied to the commuting pair $(\hat{L}^2, \hat{L}_3)$ on $L^2(S^2)$ — not from the theory of Fourier series on S^2 as such.

Parity (Proposition 6.7 (iii)): follows from the behavior of the closure state under spatial inversion $\mathbf{x} \to -\mathbf{x}$, which is a symmetry operation on the scalar–conformal transport system.

The classical results of the theory of spherical harmonics are thereby re-derived as consequences of the NUVO transport closure geometry and its associated operator algebra, rather than being imported from the theory of special functions. The external references to Edmonds [1] and Galindo-Pascual [2] serve to verify agreement with the classical literature and to supply the Rodrigues formula for the associated Legendre functions; they are not the primary logical inputs to the derivations.

Remark 9.2. *The present paper derives the spherical harmonics as angular closure eigenstates on* S^2*. Their appearance throughout the subsequent QM-series papers — as angular factors in the three-dimensional harmonic oscillator eigenstates (QM6), as basis functions for multi-particle angular momentum states (QM7), as the angular components of spin-orbit coupled states in QM8, and as partial wave basis functions in scattering theory (QM10) — follows from the completeness of the spherical harmonics on* $L^2(S^2)$ *established here. In each subsequent application, the spherical* *harmonics appear not as imported special functions but as the previously derived eigenstates of the rotational transport structure.*

9.3 The Integer versus Half-Integer Distinction

The distinction between integer orbital angular momentum (the content of the present paper) and half-integer spin angular momentum (the content of QM8) is a structural distinction within the NUVO program, not an ad hoc exclusion.

The algebraic analysis of Sec. 4 establishes that $\mu_{\max}/\Phi_0 \in \frac{1}{2}\mathbb{Z}$: the maximum $\hat{L}_3$ eigenvalue is a half-integer or integer multiple of Φ_0. The holonomy condition of Sec. 5 restricts $\mu/\Phi_0 \in \mathbb{Z}$ for admissible transport closure states on $\mathcal{H}$, thereby selecting the integer case. This restriction arises because the closure state Ψ is required to be single-valued on $\mathbb{R}^3$: after the azimuthal angle φ completes one full period $[0, 2\pi)$, the state must return to its initial value. Single-valuedness on $\mathbb{R}^3$ is a requirement of the transport closure framework because Ψ encodes the closure density $\rho = |\Psi|^2$, which must be a well-defined function on physical space.

In QM8, the transport closure structure is extended to the double cover SU(2) of the rotation group SO(3). On the double cover, the physical space of the transport is not $\mathbb{R}^3$ with single-valued functions but a bundle over $\mathbb{R}^3$ with a non-trivial topology. The single-valuedness condition is relaxed to $\Psi(\varphi + 4\pi) = \Psi(\varphi)$ (a 4π period), which selects $\mu/\Phi_0 \in \frac{1}{2}\mathbb{Z}$. The half-integer quantum numbers of spin therefore arise from the same holonomy quantization principle as the integer quantum numbers of orbital angular momentum, applied to a transport structure with a topologically non-trivial base space.

The distinction between integer and half-integer is thus a distinction between two topological structures:

- Integer orbital angular momentum: single-valued transport closure on SO(3), holonomy period 2π, $m \in \mathbb{Z}$.
- Half-integer spin: transport closure on SU(2), holonomy period 4π, $j \in \frac{1}{2}\mathbb{Z}$.

No ad hoc exclusion of half-integers from the orbital case is needed; the single-valuedness condition on $\mathbb{R}^3$ provides the exclusion naturally.

9.4 Scope of the Present Construction

The present paper establishes the complete angular momentum algebra, spectrum, spherical harmonic eigenstates, hydrogenic quantum number structure, and sum rules for the orbital angular momentum of the scalar–conformal transport system. The following results are established in the present paper and are available as inputs to subsequent QM-series papers.

Algebraic results: The commutation algebra $[\hat{L}_j, \hat{L}_k] = \mathrm{i}\Phi_0\, \epsilon_{jkl}\hat{L}_l$ and $[\hat{L}^2, \hat{L}_j] = 0$ (Theorems 3.1 and 3.3); the ladder operator commutation relations and their action on eigenstates (Lemma 4.3 and Proposition 4.4); the ladder product identities $\hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_3^2 - \Phi_0\, \hat{L}_3$ (Lemma 4.8).

Spectral results: The complete joint spectrum of $\hat{L}^2$ and $\hat{L}_3$ (Theorem 5.5); the matrix elements of the ladder operators in the $|\,\ell,\, m\,\rangle$ basis (Proposition 5.7); the resolution of the identity on $\mathcal{H}_{\text{ang}}$ (Eq. (42)).

Position-space results: The form of $\hat{L}^2$ as $-\Phi_0^2\Delta_{S^2}$ in spherical coordinates (Proposition 6.1); the spherical harmonics Y_ℓ^m as angular closure eigenstates with their normalization, orthonormality, completeness, parity, and complex conjugate properties (Theorem 6.4 and Proposition 6.7).

Physical sector results: The separation of variables for the hydrogenic Hamiltonian (Proposition 7.1); the constraint $\ell \leq n-1$ and the n^2-fold degeneracy (Theorem 7.5); the orthonormality and completeness of the hydrogenic eigenstates (Proposition 7.8).

Uncertainty results: The $\hat{L}^2$ sum rule for angular momentum eigenstates (Theorem 8.3); the consistency and saturation of the Robertson bounds (Proposition 8.5 and Corollary 8.8).

The following topics are deferred to subsequent papers.

Angular momentum addition and Clebsch-Gordan coefficients. The present paper treats the angular momentum of a single transport closure sector. The addition of angular momenta for composite systems — $\mathbf{L}_{\text{tot}} = \mathbf{L}_1 + \mathbf{L}_2$ — and the Clebsch-Gordan decomposition of the tensor product $\mathcal{H}^{(1)}_{\text{ang}} \otimes \mathcal{H}^{(2)}_{\text{ang}}$ into irreducible representations of SO(3) are developed in QM7. The Clebsch-Gordan coefficients $\langle j_1, m_1; j_2, m_2 | J, M\rangle$ that express coupled angular momentum states in terms of uncoupled states require the multi-particle Hilbert space of QM7 and are outside the scope of the present paper.

Spin angular momentum. The double-cover holonomy structure that gives rise to half-integer spin quantum numbers and the spinor representation of the rotation group are developed in QM8. The present paper establishes the integer-holonomy case only; QM8 builds on the same algebraic and holonomic framework to derive the spin structure.

Tensor operators and the Wigner-Eckart theorem. The systematic treatment of operators that transform as irreducible representations of SO(3) (tensor operators) and the Wigner-Eckart theorem that factors their matrix elements into a geometric (Clebsch-Gordan) part and a dynamical (reduced matrix element) part are beyond the scope of the present paper. This material is relevant to the selection rules for transition matrix elements in QM10 and is recorded as a structural extension of the series.

The SO(4) *accidental degeneracy.* The additional n-fold degeneracy of the hydrogen energy levels (beyond the $2\ell+1$ rotational degeneracy) was noted in Remark 7.6 as arising from the SO(4) symmetry of the Coulomb problem. The derivation of this symmetry group and the associated Runge-Lenz vector operator, and the demonstration that the full n^2 degeneracy is generated by the SO(4) algebra rather than the SO(3) algebra alone, are deferred as a structural extension of the hydrogenic analysis.

Relativistic angular momentum. The covariant generalization of the angular momentum operators to the relativistic transport sector, including the orbital angular momentum in the Dirac equation and the covariant spin-orbit coupling, is developed in QM11 and the RQM-series.

10 Conclusion

10.1 Summary of Results

The present paper has derived the complete angular momentum algebra, spectrum, and eigenstate structure of the scalar–conformal NUVO transport closure system from the canonical commutation relations of QM1 and the holonomy quantization principle of the Q-series, without postulating the quantum numbers ℓ and m, the commutation algebra, or the spherical harmonics. The eighteen principal results are as follows.

The angular momentum commutation algebra (Theorem 3.1). The relation $[\hat{L}_j, \hat{L}_k] = \mathrm{i}\Phi_0\, \epsilon_{jkl}\hat{L}_l$ is derived by explicit computation from the three canonical commutation relations $[\hat{x}^j, \hat{p}_k] = \mathrm{i}\Phi_0\, \delta^j{}_k$, $[\hat{x}^j, \hat{x}^k] = 0$, and $[\hat{p}_j, \hat{p}_k] = 0$ of QM1 Proposition 5.4. The computation expands the bracket $[\hat{L}_1, \hat{L}_2]$ into sixteen terms via the Leibniz rule, of which fourteen vanish by the commutativity of position operators, of momentum operators, and of position and momentum operators in different directions; the two surviving terms combine to give $\mathrm{i}\Phi_0\,\hat{L}_3$. The remaining two cyclic relations follow by relabelling indices.

Commutativity of $\hat{L}^2$ with each component (Theorem 3.3). The relation $[\hat{L}^2, \hat{L}_j] = 0$ is derived from the commutation algebra of Theorem 3.1 by the Leibniz rule: the contributions from $[\hat{L}_1^2, \hat{L}_3]$ and $[\hat{L}_2^2, \hat{L}_3]$ each produce an anti-commutator $\{\hat{L}_1, \hat{L}_2\}$ with opposite signs, and the cancellation reflects the rotational isotropy of $\hat{L}^2$. The pair $(\hat{L}^2, \hat{L}_3)$ constitutes a complete set of commuting observables for the angular sector.

Self-adjointness and non-negativity of $\hat{L}^2$ (Proposition 3.6). Each $\hat{L}_j$ is essentially self-adjoint on $\mathcal{H}$ (recalled from QM4 Proposition 7.2). The non-negativity $\left\langle \Psi, \hat{L}^2\Psi \right\rangle_{\mathcal{H}} = \sum_j \left\| \hat{L}_j\Psi \right\|_{\mathcal{H}}^2 \geq 0$ implies $\mu^2 \leq \lambda$ for any joint eigenstate with $\hat{L}^2$ eigenvalue λ and $\hat{L}_3$ eigenvalue μ, bounding the $\hat{L}_3$ spectrum and forcing the ladder sequence to terminate.

Ladder operators and their commutation relations (Definition 4.1 and Lemma 4.3). The raising and lowering operators $\hat{L}_+ = \hat{L}_1 + \mathrm{i}\hat{L}_2$ and $\hat{L}_- = \hat{L}_1 - \mathrm{i}\hat{L}_2$ satisfy $[\hat{L}_3, \hat{L}_+] = \Phi_0\,\hat{L}_+$, $[\hat{L}_3, \hat{L}_-] = -\Phi_0\,\hat{L}_-$, $[\hat{L}_+, \hat{L}_-] = 2\Phi_0\,\hat{L}_3$, and $[\hat{L}^2, \hat{L}_+] = [\hat{L}^2, \hat{L}_-] = 0$. These are derived from Theorem 3.1 by direct substitution.

Raising and lowering action on eigenstates (Proposition 4.4). The operator $\hat{L}_+$ maps a joint eigenstate with $\hat{L}_3$ eigenvalue μ to one with eigenvalue $\mu + \Phi_0$, preserving the $\hat{L}^2$ eigenvalue; $\hat{L}_-$ maps to $\mu - \Phi_0$. Repeated application generates the complete multiplet from any single member.

Algebraic spectral constraints from ladder termination (Theorem 4.6 and Lemma 4.8). The ladder product identities $\hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_3^2 - \Phi_0\,\hat{L}_3$ and $\hat{L}_+\hat{L}_- = \hat{L}^2 - \hat{L}_3^2 + \Phi_0\,\hat{L}_3$, applied at the termination points $\hat{L}_+\Psi_{\max} = 0$ and $\hat{L}_-\Psi_{\min} = 0$, yield $\mu_{\min} = -\mu_{\max}$ and $\lambda = \mu_{\max}(\mu_{\max} + \Phi_0)$. The algebraic analysis alone admits both integer and half-integer values of $\mu_{\max}/\Phi_0$.

Integer quantization of m from holonomy (Theorem 5.2). The representation $\hat{L}_3 = -\mathrm{i}\Phi_0\,\partial_\varphi$ in spherical coordinates (Lemma 5.1) and the single-valuedness condition $\Psi(\varphi+2\pi) = \Psi(\varphi)$ require $\mathrm{e}^{2\pi\mathrm{i}\mu/\Phi_0} = 1$, selecting $\mu = m\Phi_0$ for $m \in \mathbb{Z}$. This is the Q-series holonomy quantization applied to the azimuthal transport closure path.

Complete joint spectrum of $\hat{L}^2$ and $\hat{L}_3$ (Theorem 5.5). Combining the algebraic constraints with integer holonomy gives $\sigma(\hat{L}^2) = \{\ell(\ell+1)\Phi_0^2 : \ell = 0, 1, 2, \ldots\}$ and $\sigma(\hat{L}_3)|_\ell = \{m\Phi_0 : m = -\ell, \ldots, +\ell\}$, with $2\ell + 1$-fold degeneracy in m for each ℓ. The half-integer case is excluded by the 2π holonomy period of the orbital transport closure on SO(3).

Matrix elements of the ladder operators (Proposition 5.7). $\hat{L}_+|\,\ell, m\,\rangle = \Phi_0\sqrt{\ell(\ell+1) - m(m+1)}\,|\,\ell, m+1\,\rangle$ and the analogous lowering formula, derived by computing $\left\| \hat{L}_+|\,\ell, m\,\rangle \right\|^2$ via the ladder product identity. The square-root factors vanish at the termination points $m = \pm\ell$, confirming the ladder structure.

$\hat{L}^2$ as the Laplace–Beltrami operator (Proposition 6.1). In spherical coordinates, $\hat{L}^2 = -\Phi_0^2\Delta_{S^2}$, where Δ_{S^2} is the Laplace–Beltrami operator on the unit sphere. This separates the kinetic operator into radial and centrifugal parts: $\hat{T} = -(\Phi_0^2/2m)(1/r^2)\partial_r(r^2\partial_r) + \hat{L}^2/(2mr^2)$.

Spherical harmonics as angular closure eigenstates (Theorem 6.4). The position-space eigenfunctions of $\hat{L}^2$ and $\hat{L}_3$ on S^2 are the spherical harmonics $Y_\ell^m(\theta, \varphi) = \mathcal{N}_{\ell m} P_\ell^{|m|}(\cos\theta)\mathrm{e}^{\mathrm{i}m\varphi}$, derived as solutions of the angular eigenvalue equations: the azimuthal equation gives $\mathrm{e}^{\mathrm{i}m\varphi}$ from Theorem 5.2, and the polar equation gives the associated Legendre polynomial $P_\ell^{|m|}$ from the regularity condition at the poles of S^2.

Orthonormality, completeness, parity, and complex conjugate of spherical harmonics (Proposition 6.7). Orthonormality follows from self-adjointness of $\hat{L}^2$ and $\hat{L}_3$; completeness on $L^2(S^2)$ follows from the spectral theorem of QM1; parity $(-1)^\ell$ under $\mathbf{x} \to -\mathbf{x}$ follows from the coordinate transformation; and $\overline{Y_\ell^m} = (-1)^m Y_\ell^{-m}$ from the Condon-Shortley convention.

Separation of variables for the hydrogenic Hamiltonian (Proposition 7.1). The decomposition $\hat{H}_{\rm H} = -(\Phi_0^2/2m)(1/r^2)\partial_r(r^2\partial_r) + \hat{L}^2/(2mr^2) - e^2/(4\pi\epsilon_0 r)$ and the $\hat{L}^2$-eigenstate property of Y_ℓ^m reduce the hydrogenic eigenvalue equation to the purely radial equation Eq. (58) with ℓ as a parameter.

Hydrogenic quantum number structure and n^2 degeneracy (Theorem 7.5). The constraint $\ell \le n-1$ from the radial bound state analysis and the $2\ell+1$ values of m for each ℓ give the total degeneracy $g_n = \sum_{\ell=0}^{n-1}(2\ell+1) = n^2$. The full n-ℓ-m quantum number labeling of the hydrogen spectrum is thereby established.

Orthonormality and completeness of hydrogenic eigenstates (Proposition 7.8). The family $\{\Psi_{n\ell m} = R_{n\ell}(r)Y_\ell^m(\theta,\varphi)\}$ is complete and orthonormal in the discrete-spectrum subspace of $\mathcal{H}$ by the spectral theorem applied to $\hat{H}_{\rm H}$.

The $\hat{L}^2$ sum rule for angular momentum eigenstates (Theorem 8.3). For $|\,\ell,\,m\,\rangle$: $\Delta L_3 = 0$, $\Delta L_1 = \Delta L_2 = \Phi_0\sqrt{[\ell(\ell+1)-m^2]/2}$, and the sum rule $\sum_j[(\Delta L_j)^2 + \langle\hat{L}_j\rangle^2] = \ell(\ell+1)\Phi_0^2$ holds for all m in the multiplet. This result closes the forward reference of QM3 Remark 7.3.

Consistency and saturation of Robertson bounds (Proposition 8.5 and Corollary 8.8). The Robertson bound $\Delta L_1 \cdot \Delta L_2 \ge (\Phi_0/2)|\langle\hat{L}_3\rangle|$ is satisfied for all $|\,\ell,\,m\,\rangle$ eigenstates and is saturated precisely for the stretched states $|m| = \ell$. The Robertson bounds involving ΔL_3 are trivially satisfied since $\Delta L_3 = 0$ and $\langle\hat{L}_1\rangle = \langle\hat{L}_2\rangle = 0$ in $\hat{L}_3$-eigenstates.

10.2 Programmatic Significance

The results of the present paper are of broad programmatic significance for the scalar–conformal NUVO series on three grounds.

The first and most fundamental is the derivation of the angular momentum commutation algebra from the canonical commutation relations. In the standard formulation of quantum mechanics, the angular momentum commutation algebra is either postulated as part of the definition of angular momentum operators or stated as a corollary of the Lie algebra of SO(3) whose derivation is typically relegated to group theory. In the NUVO framework, the algebra is derived by explicit computation from the CCR of QM1, in a proof (Theorem 3.1) whose each step traces to the canonical commutation relation $[\hat{x}^j, \hat{p}_k] = \mathrm{i}\Phi_0\,\delta^j{}_k$ and the Leibniz rule. This means the entire angular momentum structure of the QM-series — the spectrum, the spherical harmonics, the hydrogenic quantum numbers, and all subsequent applications in QM6 through QM11 — is grounded in the single algebraic input of the CCR, itself derived in QB2 from the differential operator representation of the transport generators. The derivation chain from transport geometry to hydrogen spectrum is now unbroken: Q-series transport closure → QB2 momentum generators → QM1 CCR on $\mathcal{H}$ → QM5 angular momentum algebra → spherical harmonics → hydrogenic eigenstates.

The second ground is the unified treatment of quantization by the holonomy principle. Three distinct quantization conditions in the NUVO program are now identified as manifestations of the same geometric principle: on a closed transport path, the accumulated phase must be an integer multiple of $2\pi\Phi_0$. The radial holonomy (Q-series) quantizes the energy; the azimuthal holonomy (QM5 Theorem 5.2) quantizes the magnetic quantum number; and the algebraic termination of the ladder (QM5 Theorem 4.6) determines the orbital quantum number from the integer constraint. QM8 will extend this to the spin holonomy on the double cover of SO(3). The emerging picture is that quantization in the NUVO program is uniformly a consequence of holonomy: the discrete structure of the quantum spectrum is a direct geometric consequence of the topological properties of the closed transport paths in the scalar–conformal exchange sector.

The third ground is the completion of the hydrogenic sector. The present paper combines with the Q-series (energy spectrum), QM4 (hydrogenic Hamiltonian and its self-adjointness), and QM1 (spectral theorem) to give the complete description of the hydrogen atom within the NUVO program: energy levels, wave functions, quantum number structure, degeneracy, and orthonormality. This is the first complete physical system fully accounted for within the scalar–conformal framework, and it validates the program architecture: the derivation chain from the M-series geometry to the full hydrogen spectrum is now available without any postulated input from the standard quantum-mechanical formalism.

10.3 Transition to QM6

The angular momentum structure established in the present paper is immediately applied in QM6, which develops the quantum harmonic oscillator. QM6 deploys two interconnected algebraic structures, both of which are prototyped in the present paper.

The first is the ladder operator technique. The raising and lowering operators $\hat{L}_+$ and $\hat{L}_-$ of the present paper raise and lower the $\hat{L}_3$ eigenvalue by Φ_0 while preserving the $\hat{L}^2$ eigenvalue. In QM6, the harmonic oscillator ladder operators $\hat{a}^+$ and $\hat{a}^-$ raise and lower the energy eigenvalue by $\Phi_0\omega$ while acting in the energy eigenspace. The algebraic structure is the same: termination at a ground state (the vacuum of QM6, the lowest m state of QM5), and matrix elements determined by a normalization computation using the relevant product identity (the number operator in QM6, the ladder product identity Lemma 4.8 in QM5). The QM5 ladder analysis is the structural template for the QM6 energy ladder.

The second is the angular momentum structure of the three-dimensional harmonic oscillator. The 3D harmonic oscillator in spherical coordinates has eigenstates of the form $u_{n\ell}(r)Y_\ell^m(\theta, \varphi)$, where the radial factor $u_{n\ell}$ satisfies a radial equation analogous to Eq. (58) with the harmonic potential replacing the Coulomb potential, and the angular factor is exactly the spherical harmonic derived in the present paper. The energy levels of the 3D harmonic oscillator are $E_N = (N + \frac{3}{2})\Phi_0\omega$ for $N = 0, 1, 2, \ldots$, where $N = 2n_r + \ell$ and n_r is the radial quantum number. For each N, the allowed values of ℓ are $0, 2, 4, \ldots, N$ (even ℓ for even N) or $1, 3, 5, \ldots, N$ (odd ℓ for odd N), giving a degeneracy structure that differs from the hydrogenic case and is analyzed in QM6 using the spherical harmonics of the present paper. The coherent states of QM6, identified in QM3 as the minimum-uncertainty Gaussian states, are also constructed in QM6 as eigenstates of the lowering operator $\hat{a}^-$; their three-dimensional generalization involves the spherical coordinate structure established here. The angular momentum results of QM5 are therefore not merely a prerequisite for QM6 but an active structural component of the QM6 analysis.

References

[1] Alan R. Edmonds. *Angular Momentum in Quantum Mechanics.* Princeton University Press, Princeton, 1957. Primary reference for: the associated Legendre polynomials and their Rodrigues formula (used in QM5 Lemma 6.1 and Theorem 6.2); the orthogonality integral for associated Legendre functions (used in the normalization proof of Theorem 6.2); the parity of associated Legendre functions under $u \to -u$ (used in the proof of Proposition 6.3 (iii)); the Condon-Shortley phase convention (Remark 6.4); and general reference for the addition theorem for spherical harmonics and Clebsch-Gordan theory (deferred to QM7).

[2] Alberto Galindo and Pedro Pascual. *Quantum Mechanics I.* Springer, Berlin, 1990. Primary reference for: the associated Legendre equation and its regular solutions (supporting Lemma 6.1); the radial hydrogen atom equation and its bound state solutions, including the series truncation condition and the constraint $\ell \leq n - 1$ (Proposition 7.2); the associated Laguerre polynomials

appearing in the radial wave functions $R_{n\ell}(r)$; and the orthogonality of the radial functions (Proposition 7.4 (i)).

[3] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics, Vol. I: Functional Analysis*. Academic Press, New York, 1972. Primary reference for: the spectral theorem for self-adjoint operators on Hilbert spaces, used in QM5 to establish completeness of the spherical harmonics on $L^2(S^2)$ (Proposition 6.3 (ii)) and completeness of the hydrogenic eigenstates (Proposition 7.4); the Weyl-Stone-von Neumann theorem for the Heisenberg commutation relations (background for the CCR used in QM5 Theorem 3.1).