QM6 — The Quantum Harmonic Oscillator: Algebraic Structure, Coherent States, and the Three-Dimensional Extension

in Scalar–Conformal NUVO Systems *Preprint, Version 1.0*[*]

Rickey W. Austin
St Claire Scientific Research, Development, and Publishing

Notation and Conventions

- $\mathcal{M}$ denotes the spacetime manifold.
- η denotes the reference Lorentzian metric (typically Minkowski in a global chart).
- g denotes the physical metric.
- The scalar field $\Lambda : \mathcal{M} \to \mathbb{R}_{>0}$ is the NUVO modulation field.
- The physical metric is scalar–conformal:

$$g_{\mu\nu} = \Lambda^2 \, \eta_{\mu\nu}.$$

- $\Lambda_0 > 0$ denotes the baseline scalar availability level supported by the intrinsic delivery structure of the underlying field. In the absence of localized structural occupation the scalar field satisfies $\Lambda(x) = \Lambda_0$.
- The dimensionless scalar diagnostic is

$$\lambda(x) := \frac{\Lambda(x)}{\Lambda_0}.$$

- The scalar field represents the *locally available structural capacity* of the underlying delivery field. Localized structures may reduce this availability through occupation or transport, but the intrinsic delivery baseline Λ_0 remains fixed.
- Greek indices $\mu, \nu, \ldots$ range over spacetime coordinates $0, 1, 2, 3$.
- We use the Einstein summation convention unless explicitly stated otherwise.

Remark 0.1. *Unless otherwise stated, the background signature is* $(-, +, +, +)$.

[*]Bibliography is provisional. Cross-references to companion NUVO-series papers (M-, SR-, Q-, QB-, QM-series) will be updated with Zenodo DOIs in subsequent versions.

Program scope.

Abstract

The harmonic oscillator occupies a central structural position in quantum mechanics: it is the simplest non-trivial dynamical system, the foundation of quantum field theory, and the generating model for coherent states, squeezed states, and the semiclassical limit. In the scalar–conformal NUVO framework, the harmonic oscillator Hamiltonian $\hat{H}_{\rm osc} = \hat{p}^2/(2m) + \frac{1}{2}m\omega^2\hat{x}^2$ is a special case of the self-adjoint Hamiltonians established in QM4 with a Kato-class potential $V(x) = \frac{1}{2}m\omega^2 x^2$. The present paper derives its complete structure as a sequence of theorems from the canonical commutation relations of QM1 and the dynamical framework of QM4.

The central algebraic tool is the pair of ladder operators $\hat{a} = (\hat{p} + \mathrm{i}m\omega\hat{x})/\sqrt{2m\omega\Phi_0}$ and $\hat{a}^\dagger = (\hat{p} - \mathrm{i}m\omega\hat{x})/\sqrt{2m\omega\Phi_0}$, which satisfy $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$ and decompose the Hamiltonian as $\hat{H}_{\rm osc} = \Phi_0\omega(\hat{N} + \frac{1}{2}\hat{\mathbf{1}})$ where $\hat{N} = \hat{a}^\dagger\hat{a}$ is the number operator. The complete eigenvalue spectrum $E_n = (n + \frac{1}{2})\Phi_0\omega$ for $n \in \{0, 1, 2, \ldots\}$ and the energy eigenstates (Hermite-Gaussian functions) are derived from the ladder operator algebra by the same technique as the angular momentum spectrum in QM5.

The energy-time uncertainty relation of QM3 is applied to the harmonic oscillator to establish the zero-point energy $E_0 = \frac{1}{2}\Phi_0\omega$ as a structural consequence: a state of strictly zero energy would violate the uncertainty bound, so the ground state energy is necessarily positive.

Coherent states $|\alpha\rangle$ are defined as eigenstates of the annihilation operator $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ for $\alpha \in \mathbb{C}$, and are shown to be the Gaussian minimum-uncertainty states of QM3 with the oscillator zero-point width $\ell_0 = \sqrt{\Phi_0/(m\omega)}$, propagating under the harmonic oscillator dynamics without change of shape. The displacement operator representation $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$ is established, and the overcompleteness of the coherent state family is derived.

The paper closes with the three-dimensional isotropic harmonic oscillator, whose eigenstates are expressed in spherical coordinates as products of radial Laguerre functions and spherical harmonics Y_ℓ^m from QM5. The energy shell structure $E_N = (N + \frac{3}{2})\Phi_0\omega$ with degeneracy $d_N = (N + 1)(N + 2)/2$ is derived from the combined radial and angular quantum number constraints.

No new postulates are introduced. All results follow from the QM1 canonical commutation relations, the QM4 dynamical framework, the QM3 uncertainty relations, and the QM5 angular momentum structure.

1 Introduction

1.1 Position Within the QM-Series

The scalar–conformal NUVO program has now established its complete algebraic foundation (QM1 through QM3), its dynamical framework (QM4), and its first major physical sector (QM5: angular momentum and the hydrogen spectrum). The present paper, QM6, develops the second major physical sector: the quantum harmonic oscillator. The oscillator occupies a distinctive position within the QM-series that is different from the hydrogen atom. The hydrogen atom is the primary physical validation of the NUVO framework, whose energy spectrum was derived in the Q-series and whose full wave function structure was completed in QM5. The harmonic oscillator is a structural template: the simplest dynamical system with a discrete non-degenerate spectrum, the generating model for the coherent state theory of QM6 through QM11, and the foundational object from which quantum field theory builds its multi-excitation structure. In the scalar–conformal NUVO program, the oscillator Hamiltonian $\hat{H}_{\rm osc} = \hat{p}^2/(2m) + \frac{1}{2}m\omega^2\hat{x}^2$ is a special case of the self-adjoint Hamiltonians established in QM4 Theorem 4.2, with the harmonic potential $V(x) = \frac{1}{2}m\omega^2 x^2$ satisfying the Kato-class regularity conditions of QM4 Definition 3.2. Its complete structure is derived from prior results without new physical assumptions.

QM6 closes three program arcs opened in earlier papers and depends on their results in structurally specific ways. The first arc runs from QM3 to QM6. QM3 Theorem 6.1 identified the Gaussian minimum-uncertainty states as those saturating the Cauchy-Schwarz bound in the Robertson inequality, and showed that every Gaussian width $\sigma > 0$ gives a minimum-uncertainty state. The oscillator dynamics of QM4 and QM6 together select a preferred width: only the Gaussian with $\sigma = \ell_0/\sqrt{2} = \sqrt{\Phi_0/(2m\omega)}$ retains its shape under the harmonic oscillator time evolution. This dynamical selection of a preferred minimum-uncertainty state is the coherent state of QM6, and the program arc from QM3 to QM6 is precisely the arc from algebraic characterization (minimum uncertainty) to dynamical characterization (shape preservation). The second arc runs from QM4 to QM6. QM4 established the Ehrenfest theorem: the centroid $(\langle x\rangle(t), \langle p\rangle(t))$ of any closure state follows the classical equations of motion. For the harmonic oscillator, the classical equation is $\ddot{x}+\omega^2 x = 0$, so the centroid traces a classical oscillation. The distinctive feature of coherent states, established in QM6, is that not only the centroid but the entire Gaussian profile propagates classically: the widths Δx and Δp are constant in time, and the state at each time is a displaced Gaussian identical in shape to the initial state. The third arc runs from QM5 to QM6. The ladder operator technique introduced in QM5 for angular momentum — raising and lowering the $\hat{L}_3$ eigenvalue by Φ_0 while preserving the $\hat{L}^2$ eigenvalue — recurs here for energy, raising and lowering the energy eigenvalue by $\Phi_0\omega$. The algebraic structure is the same; the spectrum is different (semi-infinite rather than finite), reflecting the absence of an upper bound on the energy eigenvalue analogous to the upper bound $|m| \leq \ell$ for angular momentum. QM6 is thus the energy-sector analogue of QM5, and the two papers together establish the two canonical ladder algebras of the QM-series.

The harmonic oscillator is not merely a physical system but a structural template whose algebraic skeleton propagates forward throughout the series. QM7 treats the coupled two-oscillator system and the normal mode transformation, which is the simplest instance of a linear canonical transformation and the precursor of the multi-mode structure of quantum field theory. QM9 constructs entangled coherent states as non-factorizable two-mode coherent state superpositions, using the single-mode coherent state theory of the present paper. QM10 uses the oscillator algebra to model radiation field modes in the derivation of scattering cross-sections with photon emission and absorption. QM11 extends the oscillator to the relativistic Klein-Gordon oscillator and establishes the connection to the free field modes of the RQM-series. In each of these, the specific results of the present paper — the Fock state basis, the coherent state family, the displacement operator, and the overcompleteness relation — are the structural inputs rather than background material.

1.2 Objective of the Present Work

The central objective of the present paper is to derive the complete structure of the harmonic oscillator in the scalar–conformal NUVO transport closure framework from the canonical commutation relations of QM1, the dynamical framework of QM4, the uncertainty relations of QM3, and the angular momentum structure of QM5. Specifically, the paper establishes six claims.

1. The harmonic oscillator Hamiltonian decomposes as $\hat{H}_{\text{osc}} = \Phi_0\omega(\hat{N} + \frac{1}{2}\hat{\mathbf{1}})$, where the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$ is self-adjoint and non-negative, and the annihilation and creation operators $\hat{a} = (m\omega\hat{x} + \mathrm{i}\hat{p})/\sqrt{2m\omega\Phi_0}$ and $\hat{a}^\dagger = (\hat{a})^\dagger$ satisfy $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$, derived from the canonical commutation relation $[\hat{x}, \hat{p}] = \mathrm{i}\Phi_0$ of QM1.

2. The complete eigenvalue spectrum of $\hat{H}_{\text{osc}}$ is $\sigma(\hat{H}_{\text{osc}}) = \{(n + \frac{1}{2})\Phi_0\omega : n \in \{0, 1, 2, \ldots\}\}$, derived by the same ladder termination argument as QM5: non-negativity of $\hat{N}$ forces the eigenvalue sequence to terminate at $n = 0$ (the vacuum $\hat{a}|0\rangle = 0$), and integer steps above

give $n \in \mathbb{Z}_{\geq 0}$. Each eigenvalue is non-degenerate, and the Fock states $|n\rangle$ form a complete orthonormal basis for $\mathcal{H}$.

3. The zero-point energy $E_0 = \frac{1}{2}\Phi_0\omega$ is derived as a structural consequence of the Heisenberg uncertainty relation of QM3: any state of the oscillator satisfies $\langle \hat{H}_{\rm osc}\rangle \geq \omega\Delta x\Delta p \geq \frac{1}{2}\Phi_0\omega$, with equality achieved only by the Gaussian ground state. The non-zero ground state energy is not postulated but derived.

4. The position-space energy eigenstates are the Hermite-Gaussian functions: $\Psi_n(x) \propto H_n(x/\ell_0)\exp(-x^2/(2\ell_0^2))$ derived by applying $(\hat{a}^\dagger)^n/\sqrt{n!}$ to the ground state $\Psi_0(x) \propto \exp(-x^2/(2\ell_0^2))$ and recognizing the resulting polynomial factor as the Hermite polynomial $H_n(\xi)$. Orthonormality follows from self-adjointness of $\hat{H}_{\rm osc}$; completeness from the spectral theorem of QM1.

5. Coherent states $|\alpha\rangle$ are defined as eigenstates of $\hat{a}$: $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ for $\alpha \in \mathbb{C}$. They are shown to be equivalent to three characterizations: the Gaussian minimum-uncertainty states of QM3 with oscillator width $\sigma = \ell_0/\sqrt{2}$; the states generated from the vacuum by the displacement operator $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$; and the states that evolve under $\hat{H}_{\rm osc}$ dynamics without change of shape ($\Delta x(t)$ and $\Delta p(t)$ constant). The coherent state family is overcomplete in $\mathcal{H}$, satisfying the resolution of the identity $(1/\pi)\int_{\mathbb{C}} |\alpha\rangle\langle\alpha|\,\mathrm{d}^2\alpha = \hat{\mathbf{1}}$.

6. The three-dimensional isotropic harmonic oscillator $\hat{H}^{(3)}_{\rm osc} = \sum_{j=1}^{3}[\hat{p}_j^2/(2m) + \frac{1}{2}m\omega^2(\hat{x}^j)^2]$ has energy levels $E_N = (N + \frac{3}{2})\Phi_0\omega$ for $N \in \{0, 1, 2, \ldots\}$ with degeneracy $d_N = (N+1)(N+2)/2$, derived from the Cartesian product of three 1D spectra. In spherical coordinates, the eigenstates factorize as $u_{n_r\ell}(r)Y_\ell^m(\theta,\varphi)$ with the angular factor given by the QM5 spherical harmonics and the radial factor expressed in terms of associated Laguerre polynomials.

Claims (1) through (6) are logically ordered. The ladder decomposition of claim (1) is the algebraic input to the spectrum derivation of claim (2). The zero-point energy of claim (3) connects claim (2) to the QM3 uncertainty structure and motivates the identification of the ground state in claim (4). The Hermite-Gaussian structure of claim (4) is the position-space realization of the abstract Fock states of claim (2). The coherent states of claim (5) are identified as the distinguished subfamily of the QM3 minimum-uncertainty family that is dynamically stable under the oscillator of claims (1)–(4). The three-dimensional extension of claim (6) combines the 1D structure of claims (1)–(4) with the QM5 angular momentum structure to give the full 3D oscillator spectrum and eigenstates.

1.3 What Is Not Assumed

The present work maintains without modification the interpretive discipline established throughout the prior series. Four exclusions are of particular importance for QM6.

The energy spectrum is not postulated. In many presentations of the harmonic oscillator, the energy eigenvalues $E_n = (n + \frac{1}{2})\hbar\omega$ are stated as a result to be verified by solving the Schrödinger equation, with the zero-point energy $\frac{1}{2}\hbar\omega$ sometimes described as an empirical feature or a consequence of the uncertainty principle invoked without derivation. In the NUVO framework, the spectrum is derived from the ladder operator algebra via the termination argument of Theorem 4.1: non-negativity of the number operator $\hat{N}$ forces the spectrum to terminate at $n = 0$, and the integer-step structure of the ladder above the ground state gives the complete spectrum. The zero-point energy is then derived in Theorem 4.3 as a structural lower bound from the Heisenberg relation of QM3, not introduced as a separate assumption.

The Hermite polynomials are not introduced as known special functions. The polynomial factor $H_n(x/\ell_0)$ in the energy eigenfunction Eq. (36) emerges from applying $(\hat{a}^\dagger)^n$ to the ground state in position space; the resulting function is recognized as the n-th Hermite polynomial after the fact, not before. The Rodrigues formula for Hermite polynomials, which is the classical definition, emerges from this application of the creation operator as a consequence rather than as an input.

Coherent states are not defined by their Gaussian form. The Gaussian form of the coherent state position-space representation is derived as a consequence of the eigenvalue condition $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, not assumed. The equivalence of the three characterizations of coherent states — eigenstate of $\hat{a}$, Gaussian minimum-uncertainty state, and displaced vacuum — is established as a theorem rather than taken as a definition.

The three-dimensional oscillator spectrum is not assumed. The degeneracy $d_N = (N+1)(N+2)/2$ and the parity selection rule (ℓ and N must have the same parity) are derived from the explicit quantum number structure of the Cartesian and spherical decompositions, not stated without derivation.

1.4 Structure of the Paper

Sec. 2 recalls the harmonic oscillator Hamiltonian and its self-adjointness from QM4, the canonical commutation relation from QM1, the Gaussian minimum-uncertainty states and the uncertainty bound from QM3, the Ehrenfest theorem from QM4, and the spherical harmonics from QM5. Sec. 3 introduces the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$, derives the fundamental commutation relation $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$ from the CCR, decomposes the Hamiltonian as $\Phi_0\omega(\hat{N} + \frac{1}{2}\hat{\mathbf{1}})$, and derives the commutation relations of the number operator with $\hat{a}$ and $\hat{a}^\dagger$. Sec. 4 derives the complete spectrum $\{(n+\frac{1}{2})\Phi_0\omega : n \geq 0\}$ from the ladder algebra, establishes the zero-point energy from the uncertainty principle, and records the Fock state matrix elements. Sec. 5 derives the ground state as the Gaussian solving $\hat{a}\Psi_0 = 0$ in position space, the excited states as Hermite-Gaussian functions, and establishes their orthonormality and completeness. Sec. 6 defines coherent states as eigenstates of $\hat{a}$, derives their Fock-state expansion and Gaussian position representation, constructs them via the displacement operator, derives their shape-preserving time evolution, and establishes the overcompleteness of the coherent state family. Sec. 7 develops the three-dimensional isotropic harmonic oscillator in both Cartesian and spherical representations, derives the energy shell structure and degeneracy, identifies the spherical harmonic angular factors from QM5, and records the shell quantum number constraints and parity selection rule. Sec. 8 records the role of the oscillator as a structural template, the explicit comparison of the QM5 and QM6 ladder algebras, and the scope of the present construction. Sec. 9 summarizes the sixteen principal results, records the programmatic significance of the coherent state construction and the zero-point energy derivation, and prepares the transition to QM7.

2 Recalled Structure from Prior Papers

The present section collects the results from QM1, QM3, QM4, and QM5 that are directly required for the derivations of Secs. 3–7. Nothing in this section is new. The section serves two purposes: making the logical dependencies explicit before the main derivations begin, and recording the specific forms in which prior results will be applied so that the proofs of Secs. 3–7 can refer back to numbered equations rather than repeating the recalled content.

2.1 The Harmonic Oscillator Hamiltonian from QM4

The harmonic oscillator Hamiltonian in one spatial dimension is

$$\hat{H}_{\text{osc}} := \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2, \tag{1}$$

where $\hat{x}$ is multiplication by x, $\hat{p} = -\mathrm{i}\Phi_0\,\partial_x$, $m > 0$ is the mass parameter, and $\omega > 0$ is the angular frequency of the oscillator.

Self-adjointness (QM4 Theorem 4.2). The potential $V(x) = \frac{1}{2}m\omega^2x^2$ is a Kato-class potential (QM4 Definition 3.2): it is locally square-integrable and satisfies the Kato-Rellich bound with respect to the kinetic operator $\hat{T} = -(\Phi_0^2/2m)\partial_x^2$. By QM4 Theorem 4.2, the operator $\hat{H}_{\text{osc}}$ is self-adjoint on the Sobolev domain $H^2(\mathbb{R}^3) \cap \mathcal{D}(\hat{x}^2)$ and bounded below.

Unitary time evolution (QM4 Theorem 3.1). Stone's theorem, applied to the self-adjoint generator $\hat{H}_{\text{osc}}$, gives the strongly continuous unitary group $U(t) = \mathrm{e}^{-\mathrm{i}\hat{H}_{\text{osc}}t/\Phi_0}$ on $\mathcal{H}$, well-defined for all $t \in \mathbb{R}$. Every initial state $\Psi_0 \in \mathcal{H}$ evolves to $\Psi(t) = U(t)\Psi_0$, with $\|\Psi(t)\|_{\mathcal{H}} = 1$ for all t if $\|\Psi_0\|_{\mathcal{H}} = 1$.

The Ehrenfest equations for the oscillator (QM4 Proposition 5.1). For any normalized closure state $\Psi(t) = U(t)\Psi_0 \in \mathcal{D}(\hat{x}) \cap \mathcal{D}(\hat{p})$, the Ehrenfest theorem of QM4 applied to $\hat{H}_{\text{osc}}$ gives the coupled equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x\rangle(t) = \frac{\langle p\rangle(t)}{m}, \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle p\rangle(t) = -m\omega^2\langle x\rangle(t). \tag{3}$$

These are exactly the classical harmonic oscillator equations of motion with no quantum correction. The exactness — the absence of higher-moment corrections that appear for non-quadratic potentials — is a special property of the quadratic potential: since $\partial^2V/\partial x^2 = m\omega^2$ is constant, all higher moments decouple from the centroid equations. The general solution is

$$\langle x\rangle(t) = \langle x\rangle(0)\cos(\omega t) + \frac{\langle p\rangle(0)}{m\omega}\sin(\omega t), \tag{4}$$

$$\langle p\rangle(t) = \langle p\rangle(0)\cos(\omega t) - m\omega\langle x\rangle(0)\sin(\omega t), \tag{5}$$

valid for any initial state.

Remark 2.1. *The Ehrenfest theorem establishes that the centroid $(\langle x\rangle(t), \langle p\rangle(t))$ follows the classical trajectory for* any *initial state Ψ_0. It does not, however, determine the time evolution of the widths $\Delta x(t)$ and $\Delta p(t)$. For a general initial state, the widths evolve non-trivially and the Gaussian shape is not preserved. The special property of coherent states, established in Theorem 6.8, is that their widths are constant: $\Delta x(t) = \ell_0/\sqrt{2}$ and $\Delta p(t) = p_0/\sqrt{2}$ for all t. This shape-preservation property, combined with the Ehrenfest centroid motion, means that a coherent state at time t is a displaced copy of the coherent state at $t = 0$, with the displacement following the classical orbit. The Ehrenfest theorem is a necessary but not sufficient condition for this behavior; the full coherent state analysis of Sec. 6 is required.*

2.2 The Canonical Commutation Relation

In one spatial dimension, the canonical commutation relation established in QB2 and promoted to $\mathcal{H}$ in QM1 Proposition 5.4 takes the form

$$[\hat{x}, \hat{p}] = \mathrm{i}\Phi_0\,\hat{\mathbf{1}} \tag{6}$$

on the dense domain $\mathcal{S}(\mathbb{R}) \subset \mathcal{H}$. The two additional commutation relations are

$$[\hat{x}, \hat{x}] = 0, \qquad [\hat{p}, \hat{p}] = 0, \tag{7}$$

both trivially satisfied since $\hat{x}$ commutes with itself and $\hat{p}$ commutes with itself.

The three relations Eqs. (6)–(7) are the complete algebraic input to the derivation of $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$ in Lemma 3.3: the oscillator ladder algebra is a direct consequence of the position-momentum CCR, just as the angular momentum commutation algebra of QM5 was a direct consequence of the three-dimensional CCR.

Remark 2.2. *The connection between the CCR $[\hat{x}, \hat{p}] = \mathrm{i}\Phi_0$ and the ladder commutation relation $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$ is a canonical transformation in the algebraic sense. The annihilation operator $\hat{a} = (m\omega\hat{x} + \mathrm{i}\hat{p})/\sqrt{2m\omega\Phi_0}$ is a complex linear combination of $\hat{x}$ and $\hat{p}$ chosen so that the commutator $[\hat{a}, \hat{a}^\dagger]$ comes out as a scalar multiple of the identity — specifically as $\hat{\mathbf{1}}$ rather than $\mathrm{i}\Phi_0\,\hat{\mathbf{1}}$ — by absorbing the factor Φ_0 into the normalization of the ladder operators. This normalization choice makes the algebra of $\hat{a}$ and $\hat{a}^\dagger$ universal: it does not depend on m or ω, unlike the CCR which contains Φ_0. The energy spectrum then follows from this universal algebra combined with the specific Hamiltonian decomposition $\hat{H}_{\mathrm{osc}} = \Phi_0\omega(\hat{N} + \frac{1}{2}\hat{\mathbf{1}})$.*

2.3 Gaussian Minimum-Uncertainty States from QM3

The following results from QM3 are used directly in Secs. 4–6.

The uncertainty bound and its saturation (QM3 Theorem 3.2 and Proposition 3.4). For any normalized state $\Psi \in \mathcal{H}$:

$$\Delta x \cdot \Delta p \geq \frac{\Phi_0}{2}, \tag{8}$$

with equality if and only if $(\hat{p} - \langle p\rangle)\Psi = \mathrm{i}\mu(\hat{x} - \langle x\rangle)\Psi$ for some $\mu \in \mathbb{R}$ with $\mu > 0$ for a normalizable state (the saturation condition of QM3 Proposition 3.4).

Minimum-uncertainty states are Gaussian (QM3 Theorem 6.1). For $\mu > 0$, the unique normalized solution of the saturation condition is the Gaussian closure state

$$\Psi_{G(\sigma,\langle x\rangle,\langle p\rangle)}(x) = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left(-\frac{(x - \langle x\rangle)^2}{4\sigma^2} + \frac{\mathrm{i}\langle p\rangle\, x}{\Phi_0}\right), \tag{9}$$

with $\sigma = \sqrt{\Phi_0/(2\mu)}$, $\Delta x = \sigma$, and $\Delta p = \Phi_0/(2\sigma)$. The set of all minimum-uncertainty states is parametrized by $(\sigma, \langle x\rangle, \langle p\rangle) \in \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}$.

Energy lower bound from the uncertainty relation. For the harmonic oscillator, the expectation value of $\hat{H}_{\mathrm{osc}}$ in any state satisfies

$$\langle \hat{H}_{\mathrm{osc}}\rangle = \frac{(\Delta p)^2 + \langle p\rangle^2}{2m} + \frac{m\omega^2}{2}\left[(\Delta x)^2 + \langle x\rangle^2\right] \geq \frac{(\Delta p)^2}{2m} + \frac{m\omega^2(\Delta x)^2}{2}, \tag{10}$$

using $\langle A^2\rangle = (\Delta A)^2 + \langle A\rangle^2 \geq (\Delta A)^2$. The right-hand side of Eq. (10) is then bounded below by $\Phi_0\omega/2$ using the AM-GM inequality and Eq. (8), as established in Theorem 4.3.

Remark 2.3. *The use of QM3 in QM6 is specifically structured. QM3 characterizes the minimum-uncertainty states algebraically: for any $\sigma > 0$, the Gaussian $\Psi_{G(\sigma,\langle x\rangle,\langle p\rangle)}$ saturates the Heisenberg bound. QM6 adds a dynamical filter: among all Gaussians, the harmonic oscillator dynamics selects $\sigma = \ell_0/\sqrt{2}$ as the unique width preserved by the time evolution. The transition from QM3 to QM6 is therefore from a one-parameter family of minimum-uncertainty states (parametrized by σ) to a zero-parameter family of preferred states (the single width $\ell_0/\sqrt{2}$, with only the center $(\langle x\rangle, \langle p\rangle)$ remaining as a free parameter). This transition is the content of Theorem 6.8.*

2.4 Angular Momentum and Spherical Harmonics from QM5

The following results from QM5 are used in Sec. 7 for the three-dimensional harmonic oscillator.

Rotational symmetry and angular momentum conservation (QM4 Proposition 7.3, QM5 Theorems 3.1–3.2). For the isotropic three-dimensional harmonic oscillator $\hat{H}^{(3)}_{\rm osc} = \sum_j [\hat{p}_j^2/(2m) + \frac{1}{2}m\omega^2(\hat{x}^j)^2]$, the potential $V(x) = \frac{1}{2}m\omega^2|x|^2$ is rotationally symmetric. By QM4 Proposition 7.3: $[\hat{H}^{(3)}_{\rm osc}, \hat{L}_j] = 0$ for all j, so $[\hat{H}^{(3)}_{\rm osc}, \hat{L}^2] = 0$ and the eigenstates can be chosen to be simultaneous eigenstates of $\hat{H}^{(3)}_{\rm osc}$, $\hat{L}^2$, and $\hat{L}_3$.

$\hat{L}^2$ *in spherical coordinates* (QM5 Proposition 6.1). The decomposition of the three-dimensional Laplacian in spherical coordinates gives:

$$-\frac{\Phi_0^2}{2m}\Delta = -\frac{\Phi_0^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\hat{L}^2}{2mr^2}, \tag{11}$$

separating the kinetic operator into a purely radial part and a centrifugal part $\hat{L}^2/(2mr^2)$. This decomposition, combined with the spherical potential $V(x) = \frac{1}{2}m\omega^2 r^2$, enables the separation of variables in Sec. 7.

Spherical harmonics as angular closure eigenstates (QM5 Theorem 6.2 and Proposition 6.3). The joint eigenstates of $\hat{L}^2$ and $\hat{L}_3$ on the unit sphere S^2 are the spherical harmonics $Y_\ell^m(\theta,\varphi)$, satisfying

$$\hat{L}^2\, Y_\ell^m = \ell(\ell+1)\Phi_0^2\, Y_\ell^m, \qquad \hat{L}_3\, Y_\ell^m = m\Phi_0\, Y_\ell^m, \tag{12}$$

for $\ell \in \{0,1,2,\ldots\}$ and $m \in \{-\ell,\ldots,+\ell\}$, orthonormal on S^2, and complete in $L^2(S^2)$. Their parity under $\mathbf{x} \to -\mathbf{x}$ is $(-1)^\ell$ (QM5 Proposition 6.3 (iii)).

Remark 2.4. *The spherical harmonics from QM5 appear in QM6 not as background material but as active structural components. In the spherical representation of the 3D oscillator (Sec. 7), the angular factor of each eigenstate is exactly $Y_\ell^m(\theta,\varphi)$ from QM5 Theorem 6.2. The parity selection rule of Sec. 7 (the requirement that ℓ and the total excitation number N have the same parity) is a direct consequence of the parity $(-1)^\ell$ of the spherical harmonics and the parity of the radial oscillator functions. This is the first instance in the QM-series of a result from a physical sector paper (QM5) serving as a direct structural input to a subsequent physical sector paper (QM6), rather than merely as a formal prerequisite. The pattern will continue: QM6 results feed into QM7, QM5 and QM6 results together feed into QM8, and so forth.*

3 Ladder Operators and the Algebra of the Oscillator

The canonical commutation relation $[\hat{x},\hat{p}] = \mathrm{i}\Phi_0\,\hat{\mathbf{1}}$ of QM1 encodes the complete algebraic structure of the harmonic oscillator when rewritten in terms of two non-self-adjoint operators adapted to the oscillator's natural scales. These are the ladder operators $\hat{a}$ and $\hat{a}^\dagger$, which decompose the Hamiltonian into a form $\Phi_0\omega(\hat{N} + \frac{1}{2}\hat{\mathbf{1}})$ from which the entire spectrum can be read off algebraically. The derivations in the present section are the one-dimensional energy analogues of the angular momentum ladder derivations of QM5 Sec. 3: the same technique of converting a commutation relation among self-adjoint operators into a raising-lowering algebra among non-self-adjoint operators, and then reading the spectrum from the termination of the ladder.

3.1 Definition of the Ladder Operators

The natural scale of the harmonic oscillator in position space is the *oscillator length* $\ell_0 = \sqrt{\Phi_0/(m\omega)}$, and in momentum space is the *oscillator momentum* $p_0 = \sqrt{m\omega\Phi_0}$. These satisfy $\ell_0 \cdot p_0 = \Phi_0$, so that $\ell_0 \cdot p_0/2 = \Phi_0/2$ is the Heisenberg bound on $\Delta x \cdot \Delta p$ from QM3. The ground state saturates this bound, as established in Theorem 4.3.

Definition 3.1 (Oscillator ladder operators)**.** *The* annihilation operator *and* creation operator *are defined by*

$$\hat{a} := \frac{m\omega\hat{x} + \mathrm{i}\hat{p}}{\sqrt{2m\omega\Phi_0}} = \frac{\hat{x}}{\sqrt{2}\,\ell_0} + \frac{\mathrm{i}\hat{p}}{\sqrt{2}\,p_0}, \tag{13}$$

$$\hat{a}^\dagger := \frac{m\omega\hat{x} - \mathrm{i}\hat{p}}{\sqrt{2m\omega\Phi_0}} = \frac{\hat{x}}{\sqrt{2}\,\ell_0} - \frac{\mathrm{i}\hat{p}}{\sqrt{2}\,p_0}, \tag{14}$$

where $\ell_0 := \sqrt{\Phi_0/(m\omega)}$ is the oscillator length *and $p_0 := \sqrt{m\omega\Phi_0}$ is the* oscillator momentum scale*, both with dimensions of length and momentum respectively. The operators are defined on $\mathcal{S}(\mathbb{R}) \subset \mathcal{H}$ and satisfy $(\hat{a})^\dagger = \hat{a}^\dagger$.*

Remark 3.2. *The identity $(\hat{a})^\dagger = \hat{a}^\dagger$ follows from the self-adjointness of $\hat{x}$ and $\hat{p}$ on $\mathcal{H}$: $(\hat{a})^\dagger = (m\omega\hat{x} + \mathrm{i}\hat{p})^\dagger/\sqrt{2m\omega\Phi_0} = (m\omega\hat{x}^\dagger - \mathrm{i}\hat{p}^\dagger)/\sqrt{2m\omega\Phi_0} = (m\omega\hat{x} - \mathrm{i}\hat{p})/\sqrt{2m\omega\Phi_0} = \hat{a}^\dagger$. Neither $\hat{a}$ nor $\hat{a}^\dagger$ is self-adjoint, since $\hat{a} \neq (\hat{a})^\dagger = \hat{a}^\dagger$. This non-self-adjoint character is essential to the raising and lowering action: a self-adjoint operator commuting with $\hat{H}_{\mathrm{osc}}$ would preserve each energy eigenspace, whereas $\hat{a}$ and $\hat{a}^\dagger$ map between adjacent eigenspaces, as established in Lemma 3.7 and used in Theorem 4.1.*

The inverse relations expressing $\hat{x}$ and $\hat{p}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$ will be used in Secs. 5–6:

$$\hat{x} = \frac{\ell_0}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger), \qquad \hat{p} = \frac{\mathrm{i}p_0}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}). \tag{15}$$

These follow by adding and subtracting Eqs. (13) and (14).

3.2 The Fundamental Commutation Relation $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$

Lemma 3.3 (Fundamental commutation relation of the oscillator)**.** *On $\mathcal{S}(\mathbb{R})$:*

$$[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}. \tag{16}$$

Proof. Expand the commutator using the definitions Eqs. (13)–(14):

$$[\hat{a}, \hat{a}^\dagger] = \frac{1}{2m\omega\Phi_0}\big[m\omega\hat{x} + \mathrm{i}\hat{p},\ m\omega\hat{x} - \mathrm{i}\hat{p}\big].$$

Expand by linearity of the commutator:

$$\begin{aligned}\big[m\omega\hat{x} + \mathrm{i}\hat{p},\ m\omega\hat{x} - \mathrm{i}\hat{p}\big] = (m\omega)^2[\hat{x}, \hat{x}] - \mathrm{i}m\omega[\hat{x}, \hat{p}] \\ + \ \mathrm{i}m\omega[\hat{p}, \hat{x}] - \mathrm{i}^2[\hat{p}, \hat{p}].\end{aligned}$$

Applying Eqs. (6) and (7): $[\hat{x}, \hat{x}] = 0$, $[\hat{p}, \hat{p}] = 0$, $[\hat{x}, \hat{p}] = \mathrm{i}\Phi_0\,\hat{\mathbf{1}}$, $[\hat{p}, \hat{x}] = -\mathrm{i}\Phi_0\,\hat{\mathbf{1}}$:

$$\begin{aligned}&= 0 - \mathrm{i}m\omega(\mathrm{i}\Phi_0\,\hat{\mathbf{1}}) + \mathrm{i}m\omega(-\mathrm{i}\Phi_0\,\hat{\mathbf{1}}) - \mathrm{i}^2 \cdot 0 \\ &= m\omega\Phi_0\,\hat{\mathbf{1}} + m\omega\Phi_0\,\hat{\mathbf{1}} = 2m\omega\Phi_0\,\hat{\mathbf{1}}.\end{aligned}$$

Dividing by $2m\omega\Phi_0$: $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$. □

Remark 3.4. *The commutation relation* $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$ *differs from the angular momentum commutation relations of QM5 in one fundamental way: the right-hand side is a scalar multiple of the identity operator, not another element of the algebra. This means the oscillator ladder algebra* $\{I, \hat{a}, \hat{a}^\dagger\}$ *is the Weyl-Heisenberg algebra, while the angular momentum algebra* $\{\hat{L}_1, \hat{L}_2, \hat{L}_3\}$ *is the Lie algebra of* SO(3)*. In the Weyl-Heisenberg algebra,* $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$ *implies that repeated application of* $\hat{a}^\dagger$ *never returns to the starting state, so the spectrum is semi-infinite (bounded below, unbounded above). In the* SO(3) *algebra,* $[\hat{L}_j, \hat{L}_k] = \mathrm{i}\Phi_0\, \epsilon_{jkl}\hat{L}_l$ *implies that repeated application of* $\hat{L}_+$ *eventually reaches a maximum (because* $|m| \le \ell$*), so the spectrum is finite within each* ℓ*-multiplet. The two algebras produce qualitatively different spectral structures from the same ladder technique.*

3.3 Decomposition of the Hamiltonian

The Hamiltonian $\hat{H}_{\mathrm{osc}}$ is expressed entirely in terms of the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$.

Theorem 3.5 (Hamiltonian decomposition into the number operator)**.** *The harmonic oscillator Hamiltonian* $\hat{H}_{\mathrm{osc}} = \hat{p}^2/(2m) + \frac{1}{2}m\omega^2\hat{x}^2$ *decomposes as*

$$\hat{H}_{\mathrm{osc}} = \Phi_0\omega\left(\hat{N} + \frac{1}{2}\hat{\mathbf{1}}\right), \tag{17}$$

where $\hat{N} := \hat{a}^\dagger\hat{a}$ *is the* number operator*. The number operator is self-adjoint and non-negative on* $\mathcal{H}$*:* $\left\langle \Psi, \hat{N}\,\Psi \right\rangle_{\mathcal{H}} = \|\hat{a}\Psi\|^2_{\mathcal{H}} \ge 0$ *for all* $\Psi \in \mathcal{D}(\hat{a})$*.*

Proof. Compute $\hat{a}^\dagger\hat{a}$ directly from the definitions Eqs. (13)–(14):

$$\begin{aligned}\hat{a}^\dagger\hat{a} &= \frac{(m\omega\hat{x} - \mathrm{i}\hat{p})(m\omega\hat{x} + \mathrm{i}\hat{p})}{2m\omega\Phi_0} \\ &= \frac{(m\omega)^2\hat{x}^2 + \mathrm{i}m\omega\hat{x}\hat{p} - \mathrm{i}m\omega\hat{p}\hat{x} + \hat{p}^2}{2m\omega\Phi_0} \\ &= \frac{(m\omega)^2\hat{x}^2 + \mathrm{i}m\omega[\hat{x},\hat{p}] + \hat{p}^2}{2m\omega\Phi_0}.\end{aligned}$$

Substituting $[\hat{x}, \hat{p}] = \mathrm{i}\Phi_0\,\hat{\mathbf{1}}$:

$$\begin{aligned}\hat{a}^\dagger\hat{a} &= \frac{(m\omega)^2\hat{x}^2 + \mathrm{i}m\omega(\mathrm{i}\Phi_0\,\hat{\mathbf{1}}) + \hat{p}^2}{2m\omega\Phi_0} \\ &= \frac{(m\omega)^2\hat{x}^2 - m\omega\Phi_0\,\hat{\mathbf{1}} + \hat{p}^2}{2m\omega\Phi_0} \\ &= \frac{\hat{p}^2}{2m\Phi_0\omega} + \frac{m\omega\hat{x}^2}{2\Phi_0} - \frac{1}{2}\hat{\mathbf{1}} = \frac{\hat{H}_{\mathrm{osc}}}{\Phi_0\omega} - \frac{1}{2}\hat{\mathbf{1}}.\end{aligned}$$

Solving for $\hat{H}_{\mathrm{osc}}$: $\hat{H}_{\mathrm{osc}} = \Phi_0\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}\hat{\mathbf{1}}) = \Phi_0\omega(\hat{N} + \frac{1}{2}\hat{\mathbf{1}})$, which is Eq. (17). Self-adjointness of $\hat{N}$: since $(\hat{N})^\dagger = (\hat{a}^\dagger\hat{a})^\dagger = (\hat{a})^\dagger(\hat{a}^\dagger)^\dagger = \hat{a}^\dagger\hat{a} = \hat{N}$, the number operator is self-adjoint. Non-negativity: for any $\Psi \in \mathcal{D}(\hat{a})$, $\left\langle \Psi, \hat{N}\Psi \right\rangle_{\mathcal{H}} = \left\langle \Psi, \hat{a}^\dagger\hat{a}\Psi \right\rangle_{\mathcal{H}} = \langle \hat{a}\Psi, \hat{a}\Psi\rangle_{\mathcal{H}} = \|\hat{a}\Psi\|^2_{\mathcal{H}} \ge 0$. □

Remark 3.6. *The decomposition Eq.* (17) *is the central algebraic result of QM6. It expresses* $\hat{H}_{\mathrm{osc}}$ *entirely in terms of the number operator* $\hat{N} = \hat{a}^\dagger\hat{a}$*, making the spectrum of* $\hat{H}_{\mathrm{osc}}$ *directly readable from the spectrum of* $\hat{N}$*: if* $\hat{N}|n\rangle = n|n\rangle$*, then* $\hat{H}_{\mathrm{osc}}|n\rangle = (n + \frac{1}{2})\Phi_0\omega|n\rangle$*. The task of finding the* *spectrum of $\hat{H}_{\text{osc}}$ reduces to finding the spectrum of $\hat{N}$, which is accomplished by the ladder operator argument of Sec. 4. The $\frac{1}{2}\Phi_0\omega$ offset from the zero-point energy is an immediate consequence of the $\frac{1}{2}\hat{\mathbf{1}}$ term in Eq. (17), which in turn arises from the $\mathrm{i}m\omega[\hat{x},\hat{p}]$ term in the computation of $\hat{a}^\dagger\hat{a}$: it is the canonical commutation relation that produces the zero-point energy.*

3.4 Commutation Relations of the Number Operator

The commutation relations of $\hat{N}$ with $\hat{a}$ and $\hat{a}^\dagger$ establish the raising and lowering action of the ladder operators on the energy eigenstates.

Lemma 3.7 (Number operator commutation relations)**.** *On $\mathcal{S}(\mathbb{R})$:*

$$[\hat{N},\hat{a}] = -\hat{a}, \tag{18}$$

$$[\hat{N},\hat{a}^\dagger] = +\hat{a}^\dagger, \tag{19}$$

and consequently:

$$[\hat{H}_{\text{osc}},\hat{a}] = -\Phi_0\omega\,\hat{a}, \tag{20}$$

$$[\hat{H}_{\text{osc}},\hat{a}^\dagger] = +\Phi_0\omega\,\hat{a}^\dagger. \tag{21}$$

Proof. Equation (18)*:* Apply the Leibniz rule to $[\hat{a}^\dagger\hat{a},\hat{a}]$:

$$[\hat{a}^\dagger\hat{a},\hat{a}] = \hat{a}^\dagger[\hat{a},\hat{a}] + [\hat{a}^\dagger,\hat{a}]\hat{a} = 0 + (-\hat{\mathbf{1}})\hat{a} = -\hat{a},$$

using $[\hat{a},\hat{a}] = 0$ and $[\hat{a}^\dagger,\hat{a}] = -[\hat{a},\hat{a}^\dagger] = -\hat{\mathbf{1}}$ from Lemma 3.3.

Equation (19)*:* Apply the Leibniz rule to $[\hat{a}^\dagger\hat{a},\hat{a}^\dagger]$:

$$[\hat{a}^\dagger\hat{a},\hat{a}^\dagger] = \hat{a}^\dagger[\hat{a},\hat{a}^\dagger] + [\hat{a}^\dagger,\hat{a}^\dagger]\hat{a} = \hat{a}^\dagger\hat{\mathbf{1}} + 0 = \hat{a}^\dagger,$$

using $[\hat{a},\hat{a}^\dagger] = \hat{\mathbf{1}}$ and $[\hat{a}^\dagger,\hat{a}^\dagger] = 0$.

Equations (20) *and* (21)*:* From the Hamiltonian decomposition $\hat{H}_{\text{osc}} = \Phi_0\omega(\hat{N} + \frac{1}{2}\hat{\mathbf{1}})$:

$$[\hat{H}_{\text{osc}},\hat{a}] = \Phi_0\omega[\hat{N} + \tfrac{1}{2}\hat{\mathbf{1}},\hat{a}] = \Phi_0\omega[\hat{N},\hat{a}] = \Phi_0\omega(-\hat{a}) = -\Phi_0\omega\,\hat{a},$$

using $[\hat{\mathbf{1}},\hat{a}] = 0$. The relation for $\hat{a}^\dagger$ follows identically using Eq. (19). □

Remark 3.8. *The commutation relations Eqs.* (20) *and* (21) *have a direct dynamical interpretation via the Heisenberg equation of motion from QM4 Remark 7.1: for any operator G, $\mathrm{d}G_{\mathrm{H}}(t)/\mathrm{d}t = (\mathrm{i}/\Phi_0)[\hat{H}_{\text{osc}},G]_{\mathrm{H}}$. Applied to $\hat{a}$:*

$$\frac{\mathrm{d}\hat{a}_{\mathrm{H}}(t)}{\mathrm{d}t} = \frac{\mathrm{i}}{\Phi_0}[\hat{H}_{\text{osc}},\hat{a}] = \frac{\mathrm{i}}{\Phi_0}(-\Phi_0\omega\,\hat{a}) = -\mathrm{i}\omega\,\hat{a}_{\mathrm{H}}(t),$$

giving the solution $\hat{a}_{\mathrm{H}}(t) = \hat{a}\,\mathrm{e}^{-\mathrm{i}\omega t}$. This shows that the annihilation operator in the Heisenberg picture oscillates at frequency ω, which is the key identity used in the proof of Theorem 6.8 to show that coherent states evolve as coherent states.

Remark 3.9. *The four commutation relations of Lemma 3.7 are the energy-sector analogues of the angular momentum ladder commutation relations $[\hat{L}_3,\hat{L}_+] = \Phi_0\hat{L}_+$ and $[\hat{L}_3,\hat{L}_-] = -\Phi_0\hat{L}_-$ of QM5 Lemma 4.1. In both cases, the commutation relation of the "diagonal" operator (here $\hat{N}$, in QM5 the component $\hat{L}_3$) with the raising operator has the raising operator itself on the right-hand side with a positive coefficient; and with the lowering operator with a negative coefficient. This is the algebraic signature of a ladder structure. The difference is in the coefficient: QM5 gives $[\hat{L}_3,\hat{L}_+] = \Phi_0\hat{L}_+$ (coefficient Φ_0), while QM6 gives $[\hat{N},\hat{a}^\dagger] = \hat{a}^\dagger$ (coefficient 1, dimensionless). The dimensionless coefficient in QM6 reflects the fact that $\hat{N}$ measures eigenvalues in units of $\Phi_0\omega$ (via $\hat{H}_{\text{osc}} = \Phi_0\omega(\hat{N} + \frac{1}{2})$), while $\hat{L}_3$ measures eigenvalues in units of Φ_0 directly.*

4 Eigenvalues, Fock States, and Zero-Point Energy

The algebra established in Sec. 3 — the decomposition $\hat{H}_{\text{osc}} = \Phi_0\omega(\hat{N} + \frac{1}{2}\hat{\mathbf{1}})$ and the commutation relations $[\hat{N}, \hat{a}] = -\hat{a}$ and $[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$ — contains the complete spectral information of the harmonic oscillator. The present section extracts this information by the ladder termination argument, derives the zero-point energy as a consequence of the uncertainty principle, and records the matrix elements of the ladder operators in the Fock basis. The logical structure is identical to the angular momentum spectrum derivation of QM5 Sec. 3: non-negativity of the diagonal operator forces termination of the lowering sequence, the termination condition identifies the ground state, and the ladder above the ground state generates the complete spectrum. The difference from QM5 is that the spectrum here is semi-infinite (no upper termination), as anticipated in Remark 3.4.

4.1 The Spectrum of the Number Operator

Theorem 4.1 (Complete spectrum of the harmonic oscillator)**.** *The spectrum of the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$ is*

$$\sigma(\hat{N}) = \{0, 1, 2, \ldots\}, \tag{22}$$

and the spectrum of $\hat{H}_{\text{osc}}$ is

$$\sigma(\hat{H}_{\text{osc}}) = \left\{ E_n := \left(n + \tfrac{1}{2}\right)\Phi_0\omega \;\middle|\; n \in \{0, 1, 2, \ldots\} \right\}. \tag{23}$$

Each eigenvalue E_n is non-degenerate *(simple). The corresponding eigenstates — the* Fock states *or* number states $|n\rangle$ *— satisfy*

$$\hat{N}|n\rangle = n\,|n\rangle, \qquad \hat{H}_{\text{osc}}|n\rangle = E_n|n\rangle, \qquad \langle n', n\rangle = \delta_{n'n}. \tag{24}$$

The Fock states form a complete orthonormal basis for $\mathcal{H}$, established in Proposition 5.7.

Proof. The proof proceeds in four steps.

Step 1: Non-negativity of eigenvalues. Suppose $\hat{N}|n\rangle = n|n\rangle$ for some normalized $|n\rangle$ and $n \in \mathbb{R}$. Then

$$n = \langle n, n\, n\rangle = \left\langle n, \hat{N}\, n \right\rangle = \left\langle n, \hat{a}^\dagger \hat{a}\, n \right\rangle = \langle \hat{a}\, n, \hat{a}\, n\rangle = \|\hat{a}|n\rangle\|^2_{\mathcal{H}} \geq 0,$$

using the self-adjointness of $\hat{a}^\dagger = \hat{a}^\dagger$. Therefore every eigenvalue of $\hat{N}$ is non-negative.

Step 2: Raising and lowering action. From Lemma 3.7, $[\hat{N}, \hat{a}] = -\hat{a}$ and $[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$. By the same argument as QM5 Proposition 4.2, if $\hat{N}|n\rangle = n|n\rangle$, then (when non-zero): $\hat{N}(\hat{a}|n\rangle) = (n-1)(\hat{a}|n\rangle)$ and $\hat{N}(\hat{a}^\dagger|n\rangle) = (n+1)(\hat{a}^\dagger|n\rangle)$. Thus $\hat{a}$ lowers the eigenvalue by 1 and $\hat{a}^\dagger$ raises it by 1.

Step 3: Termination and identification of the ground state. Starting from any eigenvalue $n \geq 0$ and applying $\hat{a}$ repeatedly, the sequence of eigenvalues $n, n-1, n-2, \ldots$ must terminate at a non-negative value (by Step 1). Let $n_{\min}$ be the termination point, so $\hat{a}|n_{\min}\rangle = 0$. Then

$$\hat{N}|n_{\min}\rangle = \hat{a}^\dagger\hat{a}|n_{\min}\rangle = \hat{a}^\dagger \cdot 0 = 0,$$

giving $n_{\min} = 0$. Since the eigenvalue decreases by 1 at each step and terminates at $n_{\min} = 0$, the starting eigenvalue n must be a non-negative integer: $n \in \{0, 1, 2, \ldots\}$.

Step 4: Non-degeneracy. Each eigenvalue $n \in \{0, 1, 2, \ldots\}$ is non-degenerate. Suppose $\hat{N}\Psi = n\Psi$ for some normalized Ψ. Then $\hat{a}^n\Psi$ is an eigenstate of $\hat{N}$ with eigenvalue 0 (by applying Step 2 n times), so $\hat{a}^n\Psi \propto |0\rangle$. But the vacuum $|0\rangle$ satisfying $\hat{a}|0\rangle = 0$ is unique up to phase: in position space this condition is a first-order ODE with a one-dimensional solution space (established in Theorem 5.3). Therefore Ψ is uniquely determined (up to phase) by n, confirming non-degeneracy.

The spectrum of $\hat{H}_{\text{osc}}$ follows from Eq. (17): $E_n = \Phi_0\omega(n + \frac{1}{2})$. □

Remark 4.2. *The harmonic oscillator spectrum* $\sigma(\hat{H}_{\rm osc}) = \{(n+\frac{1}{2})\Phi_0\omega : n \geq 0\}$ *and the angular momentum spectrum* $\sigma(\hat{L}^2) = \{\ell(\ell+1)\Phi_0^2 : \ell \geq 0\}$ *of QM5 both arise from ladder termination arguments, but their structures reflect the different algebras.*

The oscillator spectrum is equally spaced: consecutive eigenvalues are separated by a fixed gap $\Phi_0\omega$ *at every level. This equal spacing is a consequence of* $[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$ *with unit coefficient: each application of* $\hat{a}^\dagger$ *raises by exactly* 1 *unit regardless of the current level.*

The angular momentum spectrum $\ell(\ell+1)\Phi_0^2$ *is* not *equally spaced: the gap between consecutive levels grows as* ℓ *increases. This is because the* SO(3) *commutation algebra* $[\hat{L}_j, \hat{L}_k] = \mathrm{i}\Phi_0\,\epsilon_{jkl}\hat{L}_l$ *involves the generators themselves on the right, so the step size effectively increases with* ℓ.

*The two spectral structures represent the two canonical types of quantum spectra: the equally-spaced oscillator spectrum (Weyl-Heisenberg algebra) and the quadratically-growing angular momentum spectrum (*SO(3) *algebra).*

4.2 The Ground State and Zero-Point Energy

The ground state $|0\rangle$ satisfies $\hat{a}|0\rangle = 0$ and has energy $E_0 = \frac{1}{2}\Phi_0\omega$. The positivity of this energy is not a coincidence or a computational artifact; it is a structural consequence of the uncertainty principle of QM3.

Theorem 4.3 (Zero-point energy from the uncertainty principle)**.** *For any normalized state* $\Psi \in \mathcal{D}(\hat{H}_{\rm osc})$*:*

$$\langle \hat{H}_{\rm osc} \rangle \geq \frac{1}{2}\Phi_0\omega = E_0, \tag{25}$$

with equality if and only if Ψ *is the harmonic oscillator ground state* Ψ_0*, which is the Gaussian minimum-uncertainty state of QM3 with width* $\sigma = \ell_0/\sqrt{2}$*. The quantity* $E_0 = \frac{1}{2}\Phi_0\omega > 0$ *is the* zero-point energy *of the oscillator.*

Proof. For any normalized Ψ:

$$\langle \hat{H}_{\rm osc} \rangle = \frac{\langle \hat{p}^2 \rangle}{2m} + \frac{m\omega^2}{2}\langle \hat{x}^2 \rangle. \tag{26}$$

Using $\langle A^2 \rangle = (\Delta A)^2 + \langle A \rangle^2 \geq (\Delta A)^2$ for any observable A:

$$\langle \hat{H}_{\rm osc} \rangle \geq \frac{(\Delta p)^2}{2m} + \frac{m\omega^2}{2}(\Delta x)^2. \tag{27}$$

Apply the AM-GM inequality to the right-hand side of Eq. (27): for any $a, b \geq 0$, $a + b \geq 2\sqrt{ab}$. Setting $a = (\Delta p)^2/(2m)$ and $b = m\omega^2(\Delta x)^2/2$:

$$\frac{(\Delta p)^2}{2m} + \frac{m\omega^2}{2}(\Delta x)^2 \geq 2\sqrt{\frac{(\Delta p)^2}{2m} \cdot \frac{m\omega^2(\Delta x)^2}{2}} = \omega\,\Delta x \cdot \Delta p. \tag{28}$$

Applying the Heisenberg uncertainty relation Eq. (8): $\Delta x \cdot \Delta p \geq \Phi_0/2$. Combining with Eqs. (27) and (28):

$$\langle \hat{H}_{\rm osc} \rangle \geq \omega\,\Delta x \cdot \Delta p \geq \omega \cdot \frac{\Phi_0}{2} = \frac{1}{2}\Phi_0\omega,$$

which is Eq. (25).

Equality conditions. Equality in Eq. (25) requires equality in both inequalities. Equality in the Heisenberg bound requires $\Delta x \cdot \Delta p = \Phi_0/2$, i.e., Ψ is a Gaussian minimum-uncertainty state (QM3 Theorem 6.1). Equality in the AM-GM step requires $a = b$, i.e.,

$$\frac{(\Delta p)^2}{2m} = \frac{m\omega^2(\Delta x)^2}{2}, \quad \text{i.e.,} \quad \Delta p = m\omega\,\Delta x.$$

Combining this with the saturation condition $\Delta x \cdot \Delta p = \Phi_0/2$:

$$\Delta x \cdot m\omega\, \Delta x = \frac{\Phi_0}{2}, \quad \Rightarrow \quad (\Delta x)^2 = \frac{\Phi_0}{2m\omega} = \frac{\ell_0^2}{2}, \quad \Rightarrow \quad \Delta x = \frac{\ell_0}{\sqrt{2}}.$$

The minimum-uncertainty state with $\sigma = \Delta x = \ell_0/\sqrt{2}$ is the Gaussian $\Psi_{G(\ell_0/\sqrt{2},\langle x\rangle,\langle p\rangle)}$ of QM3 Theorem 6.1. For this to also achieve $\langle \hat{H}_{\rm osc}\rangle = \Phi_0\omega/2$, we further need $\langle x\rangle^2 = 0$ and $\langle p\rangle^2 = 0$ (so that $\langle A^2\rangle = (\Delta A)^2$ in Eq. (26)), i.e., $\langle x\rangle = \langle p\rangle = 0$. The ground state is therefore the specific Gaussian with $\sigma = \ell_0/\sqrt{2}$, $\langle x\rangle = 0$, $\langle p\rangle = 0$, which is the state derived in Theorem 5.3. Its uniqueness (up to phase) confirms the non-degeneracy of the ground state established in Theorem 4.1. □

Remark 4.4. *Theorem 4.3 establishes the zero-point energy as a structural consequence of the Heisenberg uncertainty relation of QM3 rather than as a computational result read off from the spectrum. The derivation makes the physical content precise: a state of zero energy in the harmonic oscillator would require $\Delta x = 0$ (perfect spatial localization at the equilibrium point) and $\Delta p = 0$ (perfect rest), which would violate $\Delta x \cdot \Delta p \geq \Phi_0/2$ from QM3. The minimum non-zero energy compatible with this bound is $E_0 = \Phi_0\omega/2$, achieved by the Gaussian ground state. In the NUVO framework, the zero-point energy is therefore not a mysterious quantum feature but an algebraic consequence of the canonical commutation relation: the same CCR that gives $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$ (and hence the $\frac{1}{2}\hat{\mathbf{1}}$ in $\hat{H}_{\rm osc} = \Phi_0\omega(\hat{N} + \frac{1}{2}\hat{\mathbf{1}})$) also gives $\Delta x \cdot \Delta p \geq \Phi_0/2$, and both are manifestations of the same underlying commutation structure.*

4.3 Matrix Elements of the Ladder Operators in the Fock Basis

The matrix elements of $\hat{a}$ and $\hat{a}^\dagger$ in the Fock basis are determined by normalization, in exact parallel with the matrix elements of $\hat{L}_-$ and $\hat{L}_+$ in the $|\,\ell, m\,\rangle$ basis established in QM5 Proposition 5.3.

Proposition 4.5 (Matrix elements of the oscillator ladder operators)**.** *In the Fock basis $\{|n\rangle\}_{n\geq 0}$:*

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\;|n+1\rangle, \tag{29}$$

$$\hat{a}\,|n\rangle = \sqrt{n}\;|n-1\rangle \quad (n \geq 1), \qquad \hat{a}\,|0\rangle = 0. \tag{30}$$

The Fock state $|n\rangle$ is generated from the vacuum $|0\rangle$ by repeated application of $\hat{a}^\dagger$:

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\,|0\rangle. \tag{31}$$

Proof. Creation operator, Eq. (29)*:* By Theorem 4.1, $\hat{a}^\dagger|n\rangle$ is an eigenstate of $\hat{N}$ with eigenvalue $n+1$ (from Lemma 3.7), so $\hat{a}^\dagger|n\rangle = c_n^+|n+1\rangle$ for some constant c_n^+. Compute $|c_n^+|^2 = \left\|\hat{a}^\dagger|n\rangle\right\|_{\mathcal{H}}^2$:

$$\left\|\hat{a}^\dagger|n\rangle\right\|_{\mathcal{H}}^2 = \left\langle n, \hat{a}\hat{a}^\dagger\, n\right\rangle.$$

From $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$, we have $\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + \hat{\mathbf{1}} = \hat{N} + \hat{\mathbf{1}}$. Therefore:

$$\left\langle n, \hat{a}\hat{a}^\dagger\, n\right\rangle = \left\langle n, (\hat{N} + \hat{\mathbf{1}})\, n\right\rangle = n + 1.$$

Choosing the positive real square root: $c_n^+ = \sqrt{n+1}$, giving Eq. (29).

Annihilation operator, Eq. (30)*:* Similarly, $\hat{a}|n\rangle = c_n^-|n-1\rangle$ for $n \geq 1$. Compute $|c_n^-|^2 = \|\hat{a}|n\rangle\|_{\mathcal{H}}^2 = \left\langle n, \hat{N}\, n\right\rangle = n$. Choosing the positive real square root: $c_n^- = \sqrt{n}$, giving Eq. (30). For $n = 0$: $\hat{a}|0\rangle = 0$ by the ground state condition established in the proof of Theorem 4.1.

Generation formula, Eq. (31)*:* Apply $\hat{a}^\dagger$ repeatedly to $|0\rangle$, using Eq. (29) at each step:

$$\hat{a}^\dagger|0\rangle = \sqrt{1}\,|1\rangle, \quad \hat{a}^\dagger|1\rangle = \sqrt{2}\,|2\rangle, \quad \ldots, \quad \hat{a}^\dagger|n-1\rangle = \sqrt{n}\,|n\rangle.$$

Composing: $(\hat{a}^\dagger)^n|0\rangle = \sqrt{n!}\,|n\rangle$, which rearranges to Eq. (31). □

Remark 4.6. *The matrix elements Eqs.* (29) *and* (30) *are structurally simpler than their angular momentum analogues from QM5 Proposition 5.3. The QM5 matrix elements involve the square root* $\sqrt{\ell(\ell+1) - m(m \pm 1)}$*, which depends on both quantum numbers* ℓ *and* m *and vanishes at the termination points* $m = \pm\ell$*. The oscillator matrix elements involve simply* $\sqrt{n}$ *and* $\sqrt{n+1}$*, which depend only on the single quantum number* n *and never vanish (except* $\hat{a}|0\rangle = 0$*, the ground state). The simplicity reflects the simpler algebra: the oscillator ladder raises* n *by* 1 *unconditionally (no upper bound), while the angular momentum ladder must vanish at the multiplet boundary. The generation formula Eq.* (31) *has a particularly clean form: the* $\sqrt{n!}$ *denominator accumulates exactly the product of* $\sqrt{1}\cdot\sqrt{2}\cdots\sqrt{n}$ *from the* n *applications of* $\hat{a}^\dagger$*.*

Remark 4.7. *The inverse relations Eq.* (15)*,* $\hat{x} = (\ell_0/\sqrt{2})(\hat{a} + \hat{a}^\dagger)$ *and* $\hat{p} = (\mathrm{i}p_0/\sqrt{2})(\hat{a}^\dagger - \hat{a})$*, combined with Proposition 4.5, give the matrix elements of* $\hat{x}$ *and* $\hat{p}$ *in the Fock basis:*

$$\langle n', \hat{x}\, n\rangle = \frac{\ell_0}{\sqrt{2}}\big(\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta_{n',n+1}\big),$$

$$\langle n', \hat{p}\, n\rangle = \frac{\mathrm{i}p_0}{\sqrt{2}}\big(\sqrt{n+1}\,\delta_{n',n+1} - \sqrt{n}\,\delta_{n',n-1}\big).$$

These matrix elements show that $\hat{x}$ *and* $\hat{p}$ *connect only adjacent Fock states (*$\Delta n = \pm 1$*), which is the origin of the dipole selection rule for the harmonic oscillator: only transitions between adjacent energy levels are allowed under a linear coupling* $V = -qE\hat{x}$*. This selection rule will be used in QM10 in the analysis of radiation emission and absorption by an oscillating charge.*

5 Energy Eigenstates in Position Space

The abstract Fock states $|n\rangle$ established in Sec. 4 are elements of the abstract Hilbert space $\mathcal{H}$. Their position-space representations — the wave functions $\Psi_n(x) = \langle x|n\rangle$ — are derived in the present section by translating the abstract ladder operator conditions into differential equations on $L^2(\mathbb{R})$. The ground state is determined by solving $\hat{a}\Psi_0 = 0$ as a first-order ODE; the excited states are obtained by applying the position-space form of $\hat{a}^\dagger$ to the ground state n times, and the polynomial factor generated by these applications is recognized as the n-th Hermite polynomial. The connection to the Gaussian minimum-uncertainty states of QM3 is made explicit: Ψ_0 is the specific Gaussian with width $\sigma = \ell_0/\sqrt{2}$ identified in the proof of Theorem 4.3.

5.1 The Annihilation Operator in Position Space

Before deriving the wave functions, it is useful to record the position-space forms of $\hat{a}$ and $\hat{a}^\dagger$ that result from substituting $\hat{x} = x$ (multiplication) and $\hat{p} = -\mathrm{i}\Phi_0\,\partial_x$ into Definition 3.1.

Lemma 5.1 (Ladder operators in position space)**.** *In position space, with* $\xi := x/\ell_0$*:*

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{x}{\ell_0} + \ell_0\,\partial_x\right) = \frac{1}{\sqrt{2}}(\xi + \partial_\xi)\,, \tag{32}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x}{\ell_0} - \ell_0\,\partial_x\right) = \frac{1}{\sqrt{2}}(\xi - \partial_\xi)\,. \tag{33}$$

Proof. Substitute $\hat{x} \to x$ and $\hat{p} \to -\mathrm{i}\Phi_0\,\partial_x$ into Eq. (13):

$$\hat{a} = \frac{m\omega\,x + \mathrm{i}(-\mathrm{i}\Phi_0\,\partial_x)}{\sqrt{2m\omega\Phi_0}} = \frac{m\omega\,x + \Phi_0\,\partial_x}{\sqrt{2m\omega\Phi_0}}.$$

Factor out $\ell_0 = \sqrt{\Phi_0/(m\omega)}$:

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{m\omega\,x}{\sqrt{m\omega\Phi_0}} + \frac{\Phi_0\,\partial_x}{\sqrt{m\omega\Phi_0}}\right) = \frac{1}{\sqrt{2}}\left(\frac{x}{\ell_0} + \ell_0\,\partial_x\right),$$

using $\sqrt{m\omega\Phi_0}/(m\omega) = \sqrt{\Phi_0/(m\omega)} = \ell_0$ and $\sqrt{\Phi_0}/\sqrt{m\omega\Phi_0} = 1/\sqrt{m\omega} = \ell_0/\Phi_0 \cdot \Phi_0 = \ell_0$. Substituting $\xi = x/\ell_0$ gives the second form of Eq. (32). The creation operator Eq. (33) follows by the same computation with the sign of $\hat{p}$ reversed. □

Remark 5.2. *In terms of the dimensionless variable $\xi = x/\ell_0$, the annihilation and creation operators take the elegant forms $\hat{a} = (\xi + \partial_\xi)/\sqrt{2}$ and $\hat{a}^\dagger = (\xi - \partial_\xi)/\sqrt{2}$. The commutation relation $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$ in these coordinates becomes $[(\xi + \partial_\xi)/\sqrt{2}, (\xi - \partial_\xi)/\sqrt{2}] = \hat{\mathbf{1}}$, which is equivalent to $[\partial_\xi, \xi] = \hat{\mathbf{1}}$, the one-dimensional CCR in dimensionless form. The operator $\hat{a}^\dagger = (\xi - \partial_\xi)/\sqrt{2}$ is sometimes written as $-\mathrm{e}^{\xi^2/2}\,\partial_\xi\,\mathrm{e}^{-\xi^2/2}$, a form that makes the Rodrigues formula for Hermite polynomials emerge directly from repeated application.*

5.2 The Ground State Wave Function

Theorem 5.3 (Ground state wave function)**.** *The harmonic oscillator ground state Ψ_0 is the unique (up to overall phase) normalized solution of $\hat{a}\Psi_0 = 0$ in $L^2(\mathbb{R})$, with position-space representation*

$$\Psi_0(x) = \frac{1}{(\pi\ell_0^2)^{1/4}} \exp\left(-\frac{x^2}{2\ell_0^2}\right). \tag{34}$$

This is the Gaussian minimum-uncertainty state of QM3 Theorem 6.1 with width $\sigma = \ell_0/\sqrt{2}$, mean position $\langle x\rangle = 0$, and mean momentum $\langle p\rangle = 0$, satisfying $\Delta x = \ell_0/\sqrt{2}$, $\Delta p = p_0/\sqrt{2}$, and $\Delta x \cdot \Delta p = \Phi_0/2$.

Proof. The condition $\hat{a}\Psi_0 = 0$ in position space, using Eq. (32), becomes

$$\frac{1}{\sqrt{2}}\left(\frac{x}{\ell_0} + \ell_0\,\partial_x\right)\Psi_0(x) = 0,$$

which rearranges to the first-order ODE:

$$\frac{\mathrm{d}\Psi_0}{\mathrm{d}x} = -\frac{x}{\ell_0^2}\,\Psi_0. \tag{35}$$

This is separable: $\mathrm{d}\Psi_0/\Psi_0 = -(x/\ell_0^2)\,\mathrm{d}x$. Integrating both sides: $\ln\Psi_0(x) = -x^2/(2\ell_0^2) + C$, giving $\Psi_0(x) = A\exp(-x^2/(2\ell_0^2))$ for a constant $A \in \mathbb{C}$. The normalization condition $\int_{\mathbb{R}} |\Psi_0(x)|^2\,\mathrm{d}x = 1$ fixes $|A|^2 \int_{\mathbb{R}} \mathrm{e}^{-x^2/\ell_0^2}\,\mathrm{d}x = |A|^2\sqrt{\pi}\ell_0 = 1$, giving $|A| = (\pi\ell_0^2)^{-1/4}$. Choosing A real and positive yields Eq. (34). Uniqueness up to phase follows from the one-dimensional solution space of the ODE Eq. (35) on $L^2(\mathbb{R})$: the only square-integrable solutions are constant multiples of the Gaussian, since $\exp(-x^2/(2\ell_0^2))$ decays at $\pm\infty$ while any other independent solution of the ODE grows.

Verification of minimum-uncertainty. The standard deviations follow from the Gaussian form: $\Delta x = \sigma = \ell_0/\sqrt{2}$ (the root-mean-square width of the closure density $|\Psi_0|^2$, which is a Gaussian of variance $\ell_0^2/2$) and $\Delta p = \Phi_0/(2\sigma) = \Phi_0/(2 \cdot \ell_0/\sqrt{2}) = \Phi_0\sqrt{2}/(2\ell_0) = \Phi_0/(\sqrt{2}\ell_0) = p_0/\sqrt{2}$ (by the QM3 Theorem 6.1 momentum-space Gaussian width formula, using $\ell_0 \cdot p_0 = \Phi_0$). The product $\Delta x \cdot \Delta p = (\ell_0/\sqrt{2}) \cdot (p_0/\sqrt{2}) = \ell_0 \cdot p_0/2 = \Phi_0/2$, confirming saturation of the Heisenberg bound. □

Remark 5.4. *The ground state wave function Eq.* (34) *is consistent with all prior results in the following senses. It achieves the zero-point energy bound of Theorem 4.3 (established by the equality conditions $\langle x\rangle = \langle p\rangle = 0$ and $\sigma = \ell_0/\sqrt{2}$). It is the Gaussian minimum-uncertainty state of QM3 Theorem 6.1 (confirmed by $\Delta x{\cdot}\Delta p = \Phi_0/2$). It is the vacuum of the ladder algebra (satisfying $\hat{a}\Psi_0 = 0$ by construction). These three characterizations — energy minimizer, uncertainty minimizer, and ladder vacuum — all select the same unique state Eq.* (34)*. Their coincidence is not accidental: as noted in Remark 4.4, all three are manifestations of the same canonical commutation relation.*

5.3 Excited State Wave Functions and Hermite Polynomials

The excited states Ψ_n for $n \geq 1$ are obtained by applying $(\hat{a}^\dagger)^n/\sqrt{n!}$ to the ground state, using the generation formula Eq. (31). In position space, the operator $\hat{a}^\dagger = (\xi - \partial_\xi)/\sqrt{2}$ applied repeatedly to the Gaussian generates a family of polynomials times the Gaussian, which are identified as the Hermite polynomials.

Theorem 5.5 (Excited state wave functions and Hermite polynomials)**.** *The n-th energy eigenstate Ψ_n, with $E_n = (n + \frac{1}{2})\Phi_0\omega$, has position-space representation*

$$\Psi_n(x) = \frac{1}{\sqrt{2^n\, n!\, \sqrt{\pi}\, \ell_0}}\, H_n(x/\ell_0)\, \exp\!\left(-\frac{x^2}{2\ell_0^2}\right), \tag{36}$$

where $H_n(\xi)$ is the Hermite polynomial *of degree n in ξ, defined by the Rodrigues formula*

$$H_n(\xi) := (-1)^n\, \mathrm{e}^{\xi^2}\, \frac{\mathrm{d}^n}{\mathrm{d}\xi^n}\, \mathrm{e}^{-\xi^2}. \tag{37}$$

Proof. From the generation formula Eq. (31): $\Psi_n(x) = (\hat{a}^\dagger)^n\Psi_0(x)/\sqrt{n!}$. In terms of the dimensionless variable $\xi = x/\ell_0$, the creation operator Eq. (33) is $\hat{a}^\dagger = (\xi - \partial_\xi)/\sqrt{2}$. Apply $\hat{a}^\dagger$ once to $\Psi_0(x) = (\pi\ell_0^2)^{-1/4}\mathrm{e}^{-\xi^2/2}$:

$$\begin{aligned}
\hat{a}^\dagger\Psi_0 &= \frac{1}{\sqrt{2}}(\xi - \partial_\xi)\frac{\mathrm{e}^{-\xi^2/2}}{(\pi\ell_0^2)^{1/4}} = \frac{1}{\sqrt{2}\,(\pi\ell_0^2)^{1/4}}\Big(\xi\,\mathrm{e}^{-\xi^2/2} - (-\xi)\mathrm{e}^{-\xi^2/2}\Big) \\
&= \frac{1}{\sqrt{2}\,(\pi\ell_0^2)^{1/4}} \cdot 2\xi\,\mathrm{e}^{-\xi^2/2} = \frac{H_1(\xi)\,\mathrm{e}^{-\xi^2/2}}{\sqrt{2}\,(\pi\ell_0^2)^{1/4}},
\end{aligned}$$

since $H_1(\xi) = 2\xi$ (verified from Eq. (37): $H_1(\xi) = -\mathrm{e}^{\xi^2}\partial_\xi\mathrm{e}^{-\xi^2} = -\mathrm{e}^{\xi^2}(-2\xi)\mathrm{e}^{-\xi^2} = 2\xi$). This gives $\Psi_1(x) = \hat{a}^\dagger\Psi_0(x)/\sqrt{1}$ in agreement with Eq. (36) for $n = 1$.

For general n, the key identity is

$$(\xi - \partial_\xi)^n\,\mathrm{e}^{-\xi^2/2} = H_n(\xi)\,\mathrm{e}^{-\xi^2/2}, \tag{38}$$

which is established by noting that $\xi - \partial_\xi = -\mathrm{e}^{\xi^2/2}\partial_\xi\,\mathrm{e}^{-\xi^2/2}$ (verified by applying both sides to a test function):

$$-\mathrm{e}^{\xi^2/2}\partial_\xi\big(\mathrm{e}^{-\xi^2/2}f\big) = -\mathrm{e}^{\xi^2/2}\big(-\xi\mathrm{e}^{-\xi^2/2}f + \mathrm{e}^{-\xi^2/2}f'\big) = \xi f - f' = (\xi - \partial_\xi)f.$$

Therefore:

$$\begin{aligned}
(\xi - \partial_\xi)^n\,\mathrm{e}^{-\xi^2/2} &= \Big(-\mathrm{e}^{\xi^2/2}\partial_\xi\,\mathrm{e}^{-\xi^2/2}\Big)^n\,\mathrm{e}^{-\xi^2/2} \\
&= (-1)^n\mathrm{e}^{\xi^2/2}\,\partial_\xi^n\big(\mathrm{e}^{-\xi^2/2}\cdot\mathrm{e}^{-\xi^2/2}\big) \quad \text{[telescope the middle factors]} \\
&= (-1)^n\mathrm{e}^{\xi^2/2}\,\partial_\xi^n\mathrm{e}^{-\xi^2} = \mathrm{e}^{-\xi^2/2}\cdot(-1)^n\mathrm{e}^{\xi^2}\partial_\xi^n\mathrm{e}^{-\xi^2} = \mathrm{e}^{-\xi^2/2}\,H_n(\xi),
\end{aligned}$$

confirming Eq. (38). Using $\hat{a}^\dagger = (\xi - \partial_\xi)/\sqrt{2}$ and Eq. (31):

$$\Psi_n(x) = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\Psi_0(x) = \frac{1}{\sqrt{n!}\,(\sqrt{2})^n} \cdot \frac{(\xi-\partial_\xi)^n \mathrm{e}^{-\xi^2/2}}{(\pi\ell_0^2)^{1/4}} = \frac{H_n(\xi)\,\mathrm{e}^{-\xi^2/2}}{\sqrt{n!\,2^n}\,(\pi\ell_0^2)^{1/4}},$$

which is Eq. (36) with $\xi = x/\ell_0$. The normalization constant $(\pi^{1/2}\ell_0\,2^n\,n!)^{-1/2}$ is confirmed by the standard orthonormality integral for Hermite polynomials: $\int_{-\infty}^{\infty} H_n(\xi)^2 \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi = \sqrt{\pi}\,2^n\,n!$ [1]. □

Remark 5.6. *The Hermite polynomials $H_n(\xi)$ satisfy the recurrence relation $H_{n+1}(\xi) = 2\xi H_n(\xi) - 2nH_{n-1}(\xi)$ and the differential equation $H_n(')'(\xi) - 2\xi H_n(')(\xi) + 2nH_n(()\xi) = 0$ (the Hermite ODE), which is the position-space form of the oscillator eigenvalue equation $\hat{H}_{\mathrm{osc}}\Psi_n = E_n\Psi_n$ after substituting Eq. (36). The first three cases are $H_0(\xi) = 1$, $H_1(\xi) = 2\xi$, and $H_2(\xi) = 4\xi^2 - 2$, from which the $n = 0$, 1, 2 wave functions can be written explicitly:*

$$\Psi_0(x) = \frac{1}{(\pi\ell_0^2)^{1/4}} \exp\left(-\frac{x^2}{2\ell_0^2}\right),$$
$$\Psi_1(x) = \frac{\sqrt{2}\,x/\ell_0}{(\pi\ell_0^2)^{1/4}} \exp\left(-\frac{x^2}{2\ell_0^2}\right),$$
$$\Psi_2(x) = \frac{4(x/\ell_0)^2 - 2}{\sqrt{8}\,(\pi\ell_0^2)^{1/4}} \exp\left(-\frac{x^2}{2\ell_0^2}\right).$$

The parity of Ψ_n under $x \to -x$ is $(-1)^n$: even n states are symmetric and odd n states are antisymmetric. This follows from $H_n(-\xi) = (-1)^n H_n(\xi)$, a consequence of the Rodrigues formula. The parity selection rule for the harmonic oscillator — dipole transitions are allowed only between states of opposite parity, i.e., between adjacent n values — follows from this property combined with the $\Delta n = \pm 1$ selection rule noted in Remark 4.7.

5.4 Orthonormality and Completeness

Proposition 5.7 (Orthonormality and completeness of the Fock states)**.** *The Fock states $\{\Psi_n\}_{n\geq 0}$ form a complete orthonormal basis for $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C})$:*

$$\langle \Psi_{n'}, \Psi_n \rangle_{\mathcal{H}} = \delta_{n'n}, \qquad \sum_{n=0}^{\infty} |\Psi_n\rangle\langle\Psi_n| = \hat{\mathbf{1}}_{\mathcal{H}}. \tag{39}$$

Proof. Orthonormality. For $n \neq n'$: the states Ψ_n and $\Psi_{n'}$ are eigenstates of the self-adjoint operator $\hat{H}_{\mathrm{osc}}$ with distinct eigenvalues $E_n \neq E_{n'}$; eigenstates of a self-adjoint operator corresponding to distinct eigenvalues are orthogonal. For $n = n'$: unit normalization was verified in the proof of Theorem 5.5 via the standard Hermite orthogonality integral [1].

Completeness. The operator $\hat{H}_{\mathrm{osc}}$ is a self-adjoint operator on $\mathcal{H}$ bounded below, with purely discrete spectrum $\sigma(\hat{H}_{\mathrm{osc}}) = \{(n + \frac{1}{2})\Phi_0\omega : n \geq 0\}$ (Theorem 4.1). By the spectral theorem for self-adjoint operators (QM1 Theorem 6.1), the family of eigenstates corresponding to the complete discrete spectrum forms a complete orthonormal basis for $\mathcal{H}$. Since each eigenvalue E_n is non-degenerate and the spectrum is $\{(n+\frac{1}{2})\Phi_0\omega : n \geq 0\}$, the family $\{\Psi_n\}_{n\geq 0}$ is precisely this complete basis, giving the resolution of the identity on the right-hand side of Eq. (39). □

Remark 5.8. *The completeness of the Fock states in $\mathcal{H} = L^2(\mathbb{R})$ means that any closure state $\Psi \in \mathcal{H}$ can be expanded as $\Psi = \sum_{n=0}^{\infty} c_n\Psi_n$ with $c_n = \langle \Psi_n, \Psi \rangle_{\mathcal{H}}$ and $\sum_n |c_n|^2 = 1$. This Fock* *basis expansion is the oscillator analogue of the spherical harmonic expansion of angular closure states on S^2 established in QM5 Proposition 6.3 (ii). In both cases, completeness is grounded in the spectral theorem of QM1 applied to a self-adjoint operator with discrete spectrum rather than in the theory of special functions. The Fock basis expansion will be used directly in Sec. 6 to derive the coherent state expansion Eq.* (41) *and in Sec. 7 to construct the three-dimensional oscillator eigenstates.*

6 Coherent States

The Fock states $|n\rangle$ established in Secs. 4 and 5 are the eigenstates of the energy — they have definite energy but maximally uncertain phase — and they form the natural basis for perturbation theory and matrix mechanics. However, for the analysis of the quantum-classical correspondence and for the physical sectors of QM9 through QM11, a different family of states is more natural: the *coherent states*, which are the states closest to classical oscillation. The present section defines coherent states as eigenstates of the annihilation operator, derives their Fock-state expansion and position-space Gaussian form, constructs them via the displacement operator, establishes their shape-preserving time evolution, and proves the overcompleteness of the coherent state family. Five results are established in sequence, each building on the previous: the Fock expansion (Definition 6.1) leads to the Gaussian form (Theorem 6.4), which connects to the displacement operator construction (Proposition 6.6), which enables the time evolution analysis (Theorem 6.8), which in turn motivates the overcompleteness (Proposition 6.10).

6.1 Definition and Fock-Basis Expansion

Definition 6.1 (Coherent states as eigenstates of $\hat{a}$)**.** *A* coherent state $|\alpha\rangle$ *with parameter* $\alpha \in \mathbb{C}$ *is a normalized eigenstate of the annihilation operator:*

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle, \qquad \langle\alpha,\alpha\rangle = 1. \tag{40}$$

Remark 6.2. *Since $\hat{a}$ is not self-adjoint, its eigenvalues need not be real. The eigenvalue α in Eq.* (40) *is in general a complex number, $\alpha \in \mathbb{C}$, and every $\alpha \in \mathbb{C}$ is an eigenvalue of $\hat{a}$. This contrasts with the self-adjoint operators $\hat{H}_{\mathrm{osc}}$ and $\hat{N}$, whose eigenvalues are the real numbers $(n + \frac{1}{2})\Phi_0\omega$ and n respectively. The parametrization of coherent states by $\alpha \in \mathbb{C}$ (a two-real-parameter family) versus the parametrization of Fock states by $n \in \mathbb{Z}_{\geq 0}$ (a one-integer-parameter family) reflects the fact that the coherent state family is overcomplete: there are continuously many more coherent states than Fock states, as established in Proposition 6.10.*

The Fock-basis expansion of the coherent state is derived from the eigenvalue condition Eq. (40) combined with the matrix elements of Proposition 4.5.

Lemma 6.3 (Fock-basis expansion of coherent states)**.** *The unique normalized eigenstate of $\hat{a}$ with eigenvalue $\alpha \in \mathbb{C}$ has the Fock-basis expansion*

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle. \tag{41}$$

Proof. Write $|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$ in the Fock basis (using completeness, Proposition 5.7). Apply the annihilation operator using Eq. (30):

$$\hat{a}|\alpha\rangle = \sum_{n=1}^{\infty} c_n\sqrt{n}\,|n-1\rangle = \sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}\,|n\rangle.$$

The eigenvalue equation $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ requires $\sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}\,|n\rangle = \alpha\sum_{n=0}^{\infty} c_n|n\rangle$. Comparing coefficients of $|n\rangle$:

$$c_{n+1}\sqrt{n+1} = \alpha\, c_n, \quad \Rightarrow \quad c_{n+1} = \frac{\alpha}{\sqrt{n+1}}\, c_n. \tag{42}$$

Solving the recurrence by induction: $c_n = (\alpha^n/\sqrt{n!})\, c_0$. The normalization condition $\sum_{n=0}^{\infty} |c_n|^2 = 1$ gives

$$|c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |c_0|^2 \mathrm{e}^{|\alpha|^2} = 1,$$

so $|c_0| = \mathrm{e}^{-|\alpha|^2/2}$. Choosing c_0 real and positive (fixing the overall phase) gives $c_0 = \mathrm{e}^{-|\alpha|^2/2}$ and hence Eq. (41). □

6.2 Equivalence to Gaussian Minimum-Uncertainty States

Theorem 6.4 (Coherent states are Gaussian minimum-uncertainty states)**.** *The position-space representation of the coherent state $|\alpha\rangle$ is the Gaussian*

$$\langle x|\alpha\rangle = \frac{1}{(\pi\ell_0^2)^{1/4}} \exp\!\left(-\frac{(x-\langle x\rangle_\alpha)^2}{2\ell_0^2} + \frac{\mathrm{i}\langle p\rangle_\alpha\, x}{\Phi_0} - \frac{\mathrm{i}\langle x\rangle_\alpha\,\langle p\rangle_\alpha}{2\Phi_0}\right), \tag{43}$$

where

$$\langle x\rangle_\alpha = \ell_0\sqrt{2}\,\mathrm{Re}(\alpha), \qquad \langle p\rangle_\alpha = p_0\sqrt{2}\,\mathrm{Im}(\alpha). \tag{44}$$

This is the Gaussian minimum-uncertainty state of QM3 Theorem 6.1 with width $\sigma = \ell_0/\sqrt{2}$, satisfying $\Delta x = \ell_0/\sqrt{2}$ and $\Delta p = p_0/\sqrt{2}$ for all $\alpha \in \mathbb{C}$.

Proof. The position-space condition $\hat{a}\langle x|\alpha\rangle = \alpha\langle x|\alpha\rangle$, using the position-space form Eq. (32) of $\hat{a}$, becomes

$$\frac{1}{\sqrt{2}}\left(\frac{x}{\ell_0} + \ell_0\,\partial_x\right)\langle x|\alpha\rangle = \alpha\,\langle x|\alpha\rangle. \tag{45}$$

This is the first-order ODE:

$$\partial_x\langle x|\alpha\rangle = \left(\frac{\sqrt{2}\,\alpha}{\ell_0} - \frac{x}{\ell_0^2}\right)\langle x|\alpha\rangle.$$

Completing the square in the exponent: $\sqrt{2}\alpha/\ell_0 - x/\ell_0^2 = -(x-\sqrt{2}\alpha\ell_0)/\ell_0^2$, giving

$$\langle x|\alpha\rangle = A(\alpha)\,\exp\!\left(-\frac{(x-\sqrt{2}\alpha\ell_0)^2}{2\ell_0^2}\right),$$

where $A(\alpha)$ is a α-dependent normalization constant. Writing $\alpha = \mathrm{Re}(\alpha) + \mathrm{i}\,\mathrm{Im}(\alpha)$ and $x_\alpha = \ell_0\sqrt{2}\,\mathrm{Re}(\alpha)$, $p_\alpha = p_0\sqrt{2}\,\mathrm{Im}(\alpha)$:

$$\sqrt{2}\alpha\ell_0 = x_\alpha + \mathrm{i}\sqrt{2}\,\mathrm{Im}(\alpha)\ell_0 = x_\alpha + \frac{\mathrm{i}p_\alpha\ell_0}{p_0} = x_\alpha + \frac{\mathrm{i}p_\alpha\Phi_0}{p_0^2}\cdot\frac{p_0}{\Phi_0}\cdot\ell_0,$$

using $p_0 = m\omega\ell_0$ so that $\ell_0/p_0 = 1/(m\omega)$. Substituting and expanding the squared term:

$$-\frac{(x-\sqrt{2}\alpha\ell_0)^2}{2\ell_0^2} = -\frac{(x-x_\alpha)^2}{2\ell_0^2} + \frac{\mathrm{i}p_\alpha(x-x_\alpha)}{\Phi_0} + \frac{\mathrm{i}p_\alpha\,x_\alpha}{\Phi_0} - \frac{p_\alpha^2}{2p_0^2}.$$

The term $-p_\alpha^2/(2p_0^2)$ is absorbed into the normalization constant $A(\alpha)$; combining the x-dependent terms:

$$\langle x|\alpha\rangle = A'(\alpha)\,\exp\!\left(-\frac{(x-x_\alpha)^2}{2\ell_0^2} + \frac{\mathrm{i}p_\alpha\, x}{\Phi_0} - \frac{\mathrm{i}p_\alpha\, x_\alpha}{2\Phi_0}\right),$$

where we identified $\mathrm{i}p_\alpha(x-x_\alpha)/\Phi_0 + \mathrm{i}p_\alpha x_\alpha/\Phi_0 = \mathrm{i}p_\alpha x/\Phi_0$ and absorbed the remaining x_α phase into a redefined A'. The overall normalization $\int_{\mathbb{R}} |\langle x|\alpha\rangle|^2\,\mathrm{d}x = 1$ fixes $|A'(\alpha)| = (\pi\ell_0^2)^{-1/4}$ (since the Gaussian envelope $\mathrm{e}^{-(x-x_\alpha)^2/\ell_0^2}$ integrates to $\sqrt{\pi}\ell_0$), giving Eq. (43).

Minimum-uncertainty verification. The closure density $|\langle x|\alpha\rangle|^2 = (\pi\ell_0^2)^{-1/2}\mathrm{e}^{-(x-x_\alpha)^2/\ell_0^2}$ is a Gaussian centered at x_α with variance $\ell_0^2/2$, so $\Delta x = \ell_0/\sqrt{2}$. The momentum density $|\langle p|\alpha\rangle|^2$ is similarly Gaussian with variance $p_0^2/2$, giving $\Delta p = p_0/\sqrt{2}$. The product $\Delta x\cdot\Delta p = (\ell_0/\sqrt{2})\cdot(p_0/\sqrt{2}) = \ell_0 p_0/2 = \Phi_0/2$, confirming saturation of the Heisenberg bound independently of α. □

Remark 6.5. *Theorem 6.4 establishes the equivalence between two of the three characterizations of coherent states: eigenstate of $\hat{a}$ (Definition 6.1) and Gaussian minimum-uncertainty state (QM3 Theorem 6.1 with the specific width $\sigma = \ell_0/\sqrt{2}$). The third characterization — displaced vacuum $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$ — is established in Proposition 6.6. The equivalence of all three is summarized as:*

$$\begin{aligned} &\hat{a}|\alpha\rangle = \alpha|\alpha\rangle && \text{(Definition 6.1)}\\ \Leftrightarrow\quad &|\langle x|\alpha\rangle|^2 \text{ Gaussian with } \sigma = \ell_0/\sqrt{2} && \text{(Theorem 6.4)}\\ \Leftrightarrow\quad &|\alpha\rangle = \hat{D}(\alpha)|0\rangle && \text{(Proposition 6.6).}\end{aligned}$$

The physical content of the width $\sigma = \ell_0/\sqrt{2}$ is that it is the unique width preserved by the harmonic oscillator dynamics: if the initial state is Gaussian with any other width $\sigma \neq \ell_0/\sqrt{2}$, the width will oscillate in time (the state "breathes") rather than remaining constant, as established in Theorem 6.8.

6.3 The Displacement Operator

Proposition 6.6 (Displacement operator construction of coherent states)**.** *The* displacement operator

$$\hat{D}(\alpha) := \exp\!\big(\alpha\hat{a}^\dagger - \overline{\alpha}\hat{a}\big) \tag{46}$$

is unitary with $\hat{D}(\alpha)^\dagger = \hat{D}(-\alpha) = \hat{D}(\alpha)^{-1}$, and satisfies

$$\hat{D}(\alpha)\,\hat{a}\,\hat{D}(\alpha)^\dagger = \hat{a} + \alpha\,\hat{\mathbf{1}}. \tag{47}$$

Consequently, the coherent state $|\alpha\rangle$ is generated from the vacuum by

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle. \tag{48}$$

Proof. Unitarity. The exponent $\alpha\hat{a}^\dagger - \overline{\alpha}\hat{a}$ is anti-Hermitian: $(\alpha\hat{a}^\dagger - \overline{\alpha}\hat{a})^\dagger = \overline{\alpha}\hat{a} - \alpha\hat{a}^\dagger = -(\alpha\hat{a}^\dagger - \overline{\alpha}\hat{a})$. The exponential of an anti-Hermitian operator is unitary, so $\hat{D}(\alpha)^\dagger = \hat{D}(-\alpha) = \hat{D}(\alpha)^{-1}$.

Conjugation identity Eq. (47). Let $X = \alpha\hat{a}^\dagger - \overline{\alpha}\hat{a}$. The commutator $[X,\hat{a}] = [\alpha\hat{a}^\dagger,\hat{a}] - [\overline{\alpha}\hat{a},\hat{a}] = \alpha[\hat{a}^\dagger,\hat{a}] - 0 = \alpha(-\hat{\mathbf{1}}) = -\alpha\,\hat{\mathbf{1}}$. By the Baker-Campbell-Hausdorff (BCH) lemma for operators with $[X,\hat{a}]$ a scalar multiple of the identity:

$$\mathrm{e}^X\hat{a}\mathrm{e}^{-X} = \hat{a} + [X,\hat{a}] + \frac{1}{2!}[X,[X,\hat{a}]] + \cdots = \hat{a} + (-\alpha\,\hat{\mathbf{1}}) + 0 + \cdots = \hat{a} - \alpha\,\hat{\mathbf{1}},$$

since $[X, -\alpha\,\hat{\mathbf{1}}] = 0$. Therefore $\hat{D}(\alpha)\hat{a}\hat{D}(\alpha)^\dagger = \mathrm{e}^X\hat{a}\mathrm{e}^{-X} = \hat{a} - \alpha\,\hat{\mathbf{1}}$. Replacing $\alpha \to -\alpha$ (equivalently, computing $\hat{D}(\alpha)^\dagger\hat{a}\hat{D}(\alpha)$ instead):

$$\hat{D}(\alpha)^\dagger\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha\,\hat{\mathbf{1}}. \tag{49}$$

Multiplying both sides on the left by $\hat{D}(\alpha)$ and on the right by $\hat{D}(\alpha)^\dagger$: $\hat{a}\hat{D}(\alpha) = \hat{D}(\alpha)(\hat{a} + \alpha\,\hat{\mathbf{1}})$, which rearranges to Eq. (47).

Generation of coherent states, Eq. (48). From Eq. (49), applying $\hat{D}(\alpha)|0\rangle$:

$$\hat{a}(\hat{D}(\alpha)|0\rangle) = \hat{D}(\alpha)(\hat{a} + \alpha\,\hat{\mathbf{1}})|0\rangle = \hat{D}(\alpha)(0 + \alpha|0\rangle) = \alpha(\hat{D}(\alpha)|0\rangle),$$

so $\hat{D}(\alpha)|0\rangle$ is a normalized eigenstate of $\hat{a}$ with eigenvalue α. By Lemma 6.3 (uniqueness of the coherent state for each α), $\hat{D}(\alpha)|0\rangle = \mathrm{e}^{\mathrm{i}\vartheta}|\alpha\rangle$ for some phase ϑ. The phase is fixed to $\vartheta = 0$ by comparing with the Fock expansion Eq. (41) at $x = 0$ (or by the BCH factorization $\hat{D}(\alpha) = \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\alpha\hat{a}^\dagger}\mathrm{e}^{-\overline{\alpha}\hat{a}}$, which gives the same expansion directly). □

Remark 6.7. *In terms of the position and momentum operators, using the inverse relations Eq.* (15)*:*

$$\alpha\hat{a}^\dagger - \overline{\alpha}\hat{a} = \frac{\mathrm{i}}{\Phi_0}\big(\langle p\rangle_\alpha\,\hat{x} - \langle x\rangle_\alpha\,\hat{p}\big),$$

so the displacement operator takes the form $\hat{D}(\alpha) = \exp[(\mathrm{i}/\Phi_0)(\langle p\rangle\,\hat{x} - \langle x\rangle\,\hat{p})]$. This is the unitary operator that displaces the vacuum $|0\rangle$ (a Gaussian centered at the origin) to a Gaussian centered at $(\langle x\rangle_\alpha, \langle p\rangle_\alpha)$ in phase space. The action of $\hat{D}(\alpha)$ is therefore a phase-space translation: it moves the center of the Gaussian without changing its shape, which is consistent with the fact that all coherent states have the same width $\sigma = \ell_0/\sqrt{2}$, independently of α.

6.4 Shape-Preserving Time Evolution

Theorem 6.8 (Shape-preserving time evolution of coherent states)**.** *Under the harmonic oscillator time evolution $U(t) = \mathrm{e}^{-\mathrm{i}\hat{H}_{\mathrm{osc}}t/\Phi_0}$, a coherent state evolves as*

$$U(t)\,|\alpha\rangle = \mathrm{e}^{-\mathrm{i}\omega t/2}\,|\alpha(t)\rangle, \tag{50}$$

where $\alpha(t) = \alpha\,\mathrm{e}^{-\mathrm{i}\omega t}$. The time-evolved state is again a coherent state with the same Gaussian shape: the widths

$$\Delta x(t) = \frac{\ell_0}{\sqrt{2}}, \qquad \Delta p(t) = \frac{p_0}{\sqrt{2}} \tag{51}$$

are constant for all t. The centroid follows the classical trajectory:

$$\langle x\rangle(t) = \ell_0\sqrt{2}\,|\alpha|\cos(\omega t - \arg\alpha), \tag{52}$$

$$\langle p\rangle(t) = -p_0\sqrt{2}\,|\alpha|\sin(\omega t - \arg\alpha), \tag{53}$$

where $\arg\alpha = \arg\alpha$.

Proof. Time evolution in the Heisenberg picture. From Remark 3.8, the Heisenberg-picture annihilation operator satisfies $\hat{a}_{\mathrm{H}}(t) = \hat{a}\,\mathrm{e}^{-\mathrm{i}\omega t}$, equivalently $U(t)^\dagger\hat{a}\,U(t) = \hat{a}\,\mathrm{e}^{-\mathrm{i}\omega t}$.

$U(t)|\alpha\rangle$ is a coherent state. Apply $\hat{a}$ to $U(t)|\alpha\rangle$:

$$\begin{aligned}\hat{a}\big(U(t)|\alpha\rangle\big) &= U(t)\big(U(t)^\dagger\hat{a}\,U(t)\big)|\alpha\rangle\\ &= U(t)\big(\hat{a}\,\mathrm{e}^{-\mathrm{i}\omega t}\big)|\alpha\rangle\\ &= U(t)\big(\mathrm{e}^{-\mathrm{i}\omega t}\alpha\big)|\alpha\rangle\\ &= \alpha(t)\big(U(t)|\alpha\rangle\big),\end{aligned}$$

where $\alpha(t) = \alpha e^{-i\omega t}$. Therefore $U(t)|\alpha\rangle$ is an eigenstate of $\hat{a}$ with eigenvalue $\alpha(t)$, hence a coherent state $|\alpha(t)\rangle$ up to an overall phase.

The overall phase. Evaluate the phase using the Fock expansion Eq. (41):

$$\begin{aligned} U(t)|\alpha\rangle &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} U(t)|n\rangle \\ &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i(n+1/2)\omega t}|n\rangle \\ &= e^{-i\omega t/2} e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}}|n\rangle \\ &= e^{-i\omega t/2}\, |\alpha(t)\rangle, \end{aligned}$$

confirming Eq. (50). The factor $e^{-i\omega t/2}$ arises from the zero-point energy $E_0 = \frac{1}{2}\Phi_0\omega$.

Constant widths. Since $U(t)|\alpha\rangle = e^{-i\omega t/2}|\alpha(t)\rangle$ is a coherent state (up to an overall phase), Theorem 6.4 applies and gives $\Delta x(t) = \ell_0/\sqrt{2}$ and $\Delta p(t) = p_0/\sqrt{2}$ for all t, confirming Eq. (51).

Centroid trajectory. From Eq. (44) applied to $\alpha(t) = |\alpha| e^{i(\arg\alpha - \omega t)}$: $\langle x\rangle(t) = \ell_0\sqrt{2}\,\mathrm{Re}(\alpha(t)) = \ell_0\sqrt{2}\,|\alpha|\cos(\arg\alpha - \omega t)$ and $\langle p\rangle(t) = p_0\sqrt{2}\,\mathrm{Im}(\alpha(t)) = -p_0\sqrt{2}\,|\alpha|\sin(\arg\alpha - \omega t)$, which are Eqs. (52) and (53). These are the classical harmonic oscillator trajectories Eqs. (4)–(5), confirming consistency with the Ehrenfest theorem. □

Remark 6.9. *Theorem 6.8 establishes the dynamical characterization of coherent states that completes the program arc from QM3. For a general Gaussian state with width $\sigma \neq \ell_0/\sqrt{2}$, the harmonic oscillator dynamics cause the width to oscillate: $\Delta x(t)$ alternates between σ and $\Phi_0/(2m\omega\sigma) = \ell_0^2/(2\sigma)$ at twice the oscillator frequency. Only for $\sigma = \ell_0/\sqrt{2}$ is $\ell_0^2/(2\sigma) = \sigma$, so that the two extremes coincide and the width is constant. The QM3 minimum-uncertainty family (all Gaussians) is dynamically filtered by the oscillator to the coherent state family (the single width $\ell_0/\sqrt{2}$, with α as the remaining free parameter encoding the classical initial conditions). This is the precise content of the program arc from QM3 to QM6 identified in Sec. 1.1.*

6.5 Overcompleteness of the Coherent State Family

Proposition 6.10 (Overcompleteness of the coherent state family)**.** *The coherent states are not mutually orthogonal:*

$$\langle\alpha'|\alpha\rangle = \exp\left(\overline{\alpha'}\alpha - \frac{|\alpha'|^2}{2} - \frac{|\alpha|^2}{2}\right), \tag{54}$$

and the family $\{|\alpha\rangle : \alpha \in \mathbb{C}\}$ satisfies the resolution of the identity:

$$\frac{1}{\pi}\int_{\mathbb{C}} |\alpha\rangle\langle\alpha|\, d^2\alpha = \hat{\mathbf{1}}_{\mathcal{H}}, \tag{55}$$

where $d^2\alpha = d(\mathrm{Re}\,\alpha)\, d(\mathrm{Im}\,\alpha)$. The coherent state family is therefore overcomplete*: it spans $\mathcal{H}$ (via Eq. (55)) but is not linearly independent (each coherent state has non-zero overlap with all others via Eq. (54)).*

Proof. Overlap, Eq. (54)*:* Using the Fock expansions of $|\alpha'\rangle$ and $|\alpha\rangle$ from Eq. (41):

$$\langle\alpha'|\alpha\rangle = \mathrm{e}^{-(|\alpha'|^2+|\alpha|^2)/2} \sum_{n',n=0}^{\infty} \frac{\overline{\alpha'}^{n'} \alpha^n}{\sqrt{n'!\,n!}} \underbrace{\langle n', n\rangle}_{\delta_{n'n}}$$

$$= \mathrm{e}^{-(|\alpha'|^2+|\alpha|^2)/2} \sum_{n=0}^{\infty} \frac{(\overline{\alpha'}\alpha)^n}{n!} = \mathrm{e}^{-(|\alpha'|^2+|\alpha|^2)/2}\mathrm{e}^{\overline{\alpha'}\alpha},$$

which is Eq. (54).

Resolution of identity, Eq. (55)*:* Compute the matrix element $\langle n'|(1/\pi)\int|\alpha\rangle\langle\alpha|\,\mathrm{d}^2\alpha\,|n\rangle$:

$$\frac{1}{\pi}\int_{\mathbb{C}} \langle n'|\alpha\rangle\langle\alpha|n\rangle\,\mathrm{d}^2\alpha = \frac{\mathrm{e}^0}{\pi}\int_{\mathbb{C}} \frac{\mathrm{e}^{-|\alpha|^2}\overline{\alpha}^{n'}\alpha^n}{\sqrt{n'!\,n!}}\,\mathrm{d}^2\alpha.$$

Converting to polar coordinates $\alpha = r\mathrm{e}^{\mathrm{i}\varphi}$ with $\mathrm{d}^2\alpha = r\,\mathrm{d}r\,\mathrm{d}\varphi$:

$$\frac{1}{\pi\sqrt{n'!\,n!}}\int_0^{\infty}\int_0^{2\pi} r^{n'+n+1}\mathrm{e}^{-r^2}\mathrm{e}^{\mathrm{i}(n-n')\varphi}\,\mathrm{d}\varphi\,\mathrm{d}r.$$

The angular integral gives $\int_0^{2\pi}\mathrm{e}^{\mathrm{i}(n-n')\varphi}\,\mathrm{d}\varphi = 2\pi\delta_{n'n}$. For $n'=n$, the radial integral: $\int_0^{\infty} r^{2n+1}\mathrm{e}^{-r^2}\,\mathrm{d}r = n!/2$ (substituting $u=r^2$). Therefore:

$$\frac{1}{\pi}\int_{\mathbb{C}} \langle n|\alpha\rangle\langle\alpha|n'\rangle\,\mathrm{d}^2\alpha = \frac{2\pi\delta_{n'n}}{2\pi n!}\cdot n! = \delta_{n'n},$$

which is the matrix element of $\hat{\mathbf{1}}_{\mathcal{H}}$ in the Fock basis, confirming Eq. (55). □

Remark 6.11. *The resolution of the identity Eq.* (55) *differs structurally from the orthonormal resolutions established in earlier papers. The Fock state resolution* $\sum_{n=0}^{\infty}|n\rangle\langle n| = \hat{\mathbf{1}}$ *(Proposition 5.7) is a discrete sum over orthonormal states. The spherical harmonic resolution* $\sum_{\ell,m}|Y_\ell^m\rangle\langle Y_\ell^m| = \hat{\mathbf{1}}_{L^2(S^2)}$ *(QM5 Proposition 6.3 (ii)) is a discrete sum over orthonormal angular eigenstates. The coherent state resolution Eq.* (55) *is a continuous integral with the factor* $1/\pi$ *(rather than a sum) over a non-orthogonal family with the phase-space area measure* $\mathrm{d}^2\alpha$*. The* $1/\pi$ *factor arises because the coherent states are overcomplete by exactly a factor of* π *per unit phase-space area, reflecting the Heisenberg phase-space uncertainty cell area* $\Delta x\cdot\Delta p = \Phi_0/2$ *whose "phase-space volume" in the* $(\mathrm{Re}\,\alpha, \mathrm{Im}\,\alpha)$ *plane is* $1/(2) =$ *half a unit, accounting for the* $1/\pi$ *normalization. The overcompleteness relation Eq.* (55) *is the foundation of the coherent-state path integral and the Bargmann-Fock representation, both of which will be used in QM10 and the field-theoretic extensions.*

7 The Three-Dimensional Isotropic Oscillator

The one-dimensional harmonic oscillator of Secs. 3–6 is now extended to three spatial dimensions. The three-dimensional isotropic harmonic oscillator $\hat{H}_{\mathrm{osc}}^{(3)} = \sum_{j=1}^{3}[\hat{p}_j^2/(2m) + \frac{1}{2}m\omega^2(\hat{x}^j)^2]$ admits two equivalent treatments: the Cartesian product of three independent one-dimensional oscillators, which makes the spectrum and degeneracy immediately transparent; and the spherical coordinate separation, which exploits the rotational symmetry of the isotropic potential and expresses the eigenstates as products of radial functions and the QM5 spherical harmonics $Y_\ell^m(\theta,\varphi)$. The two representations are complementary: the Cartesian representation counts states by quantum number triples (n_x, n_y, n_z), while the spherical representation labels them by the physically meaningful angular momentum quantum numbers (n_r, ℓ, m). Both yield the same spectrum $E_N = (N+\frac{3}{2})\Phi_0\omega$ and the same degeneracy $d_N = (N+1)(N+2)/2$, providing a consistency check on both derivations.

7.1 The Three-Dimensional Hamiltonian and Its Symmetry

Proposition 7.1 (Rotational symmetry of the isotropic oscillator)**.** *The three-dimensional isotropic harmonic oscillator Hamiltonian*

$$\hat{H}_{\text{osc}}^{(3)} := \sum_{j=1}^{3} \left[\frac{\hat{p}_j^2}{2m} + \frac{1}{2} m\omega^2 (\hat{x}^j)^2 \right] = -\frac{\Phi_0^2}{2m}\Delta + \frac{1}{2} m\omega^2 |\mathbf{x}|^2 \tag{56}$$

is self-adjoint on $\mathcal{H}^{(3)} = L^2(\mathbb{R}^3, \mathbb{C})$ *by QM4 Theorem 4.2. Its potential* $V(\mathbf{x}) = \frac{1}{2} m\omega^2 |\mathbf{x}|^2 = \frac{1}{2} m\omega^2 r^2$ *is rotationally symmetric, so by QM4 Proposition 7.3:*

$$[\hat{H}_{\text{osc}}^{(3)}, \hat{L}_j] = 0, \qquad j = 1, 2, 3, \tag{57}$$

and consequently $[\hat{H}_{\text{osc}}^{(3)}, \hat{L}^2] = 0$*. The eigenstates of* $\hat{H}_{\text{osc}}^{(3)}$ *can therefore be chosen to be simultaneous eigenstates of* $\hat{H}_{\text{osc}}^{(3)}$*,* $\hat{L}^2$*, and* $\hat{L}_3$*.*

Proof. Self-adjointness follows from QM4 Theorem 4.2 applied to the three-dimensional Kato-class potential $V(\mathbf{x}) = \frac{1}{2} m\omega^2 |\mathbf{x}|^2$ (which is locally square-integrable and satisfies the Kato-Rellich bound). The Cartesian form of $\hat{H}_{\text{osc}}^{(3)}$ decomposes as $\hat{H}_{\text{osc}}^{(3)} = \hat{H}_{\text{osc}}^{(x)} + \hat{H}_{\text{osc}}^{(y)} + \hat{H}_{\text{osc}}^{(z)}$, the sum of three commuting one-dimensional oscillators. Rotational symmetry: the potential $V(r) = \frac{1}{2} m\omega^2 r^2$ depends only on $r = |\mathbf{x}|$, so by QM4 Proposition 7.3, $[\hat{H}_{\text{osc}}^{(3)}, \hat{L}_j] = 0$ for all j. Simultaneous diagonalizability follows from $[\hat{H}_{\text{osc}}^{(3)}, \hat{L}^2] = 0$ (from $[\hat{H}_{\text{osc}}^{(3)}, \hat{L}_j] = 0$ and the identity $\hat{L}^2 = \sum_j \hat{L}_j^2$) and $[\hat{L}^2, \hat{L}_3] = 0$ (QM5 Theorem 3.2). □

The Laplacian decomposes in spherical coordinates as (QM5 Proposition 6.1, recalled):

$$-\frac{\Phi_0^2}{2m}\Delta = -\frac{\Phi_0^2}{2m} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{\hat{L}^2}{2mr^2}, \tag{58}$$

giving the spherical-coordinate form of $\hat{H}_{\text{osc}}^{(3)}$:

$$\hat{H}_{\text{osc}}^{(3)} = -\frac{\Phi_0^2}{2m} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{\hat{L}^2}{2mr^2} + \frac{1}{2} m\omega^2 r^2. \tag{59}$$

This decomposition separates the kinetic energy into a radial part and a centrifugal term $\hat{L}^2/(2mr^2)$, enabling the spherical representation of Sec. 7.3.

7.2 Cartesian Representation: Three Independent Oscillators

Proposition 7.2 (Cartesian eigenstructure of the 3D oscillator)**.** *Since* $\hat{H}_{\text{osc}}^{(3)} = \hat{H}_{\text{osc}}^{(x)} + \hat{H}_{\text{osc}}^{(y)} + \hat{H}_{\text{osc}}^{(z)}$ *with* $[\hat{H}_{\text{osc}}^{(j)}, \hat{H}_{\text{osc}}^{(k)}] = 0$ *for* $j \neq k$*, the three-dimensional eigenstates in Cartesian coordinates are product states:*

$$\Psi_{n_x n_y n_z}(x, y, z) = \Psi_{n_x}(x)\, \Psi_{n_y}(y)\, \Psi_{n_z}(z), \tag{60}$$

where Ψ_{n_j} *are the one-dimensional Hermite-Gaussian eigenstates of Theorem 5.5. The corresponding energy eigenvalue is*

$$E_{n_x, n_y, n_z} = \left(n_x + n_y + n_z + \frac{3}{2} \right) \Phi_0 \omega = \left(N + \frac{3}{2} \right) \Phi_0 \omega, \tag{61}$$

where $N := n_x + n_y + n_z \in \{0, 1, 2, \ldots\}$ *is the* total excitation number*.*

Proof. The Hamiltonian $\hat{H}^{(3)}_{\rm osc}$ is a sum of three commuting self-adjoint operators each acting on a different coordinate. The eigenvalue equation $\hat{H}^{(3)}_{\rm osc}\Psi = E\Psi$ on the product domain separates into three independent eigenvalue equations $\hat{H}^{(j)}_{\rm osc}\Psi_{n_j} = E_{n_j}\Psi_{n_j}$ for $j = x, y, z$. The product $\Psi_{n_x n_y n_z} = \Psi_{n_x}\Psi_{n_y}\Psi_{n_z}$ satisfies $\hat{H}^{(3)}_{\rm osc}\Psi_{n_x n_y n_z} = (E_{n_x} + E_{n_y} + E_{n_z})\Psi_{n_x n_y n_z} = [(n_x + \frac{1}{2}) + (n_y + \frac{1}{2}) + (n_z + \frac{1}{2})]\Phi_0\omega\,\Psi_{n_x n_y n_z} = (N + \frac{3}{2})\Phi_0\omega\,\Psi_{n_x n_y n_z}$, giving Eq. (61). The product states form a complete orthonormal basis for $\mathcal{H}^{(3)}$ by the completeness of each one-dimensional Fock basis (Proposition 5.7). □

Proposition 7.3 (Degeneracy of the 3D oscillator energy shells)**.** *The number of linearly independent eigenstates with total excitation number N is*

$$d_N = \frac{(N+1)(N+2)}{2}. \tag{62}$$

Proof. d_N is the number of non-negative integer solutions (n_x, n_y, n_z) of $n_x + n_y + n_z = N$. By the stars-and-bars formula from combinatorics, this count is $\binom{N+2}{2} = (N+1)(N+2)/2$. Explicitly for the first few shells: $d_0 = 1$, $d_1 = 3$, $d_2 = 6$, $d_3 = 10$, consistent with Eq. (62). □

7.3 Spherical Representation: Radial and Angular Separation

The spherical representation exploits the rotational symmetry established in Proposition 7.1 and the QM5 angular momentum eigenstate structure.

Proposition 7.4 (Separation of variables in the 3D oscillator)**.** *The eigenvalue equation $\hat{H}^{(3)}_{\rm osc}\Psi = E_N\Psi$ separates in spherical coordinates via the ansatz*

$$\Psi_{N\ell m}(r, \theta, \varphi) = u_{n_r\ell}(r)\,Y_\ell^m(\theta, \varphi), \tag{63}$$

where Y_ℓ^m is the spherical harmonic of QM5 Theorem 6.2 and $u_{n_r\ell}(r)$ satisfies the reduced radial equation:

$$\left[-\frac{\Phi_0^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\ell(\ell+1)\Phi_0^2}{2mr^2} + \frac{1}{2}m\omega^2 r^2\right]u_{n_r\ell}(r) = E_N\,u_{n_r\ell}(r), \tag{64}$$

obtained by the substitution $u_{n_r\ell}(r) = r\,R_{n_r\ell}(r)$ and using $\hat{L}^2 Y_\ell^m = \ell(\ell+1)\Phi_0^2 Y_\ell^m$ (QM5 Theorem 6.2).

Proof. Substitute the ansatz Eq. (63) into $\hat{H}^{(3)}_{\rm osc}\Psi = E_N\Psi$ using the spherical-coordinate form Eq. (59). Since $\hat{L}^2$ acts only on the angular variables and Y_ℓ^m is its eigenfunction (QM5 Theorem 6.2), the operator $\hat{L}^2$ acting on $u_{n_r\ell}(r)Y_\ell^m(\theta, \varphi)$ yields $\ell(\ell+1)\Phi_0^2$ times the same product. The radial derivative operator acts only on $u_{n_r\ell}(r)$. Dividing through by $Y_\ell^m \neq 0$ and substituting $u_{n_r\ell}(r) = rR(r)$ to convert the spherical Laplacian to the one-dimensional form gives Eq. (64), which is independent of m, confirming the $(2\ell+1)$-fold degeneracy in m for each ℓ. □

Theorem 7.5 (Complete eigenstructure of the 3D isotropic oscillator)**.** *The radial equation Eq.* (64) *has normalizable solutions if and only if $E_N = (N + \frac{3}{2})\Phi_0\omega$ with*

$$N = 2n_r + \ell, \quad n_r \in \{0, 1, 2, \ldots\}, \quad \ell \in \{0, 1, 2, \ldots\}, \quad \ell \equiv N \pmod 2. \tag{65}$$

The normalizable radial functions are

$$u_{n_r\ell}(r) = \mathcal{N}_{n_r\ell}\left(\frac{r}{\ell_0}\right)^{\ell} L_{n_r}^{\ell+1/2}\!\left(\frac{r^2}{\ell_0^2}\right)\exp\!\left(-\frac{r^2}{2\ell_0^2}\right), \tag{66}$$

where $L_{n_r}^{\ell+1/2}$ *is the associated Laguerre polynomial of degree* n_r *and parameter* $\ell + \frac{1}{2}$*, and* $\mathcal{N}_{n_r\ell}$ *is the normalization constant. The complete eigenstructure is: eigenvalues* $E_N = (N + \frac{3}{2})\Phi_0\omega$ *for* $N \in \{0, 1, 2, \ldots\}$*; eigenstates* $\Psi_{N\ell m} = u_{n_r\ell}(r)Y_\ell^m(\theta, \varphi)$ *with* $n_r = (N - \ell)/2$ *and* $m \in \{-\ell, \ldots, +\ell\}$*; degeneracy* $d_N = (N + 1)(N + 2)/2$*.*

Proof. Energy quantization. The radial equation Eq. (64) is the one-dimensional Schrödinger equation with the effective potential $V_{\text{eff}}(r) = \ell(\ell+1)\Phi_0^2/(2mr^2) + \frac{1}{2}m\omega^2 r^2$. Substituting $\rho = r^2/\ell_0^2$ and $v(\rho) = u_{n_r\ell}(r)/r^{\ell+1}\mathrm{e}^{-\rho/2}$ converts Eq. (64) to the associated Laguerre ODE:

$$\rho\, v'' + (\ell + \tfrac{3}{2} - \rho)v' + \left(\frac{E_N}{2\Phi_0\omega} - \frac{\ell}{2} - \frac{3}{4}\right) v = 0.$$

For normalizability, the power series in ρ must terminate, requiring the parameter $n_r := E_N/(2\Phi_0\omega) - \ell/2 - 3/4$ to be a non-negative integer: $n_r \in \{0, 1, 2, \ldots\}$. Solving for E_N: $E_N = (2n_r + \ell + \frac{3}{2})\Phi_0\omega = (N + \frac{3}{2})\Phi_0\omega$ with $N = 2n_r + \ell$, giving Eq. (65).

Normalizable solutions. The terminating series gives the associated Laguerre polynomial $L_{n_r}^{\ell+1/2}(r^2/\ell_0^2)$, and the normalizable radial function is Eq. (66).

Parity constraint. Since $N = 2n_r + \ell$ and $n_r \geq 0$, ℓ must satisfy $0 \leq \ell \leq N$ with ℓ and N having the same parity.

Degeneracy. For each N, the allowed values of (n_r, ℓ) with $N = 2n_r + \ell$ and $\ell \equiv N \pmod 2$ are $(n_r, \ell) \in \{(0, N), (1, N - 2), \ldots, (\lfloor N/2 \rfloor, N \bmod 2)\}$. For each ℓ, there are $2\ell + 1$ values of m. The total count:

$$d_N = \sum_{\substack{\ell=0 \\ \ell \equiv N}}^{N} (2\ell + 1).$$

For even $N = 2K$: $\ell \in \{0, 2, 4, \ldots, 2K\}$ and $d_{2K} = \sum_{j=0}^{K}(4j + 1) = K(2K + 2) + K + 1 = 2K^2 + 3K + 1 = (2K + 1)(K + 1) = (N + 1)(N/2 + 1) = (N + 1)(N + 2)/2$. For odd $N = 2K + 1$: $\ell \in \{1, 3, 5, \ldots, 2K + 1\}$ and $d_{2K+1} = \sum_{j=0}^{K}(4j + 3) = 2K(K + 1) + 3(K + 1) = (K + 1)(2K + 3) = (N + 1)(N + 2)/2$. In both cases $d_N = (N + 1)(N + 2)/2$, agreeing with Proposition 7.3. □

7.4 Shell Structure, Parity, and Connection to QM5

The quantum number structure of Theorem 7.5 is recorded shell by shell for the lowest four energy levels.

Remark 7.6 (Shell structure of the 3D isotropic oscillator)**.** *The first four energy shells and their angular momentum content are:*

N	$E_N/(\Phi_0\omega)$	d_N	(n_r, ℓ) *values*	*Angular content*
0	3/2	1	$(0, 0)$	$\ell = 0$*: one s-state*
1	5/2	3	$(0, 1)$	$\ell = 1$*: three p-states*
2	7/2	6	$(1, 0), (0, 2)$	$\ell = 0$*: one s;* $\ell = 2$*: five d*
3	9/2	10	$(1, 1), (0, 3)$	$\ell = 1$*: three p;* $\ell = 3$*: seven f*

The parity of the N*-th shell is* $(-1)^N$*: states in even shells* ($N = 0, 2, 4, \ldots$) *are parity-even and states in odd shells* ($N = 1, 3, 5, \ldots$) *are parity-odd.*

Remark 7.7. *The parity of the eigenstate* $\Psi_{N\ell m}(\mathbf{x})$ *under* $\mathbf{x} \to -\mathbf{x}$ *is* $(-1)^\ell$*, arising from the spherical harmonic parity of QM5 Proposition 6.3 (iii):* $Y_\ell^m(\pi - \theta, \varphi + \pi) = (-1)^\ell Y_\ell^m(\theta, \varphi)$*. The radial function* $u_{n_r\ell}(r)$ *depends only on* $r = |\mathbf{x}|$ *and is parity-even. Since* $\ell \equiv N \pmod 2$ *(from Eq.* (65)*), the parity of the full eigenstate is* $(-1)^\ell = (-1)^N$*, so all states in a given shell have the same parity. This uniform shell parity is a direct consequence of the* $\ell \equiv N \pmod 2$ *constraint and the QM5 spherical harmonic parity — two results from different papers combining to give a unified physical statement.*

Remark 7.8. *The degeneracy* $d_N = (N+1)(N+2)/2$ *of the 3D isotropic oscillator is larger than the angular momentum degeneracy* $2\ell+1$ *of a single* ℓ*-multiplet, because multiple* ℓ *values contribute to each shell. For example, the* $N = 2$ *shell has* $d_2 = 6$ *states: one s-state (*$\ell = 0$*,* $m = 0$*) and five d-states (*$\ell = 2$*,* $m = -2, -1, 0, 1, 2$*). This multi-*ℓ *degeneracy within each shell is analogous to the accidental (*n^2*) degeneracy of the hydrogen atom established in QM5 Theorem 7.2, and has a similar origin: the isotropic harmonic oscillator potential* $V \propto r^2$*, like the Coulomb potential* $V \propto 1/r$*, has a higher symmetry than* $\mathrm{SO}(3)$*. For the 3D oscillator, this higher symmetry is* $\mathrm{SU}(3)$ *(the special unitary group in three dimensions), which generates transitions between different* ℓ *values within the same shell* N*. The full derivation of the* $\mathrm{SU}(3)$ *symmetry and its consequences for the oscillator spectrum are beyond the scope of the present paper and are deferred as a structural extension of the series.*

Remark 7.9. *The three-dimensional oscillator analysis is the first instance in the QM-series in which the results of QM5 (spherical harmonics, angular momentum eigenvalues, and parity) are used as direct structural inputs to the derivation of a physical spectrum, rather than merely as formal prerequisites. The angular factor* $Y_\ell^m(\theta, \varphi)$ *in the eigenstate Eq.* (63)*, the centrifugal term* $\ell(\ell+1)\Phi_0^2/(2mr^2)$ *in the radial equation Eq.* (64)*, and the parity* $(-1)^\ell$ *in Remark 7.7 are all QM5 results applied here without re-derivation. This cross-paper dependence is the normal pattern from QM6 onward: each new physical sector paper builds directly on the structural results established in prior sector papers, and the program accumulates physical results by layering new derivations on top of established ones rather than re-deriving from first principles.*

8 Interpretive Clarifications and Scope

The present section collects the interpretive constraints governing the harmonic oscillator analysis of the preceding sections and records the scope of the present construction relative to the remainder of the QM-series. Three items are addressed: the role of the oscillator as a structural template throughout the series and beyond, the explicit comparison between the QM5 angular momentum ladder algebra and the QM6 energy ladder algebra, and the precise boundary between what the present paper establishes and what is deferred.

8.1 The Oscillator as a Structural Template

The harmonic oscillator occupies a position in the QM-series that is qualitatively different from the hydrogen atom of QM5. The hydrogen atom is the primary physical validation target of the scalar–conformal NUVO framework: its energy spectrum was derived in the Q-series and its full eigenstate structure was completed in QM5, constituting the first complete physical system derived within the program. The harmonic oscillator is instead a structural template: its algebraic skeleton — the ladder algebra $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$, the number operator decomposition $\hat{H}_{\text{osc}} = \Phi_0\omega(\hat{N} + \frac{1}{2})$, and the coherent state family — recurs throughout every subsequent paper in the QM-series and in the field-theoretic extensions beyond it.

Four specific structural roles are worth recording explicitly.

The oscillator ladder as the energy-sector template. The ladder technique introduced in QM5 for angular momentum — raising and lowering the $\hat{L}_3$ eigenvalue by Φ_0 while preserving the $\hat{L}^2$ eigenvalue — reappears here as raising and lowering the energy eigenvalue by $\Phi_0\omega$. In both cases the technique extracts the complete spectrum from a commutation relation without solving a differential equation. In QM8, the same technique will be applied to the spin algebra $[\hat{S}_j, \hat{S}_k] = \mathrm{i}\Phi_0\, \epsilon_{jkl}\hat{S}_l$ to derive the spin spectrum. In QM10, a modified ladder technique (the Lippmann-Schwinger equation) will be applied to the scattering sector. The present paper establishes the oscillator version as the simplest non-trivial instance of the ladder technique with a semi-infinite spectrum; subsequent papers adapt and extend it.

Coherent states and the quantum-classical correspondence. The Ehrenfest theorem of QM4 established that the centroid of any closure state follows the classical equations of motion. For the harmonic oscillator, Theorem 6.8 goes further: the coherent states are the unique family for which not only the centroid but the entire Gaussian profile propagates classically, with the width $\Delta x(t) = \ell_0/\sqrt{2}$ constant in time. This is the sharpest instance in the QM-series of the quantum-classical correspondence: a coherent state is, in a precise sense, as classical as a quantum state can be. The coherent states will appear in QM9 as the building blocks of *entangled coherent states* (non-factorizable superpositions of coherent states in two-mode systems), in QM10 as the reference states for scattering cross-sections involving coherent radiation fields, and in QM11 as the classical limit of relativistic field configurations.

The oscillator as the foundation of quantum field theory. The scalar–conformal NUVO program develops quantum mechanics in the QM-series; the RQM-series and the field-theoretic extensions beyond it treat the relativistic and multi-mode cases. In that broader context, the harmonic oscillator is the elementary object from which all of quantum field theory is built: each mode of a free quantum field is an independent harmonic oscillator, the vacuum of the field is the tensor product $\bigotimes_k |0_k\rangle$ of the ground states of all modes, and a field excitation is a Fock state $\hat{a}_k^\dagger|0_k\rangle$ for a specific mode k. The Fock basis established in the present paper, the creation and annihilation operators, and the coherent state overcompleteness relation Eq. (55) are the tools used throughout quantum field theory. Recording this role explicitly here allows the connections to be made precisely when the field-theoretic extension is developed, without retroactively re-deriving the oscillator structure.

The zero-point energy and the uncertainty principle. Theorem 4.3 established the zero-point energy $E_0 = \frac{1}{2}\Phi_0\omega$ as a structural consequence of the Heisenberg uncertainty relation: a state of zero energy would require exact position and exact momentum simultaneously, violating $\Delta x\cdot\Delta p \geq \Phi_0/2$. In quantum field theory, the vacuum energy of each field mode is its zero-point energy $\Phi_0\omega_k/2$, and the total vacuum energy is $\sum_k \Phi_0\omega_k/2$. The structure of the zero-point energy and its connection to the uncertainty principle, established here for the single-mode oscillator, therefore underlies one of the most fundamental — and most computationally significant — features of quantum field theory.

8.2 Comparison of the Oscillator and Angular Momentum Ladders

The angular momentum ladder of QM5 and the oscillator ladder of the present paper are the two canonical ladder algebras of the QM-series. Their structural similarities and differences are recorded here in a form that makes both papers easier to use as references.

Property	QM5 (angular momentum)	QM6 (oscillator)
Diagonal operator	$\hat{L}_3$ (self-adjoint)	$\hat{N} = \hat{a}^\dagger \hat{a}$ (self-adjoint)
Ladder operators	$\hat{L}_+ = \hat{L}_1 + \mathrm{i}\hat{L}_2$, $\hat{L}_-$	$\hat{a}^\dagger$, $\hat{a}$
Fundamental relation	$[\hat{L}_+, \hat{L}_-] = 2\Phi_0\, \hat{L}_3$	$[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$
Ladder commutators	$[\hat{L}_3, \hat{L}_+] = \Phi_0\, \hat{L}_+$	$[\hat{N}, \hat{a}^\dagger] = +\hat{a}^\dagger$
	$[\hat{L}_3, \hat{L}_-] = -\Phi_0\, \hat{L}_-$	$[\hat{N}, \hat{a}] = -\hat{a}$
Spectrum type	Finite: $m \in \{-\ell, \ldots, +\ell\}$	Semi-infinite: $n \in \{0, 1, 2, \ldots\}$
Upper termination	Yes: $\hat{L}_+ \vert\, \ell,\, \ell\,\rangle = 0$	No: $\hat{a}^\dagger \vert n\rangle \neq 0$ for all n
Lower termination	Yes: $\hat{L}_- \vert\, \ell,\, -\ell\,\rangle = 0$	Yes: $\hat{a}\vert 0\rangle = 0$ (vacuum)
Quantization source	Integer holonomy (QM5 Theorem 5.1)	$a\vert 0\rangle = 0$ (termination condition)
Eigenvalue step size	Φ_0 per step	1 per step (dimensionless)
Degeneracy per eigenvalue	$2\ell + 1$ (for each $\hat{L}^2$ eigenvalue)	1 (non-degenerate)
Underlying algebra	SO(3) (Lie algebra)	Weyl-Heisenberg algebra
Non-self-adjoint operators	$\hat{L}_+^\dagger = \hat{L}_-$	$(\hat{a})^\dagger = \hat{a}^\dagger$

The most significant structural difference between the two algebras is the presence or absence of upper termination. In the angular momentum case, the operator $\hat{L}^2$ provides an upper bound on the $\hat{L}_3$ eigenvalue: since $\hat{L}^2 = \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2$ and all terms are non-negative, the $\hat{L}_3$ eigenvalue $m\Phi_0$ is bounded by $|m|\Phi_0 \leq \sqrt{\ell(\ell+1)}\Phi_0$. This forces both upper and lower termination, giving the finite multiplet structure. In the oscillator case, there is no operator analogous to $\hat{L}^2$ that bounds $\hat{N}$ from above: the number operator has no upper bound, so the ladder extends infinitely upward. Only the non-negativity of $\hat{N}$ (which bounds it from below) forces the existence of the vacuum state.

The absence of holonomy quantization in QM6 is also structurally significant. In QM5, the integer character of the angular momentum quantum numbers was derived from the holonomy condition: the azimuthal transport closure state must be single-valued under $\varphi \to \varphi + 2\pi$, selecting $m \in \mathbb{Z}$. In QM6, no such geometric condition is required: the eigenvalues of $\hat{N}$ are forced to be non-negative integers by the non-negativity of $\hat{N}$ (Step 1 of the proof of Theorem 4.1) and the unit step size of the ladder (the CCR $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$), without any topological input. This is consistent with the physical picture: the oscillator quantum number n is not a holonomy winding number but a count of excitation quanta.

8.3 Scope of the Present Construction

The present paper establishes the complete algebraic and spectral structure of the one-dimensional harmonic oscillator, the coherent state theory, and the three-dimensional isotropic oscillator eigenstructure. The following results are established and available as inputs to subsequent QM-series papers.

Algebraic structure: The ladder operator commutation relation $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$ (Lemma 3.3), the Hamiltonian decomposition $\hat{H}_{\rm osc} = \Phi_0\omega(\hat{N} + \frac{1}{2}\hat{\mathbf{1}})$ (Theorem 3.5), and the number operator commutation relations (Lemma 3.7).

Spectral structure: The complete spectrum $\sigma(\hat{H}_{\rm osc}) = \{(n+\frac{1}{2})\Phi_0\omega : n \geq 0\}$ with non-degenerate eigenvalues (Theorem 4.1), the zero-point energy $E_0 = \frac{1}{2}\Phi_0\omega$ from the uncertainty principle (Theorem 4.3), and the Fock state matrix elements $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ (Proposition 4.5).

Position-space eigenstates: The ground state Gaussian $\Psi_0(x) \propto \mathrm{e}^{-x^2/(2\ell_0^2)}$ (Theorem 5.3), the excited state Hermite-Gaussians $\Psi_n(x) \propto H_n(x/\ell_0)\mathrm{e}^{-x^2/(2\ell_0^2)}$ (Theorem 5.5), and the completeness of the Fock basis in $L^2(\mathbb{R})$ (Proposition 5.7).

Coherent state theory: The eigenstate definition $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ (Definition 6.1), the Fock expansion (Lemma 6.3), the Gaussian minimum-uncertainty equivalence with width $\sigma = \ell_0/\sqrt{2}$ (Theorem 6.4), the displacement operator construction $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$ (Proposition 6.6), the shape-preserving time evolution $U(t)|\alpha\rangle = \mathrm{e}^{-\mathrm{i}\omega t/2}|\alpha\mathrm{e}^{-\mathrm{i}\omega t}\rangle$ (Theorem 6.8), and the overcompleteness resolution of the identity $(1/\pi)\int |\alpha\rangle\langle\alpha|\,\mathrm{d}^2\alpha = \hat{\mathbf{1}}$ (Proposition 6.10).

Three-dimensional oscillator: The rotational symmetry $[\hat{H}^{(3)}_{\mathrm{osc}}, \hat{L}_j] = 0$ (Proposition 7.1), the Cartesian product eigenstructure (Propositions 7.2 and 7.3), the spherical separation with $u_{n_r\ell}(r)Y_\ell^m(\theta,\varphi)$ eigenstates and the complete quantum number structure $N = 2n_r + \ell$, $E_N = (N + \frac{3}{2})\Phi_0\omega$, $d_N = (N+1)(N+2)/2$ (Theorem 7.5); and the parity $(-1)^N$ of each energy shell.

The following topics are outside the scope of the present paper and are deferred.

Squeezed states. A squeezed state is a state with $\Delta x < \ell_0/\sqrt{2}$ (and correspondingly $\Delta p > p_0/\sqrt{2}$), achieved by a Bogoliubov transformation that mixes $\hat{a}$ and $\hat{a}^\dagger$. Such states are minimum-uncertainty states in the QM3 sense but lie outside the coherent state family because their width differs from $\ell_0/\sqrt{2}$. Their derivation requires the squeeze operator $\hat{S}(\xi) = \exp[\frac{1}{2}(\xi^*\hat{a}^2 - \xi\hat{a}^{\dagger 2})]$ and the associated Bogoliubov transformation, which is developed in the context of quantum optics beyond the current scope.

Anharmonic corrections. The harmonic oscillator is the leading-order approximation to any potential near a stable equilibrium. Anharmonic corrections — cubic and quartic perturbations to the potential — shift the energy levels away from the equally-spaced structure of Theorem 4.1. Their treatment requires the perturbation theory developed in a subsequent paper and is outside the scope of the present derivation.

Field-mode quantization. The application of the oscillator algebra to quantize each mode of a free classical field — giving the Fock space of quantum field theory — is the central step in the transition from the QM-series to the RQM-series and the field-theoretic extensions. The present paper establishes the single-mode oscillator structure that each field mode will inherit; the multi-mode tensor product construction is the content of the field-theoretic extension.

The Jaynes-Cummings model. The coupling of a two-level atom to a single field mode — the Jaynes-Cummings model of quantum optics — involves the interaction between the oscillator ladder algebra of QM6 and the spin algebra of QM8. Its analysis requires both the coherent state theory of the present paper and the spin structure of QM8, and is deferred to the appropriate point in the series.

Remark 8.1. *The present paper completes the transition from the algebraic foundation papers (QM1-QM3) and the first physical sector (QM4-QM5) to the dynamical and multi-particle sector (QM6-QM11). QM1-QM3 established the state space, superposition structure, and uncertainty relations without dynamics. QM4-QM5 established the dynamics (Schrödinger equation) and the rotational sector (angular momentum, hydrogen spectrum). QM6 establishes the oscillator sector and, through the coherent state theory, the most precise expression of the quantum-classical correspondence available within the non-relativistic framework. The results of QM5 (spherical harmonics) and QM6 (Fock states, coherent states) together provide the structural inputs for all subsequent physical sector papers.*

9 Conclusion

9.1 Summary of Results

The present paper has derived the complete structure of the scalar–conformal NUVO harmonic oscillator from the canonical commutation relations of QM1 and the dynamical framework of QM4, without postulating the spectrum, the Hermite polynomial eigenfunctions, or the coherent state properties. The sixteen principal results are as follows.

Ladder operators from the CCR (Definition 3.1 and Lemma 3.3). The annihilation and creation operators $\hat{a} = (m\omega\hat{x} + \mathrm{i}\hat{p})/\sqrt{2m\omega\Phi_0}$ and $\hat{a}^\dagger = (\hat{a})^\dagger$ are non-self-adjoint operators on $\mathcal{S}(\mathbb{R}) \subset \mathcal{H}$ satisfying $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$, derived by direct computation from the canonical commutation relation $[\hat{x}, \hat{p}] = \mathrm{i}\Phi_0$ of QM1.

Hamiltonian decomposition and the number operator (Theorem 3.5 and Lemma 3.7). The harmonic oscillator Hamiltonian decomposes as $\hat{H}_{\mathrm{osc}} = \Phi_0\omega(\hat{N} + \frac{1}{2}\hat{\mathbf{1}})$, where $\hat{N} = \hat{a}^\dagger\hat{a}$ is the self-adjoint non-negative number operator. The commutation relations $[\hat{N}, \hat{a}] = -\hat{a}$, $[\hat{N}, \hat{a}^\dagger] = +\hat{a}^\dagger$, and their Hamiltonian equivalents $[\hat{H}_{\mathrm{osc}}, \hat{a}] = -\Phi_0\omega\hat{a}$, $[\hat{H}_{\mathrm{osc}}, \hat{a}^\dagger] = +\Phi_0\omega\hat{a}^\dagger$ establish the raising and lowering action. The $\frac{1}{2}\Phi_0\omega$ offset in the decomposition arises directly from the CCR.

Complete spectrum of the harmonic oscillator (Theorem 4.1). The spectrum is $\sigma(\hat{H}_{\mathrm{osc}}) = \{(n + \frac{1}{2})\Phi_0\omega : n \in \{0, 1, 2, \ldots\}\}$, derived by the four-step ladder termination argument: non-negativity of $\hat{N}$ bounds the spectrum from below, the lower termination condition $\hat{a}|0\rangle = 0$ identifies the ground state, the unit step of the ladder gives integer eigenvalues, and non-degeneracy follows from the uniqueness of the vacuum. Each eigenvalue is non-degenerate.

Zero-point energy from the uncertainty principle (Theorem 4.3). For any normalized state, $\langle\hat{H}_{\mathrm{osc}}\rangle \geq \frac{1}{2}\Phi_0\omega$, with equality if and only if Ψ is the ground state. The bound is derived by combining $\langle A^2\rangle \geq (\Delta A)^2$ with the AM-GM inequality and the Heisenberg bound $\Delta x \cdot \Delta p \geq \Phi_0/2$ of QM3. The equality condition $\Delta p = m\omega\Delta x$ combined with Robertson saturation uniquely identifies the ground state as the Gaussian with $\sigma = \ell_0/\sqrt{2}$, $\langle x\rangle = \langle p\rangle = 0$.

Fock state matrix elements and the generation formula (Proposition 4.5). $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n{+}1\rangle$ and $\hat{a}|n\rangle = \sqrt{n}|n{-}1\rangle$ for $n \geq 1$, with $\hat{a}|0\rangle = 0$, derived by normalization computations using the identity $\hat{a}\hat{a}^\dagger = \hat{N} + \hat{\mathbf{1}}$. The generation formula $|n\rangle = (\hat{a}^\dagger)^n|0\rangle/\sqrt{n!}$ follows by induction.

Ladder operators in position space (Lemma 5.1). $\hat{a} = (\xi + \partial_\xi)/\sqrt{2}$ and $\hat{a}^\dagger = (\xi - \partial_\xi)/\sqrt{2}$ in terms of the dimensionless variable $\xi = x/\ell_0$, derived by substituting the position-space momentum $\hat{p} = -\mathrm{i}\Phi_0\,\partial_x$ into Definition 3.1.

Ground state wave function (Theorem 5.3). The condition $\hat{a}\Psi_0 = 0$ in position space is a first-order ODE $\partial_x\Psi_0 = -(x/\ell_0^2)\Psi_0$, whose unique normalized solution is the Gaussian $\Psi_0(x) = (\pi\ell_0^2)^{-1/4}\exp(-x^2/(2\ell_0^2))$. This is the QM3 minimum-uncertainty Gaussian with $\sigma = \ell_0/\sqrt{2}$, $\langle x\rangle = \langle p\rangle = 0$, saturation $\Delta x \cdot \Delta p = \Phi_0/2$.

Excited state wave functions and Hermite polynomials (Theorem 5.5). The n-th eigenstate is $\Psi_n(x) = (2^n\,n!\,\sqrt{\pi}\,\ell_0)^{-1/2}H_n(x/\ell_0)\exp(-x^2/(2\ell_0^2))$, where the Hermite polynomial $H_n(\xi)$ is generated by the identity $(\xi - \partial_\xi)^n\mathrm{e}^{-\xi^2/2} = H_n(\xi)\mathrm{e}^{-\xi^2/2}$, derived from the factorization $\xi - \partial_\xi = -\mathrm{e}^{\xi^2/2}\partial_\xi\mathrm{e}^{-\xi^2/2}$. The Rodrigues formula $H_n(\xi) = (-1)^n\mathrm{e}^{\xi^2}\partial_\xi^n\mathrm{e}^{-\xi^2}$ emerges from this factorization.

Completeness and orthonormality of the Fock basis (Proposition 5.7). $\langle\Psi_{n'}, \Psi_n\rangle_{\mathcal{H}} = \delta_{n'n}$ from self-adjointness of $\hat{H}_{\mathrm{osc}}$ and distinct eigenvalues; $\sum_{n=0}^{\infty}|n\rangle\langle n| = \hat{\mathbf{1}}_{\mathcal{H}}$ from the spectral theorem of QM1 applied to the purely discrete spectrum of $\hat{H}_{\mathrm{osc}}$.

Coherent states as eigenstates of $\hat{a}$ (Definition 6.1 and Lemma 6.3). The coherent state $|\alpha\rangle$ with $\alpha \in \mathbb{C}$ satisfies $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ and has the unique normalized Fock expansion $|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2}\sum_n(\alpha^n/\sqrt{n!})|n\rangle$, derived from the recurrence $c_{n+1}\sqrt{n+1} = \alpha\,c_n$ and normalization.

Coherent states are Gaussian minimum-uncertainty states (Theorem 6.4). The position-space representation of $|\alpha\rangle$ is the Gaussian of QM3 Theorem 6.1 with width $\sigma = \ell_0/\sqrt{2}$, mean position $\langle x\rangle_\alpha = \ell_0\sqrt{2}\,\mathrm{Re}(\alpha)$, and mean momentum $\langle p\rangle_\alpha = p_0\sqrt{2}\,\mathrm{Im}(\alpha)$, derived by solving the ODE Eq. (45) and normalizing. The widths $\Delta x = \ell_0/\sqrt{2}$ and $\Delta p = p_0/\sqrt{2}$ are independent of α.

Displacement operator construction (Proposition 6.6). The unitary displacement operator $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \overline{\alpha}\hat{a})$ satisfies $\hat{D}(\alpha)\hat{a}\hat{D}(\alpha)^\dagger = \hat{a} - \alpha\hat{\mathbf{1}}$ (by the BCH lemma applied to the commutator $[\alpha\hat{a}^\dagger - \overline{\alpha}\hat{a}, \hat{a}] = -\alpha\hat{\mathbf{1}}$), and generates coherent states from the vacuum: $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$.

Shape-preserving time evolution of coherent states (Theorem 6.8). $U(t)|\alpha\rangle = \mathrm{e}^{-\mathrm{i}\omega t/2}|\alpha\mathrm{e}^{-\mathrm{i}\omega t}\rangle$, derived via the Heisenberg-picture identity $U^\dagger(t)\hat{a}U(t) = \hat{a}\mathrm{e}^{-\mathrm{i}\omega t}$ and confirmed by the Fock expansion. The widths $\Delta x(t) = \ell_0/\sqrt{2}$ and $\Delta p(t) = p_0/\sqrt{2}$ are constant; the centroid follows the classical trajectory. The overall phase $\mathrm{e}^{-\mathrm{i}\omega t/2}$ arises from the zero-point energy.

Overcompleteness of the coherent state family (Proposition 6.10). The overlap $\langle\alpha'|\alpha\rangle = \exp(\overline{\alpha'}\alpha - |\alpha'|^2/2 - |\alpha|^2/2)$ shows coherent states are not orthogonal. The resolution of the identity $(1/\pi)\int_{\mathbb{C}} |\alpha\rangle\langle\alpha|\,\mathrm{d}^2\alpha = \hat{\mathbf{1}}_{\mathcal{H}}$ is established by computing the Fock matrix elements in polar coordinates and using the Gaussian integral $\int_0^\infty r^{2n+1}\mathrm{e}^{-r^2}\,\mathrm{d}r = n!/2$.

Three-dimensional isotropic oscillator (Propositions 7.1–7.3 and Theorem 7.5). The Hamiltonian $\hat{H}^{(3)}_{\mathrm{osc}} = \sum_j [\hat{p}_j^2/(2m) + \frac{1}{2}m\omega^2(\hat{x}^j)^2]$ commutes with all $\hat{L}_j$ by rotational symmetry. The Cartesian product structure gives $E_N = (N+\frac{3}{2})\Phi_0\omega$ with $d_N = (N+1)(N+2)/2$ from the stars-and-bars counting. The spherical separation gives eigenstates $u_{n_r\ell}(r)Y_\ell^m(\theta,\varphi)$ with $N = 2n_r + \ell$ and parity $(-1)^N$ per shell.

9.2 Programmatic Significance

The results of the present paper are of broad programmatic significance for the scalar–conformal NUVO series on three distinct grounds.

The first is the completion of the QM3-to-QM6 program arc. QM3 identified the Gaussian minimum-uncertainty states algebraically: for any width $\sigma > 0$, the Gaussian closure state $\Psi_{G(\sigma,\langle x\rangle,\langle p\rangle)}$ saturates the Heisenberg bound $\Delta x \cdot \Delta p = \Phi_0/2$. The algebraic characterization established a one-parameter family of optimal states without selecting among them. The harmonic oscillator dynamics of the present paper acts as a dynamical filter on this family: of all the Gaussians, only those with the specific width $\sigma = \ell_0/\sqrt{2}$ retain their Gaussian form under the time evolution $U(t)$ (Theorem 6.8). All other Gaussians undergo "breathing" oscillation of their width at frequency 2ω. The dynamical selection of $\sigma = \ell_0/\sqrt{2}$ is therefore the dynamical characterization that completes the algebraic characterization of QM3: the coherent states are the minimum-uncertainty states that are also dynamically stable under the harmonic oscillator. The two characterizations (algebraic: saturation of Cauchy-Schwarz; dynamical: shape preservation) together determine the coherent states uniquely, and the program arc from QM3 to QM6 is the arc from the algebraic to the dynamical characterization.

The second ground of programmatic significance is the derivation of the zero-point energy from the uncertainty principle. Theorem 4.3 establishes that no state of the harmonic oscillator can have energy less than $\Phi_0\omega/2$. The proof uses no spectral information about $\hat{H}_{\mathrm{osc}}$; it derives the bound from the Heisenberg relation of QM3 alone, via the AM-GM inequality. The zero-point energy is therefore not a computational artifact of the energy eigenvalues but a structural consequence of the canonical commutation relation: the same CCR that gives $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$ (and the $\frac{1}{2}\hat{\mathbf{1}}$ in $\hat{H}_{\mathrm{osc}} = \Phi_0\omega(\hat{N} + \frac{1}{2}\hat{\mathbf{1}})$) also gives $\Delta x \cdot \Delta p \geq \Phi_0/2$, which forces $\langle\hat{H}_{\mathrm{osc}}\rangle \geq \Phi_0\omega/2$. This is one of the NUVO program's cleanest demonstrations that quantum phenomena arise from transport closure geometry — specifically from the holonomy structure encoded in the CCR — rather than from postulates about state spaces.

The third ground is the first active use of QM5 results in a subsequent physical sector paper. The three-dimensional oscillator analysis of Sec. 7 does not merely require QM5 as a logical prerequisite; it uses the spherical harmonics Y_ℓ^m as the explicit angular eigenstates in the eigenstate factorization $\Psi_{N\ell m} = u_{n_r\ell}(r)Y_\ell^m(\theta,\varphi)$, and uses the parity $(-1)^\ell$ of the spherical harmonics from QM5 Proposition 6.3 (iii) to derive the shell parity $(-1)^N$ of the 3D oscillator. The quantum number constraint $N = 2n_r + \ell$ arises from the interaction between the oscillator's energy structure (the radial quantum number n_r) and the angular momentum structure (the orbital quantum number ℓ). This interaction between dynamical and rotational structure is the pattern that recurs throughout the remainder of the QM-series: QM7 couples oscillators, QM8 couples orbital and spin angular momentum, QM9 entangles multi-particle states, and in each case the QM5 angular momentum structure and the QM6 oscillator structure are both active ingredients.

9.3 Transition to QM7

The present paper completes the treatment of single-particle quantum mechanics for the two canonical physical systems of the non-relativistic NUVO framework: the hydrogen atom (QM4-QM5) and the harmonic oscillator (QM4-QM6). The next paper, QM7, opens the multi-particle sector.

The fundamental new object in QM7 is the tensor product Hilbert space $\mathcal{H}^{(2)} = \mathcal{H} \otimes \mathcal{H}$ for two-particle systems, or more generally $\mathcal{H}^{(N)} = \bigotimes_{j=1}^N \mathcal{H}$ for N-particle systems. The tensor product construction is not merely a formal extension of the single-particle framework: it introduces genuinely new physical content through the entanglement structure of states that do not factorize as products $\Psi_1 \otimes \Psi_2$, which is the subject of QM9.

The harmonic oscillator of the present paper enters QM7 in a specific and concrete way: through the coupled harmonic oscillator system with Hamiltonian $\hat{H}_{\text{osccoupled}} = \hat{H}_{\text{osc}}^{(1)} + \hat{H}_{\text{osc}}^{(2)} + \kappa\,\hat{x}_1\hat{x}_2$. Here $\hat{H}_{\text{osc}}^{(1)}$ and $\hat{H}_{\text{osc}}^{(2)}$ are the individual oscillator Hamiltonians and $\kappa\,\hat{x}_1\hat{x}_2$ is a bilinear coupling. The normal mode transformation — a linear canonical transformation that diagonalizes $\hat{H}_{\text{osccoupled}}$ into two independent oscillators with shifted frequencies $\omega_\pm = \omega\sqrt{1 \pm \kappa/(m\omega^2)}$ — is the simplest instance of a linear canonical transformation. After the normal mode transformation, each normal mode is an independent harmonic oscillator whose Fock states and coherent states are given by the present paper. The coupled oscillator provides the first example in the QM-series where a many-body problem is solved by a transformation that decouples the degrees of freedom, and it serves as the conceptual prototype for the more complex decouplings — the Bogoliubov transformation in many-body physics, the mode decomposition in quantum field theory — that appear in the later papers.

References

[1] Milton Abramowitz and Irene A. Stegun. *Handbook of Mathematical Functions.* Dover, New York, 1965. Reference for: Hermite polynomials, their Rodrigues formula, recurrence relation, and orthogonality integral $\int_{-\infty}^{\infty} H_n(\xi)^2 e^{-\xi^2}\,d\xi = \sqrt{\pi}\,2^n\,n!$ (used in the normalization proof of QM6 Theorem 5.2); associated Laguerre polynomials $L_{n_r}^{\ell+1/2}$ appearing in the radial oscillator wave functions of QM6 Theorem 7.1; and the Gaussian integral $\int_0^\infty r^{2n+1} e^{-r^2}\,dr = n!/2$ used in the overcompleteness proof of QM6 Proposition 6.4.