QM7 — Multi-Particle Systems: Tensor Products, Identical Particles, and the Coupled Oscillator

in Scalar–Conformal NUVO Systems *Preprint, Version 1.0*[*]

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Notation and Conventions

- $\mathcal{M}$ denotes the spacetime manifold.
- η denotes the reference Lorentzian metric (typically Minkowski in a global chart).
- g denotes the physical metric.
- The scalar field $\Lambda : \mathcal{M} \to \mathbb{R}_{>0}$ is the NUVO modulation field.
- The physical metric is scalar–conformal:

$$g_{\mu\nu} = \Lambda^2 \eta_{\mu\nu}.$$

- $\Lambda_0 > 0$ denotes the baseline scalar availability level supported by the intrinsic delivery structure of the underlying field. In the absence of localized structural occupation the scalar field satisfies $\Lambda(x) = \Lambda_0$.
- The dimensionless scalar diagnostic is

$$\lambda(x) := \frac{\Lambda(x)}{\Lambda_0}.$$

- The scalar field represents the *locally available structural capacity* of the underlying delivery field. Localized structures may reduce this availability through occupation or transport, but the intrinsic delivery baseline Λ_0 remains fixed.
- Greek indices $\mu, \nu, \ldots$ range over spacetime coordinates $0, 1, 2, 3$.
- We use the Einstein summation convention unless explicitly stated otherwise.

Remark 0.1. *Unless otherwise stated, the background signature is* $(-, +, +, +)$.

[*]Bibliography is provisional. Cross-references to companion NUVO-series papers (M-, SR-, Q-, QB-, QM-series) will be updated with Zenodo DOIs in subsequent versions.

Program scope.

Abstract

The single-particle framework established in QM1 through QM6 describes the quantum mechanics of one transport closure configuration in the scalar–conformal exchange sector. The present paper extends this framework to systems of multiple transport closure configurations, introducing the tensor product Hilbert space $\mathcal{H}^{(2)} = \mathcal{H} \otimes \mathcal{H}$ for two particles and $\mathcal{H}^{(N)} = \bigotimes_{j=1}^{N} \mathcal{H}$ for N particles, and deriving the physical constraints that arise when the constituent configurations are indistinguishable.

The tensor product construction is established as the canonical mathematical structure for multi-particle quantum states, with the inner product, completeness, and observable algebra of the product space derived from those of the constituent single-particle spaces. The action of observables on the product space is defined through the tensor extension: single-particle observables $A^{(1)}$ and $B^{(2)}$ are extended to $A^{(1)} \otimes \hat{\mathbf{1}}$ and $\hat{\mathbf{1}} \otimes B^{(2)}$ on $\mathcal{H}^{(2)}$, and their commutation structure is derived.

When the N transport closure configurations are *indistinguishable* — when no physical quantity of the scalar–conformal transport system can distinguish configuration j from configuration k — the holonomy structure of the transport system imposes a constraint on the exchange symmetry of the state. Under exchange of two indistinguishable configurations, the closure state must either remain unchanged (symmetric, for *bosonic* transport configurations) or change sign (antisymmetric, for *fermionic* transport configurations). This dichotomy is the exchange symmetry principle, derived from the holonomy structure of the transport closure system rather than postulated.

The Fock space $\mathcal{F} = \bigoplus_{N=0}^{\infty} \mathcal{H}^{(N)}$ is established as the natural state space for systems in which the number of transport closure configurations is variable, with the vacuum $|0\rangle$, the bosonic Fock space $\mathcal{F}_+$, and the fermionic Fock space $\mathcal{F}_-$ each derived from the tensor product structure and the exchange symmetry.

The framework is applied to the coupled two-particle harmonic oscillator with Hamiltonian $\hat{H}_{\text{coup}} = \hat{H}^{(1)}_{\text{osc}} + \hat{H}^{(2)}_{\text{osc}} + \kappa\, \hat{x}^{(1)} \hat{x}^{(2)}$, which is decoupled by the center-of-mass and relative coordinate transformation. The normal mode frequencies $\omega_+ = \omega\sqrt{1 + \kappa/(m\omega^2)}$ and $\omega_- = \omega\sqrt{1 - \kappa/(m\omega^2)}$ are derived, the complete spectrum of the coupled system is obtained, and the role of the coupling in generating entanglement between the two subsystems is analyzed.

No new postulates are introduced beyond the tensor product construction and the exchange symmetry principle derived from holonomy.

1 Introduction

1.1 Position Within the QM-Series

The scalar–conformal NUVO program has now established, through QM1 to QM6, the complete framework for a single transport closure configuration in the exchange sector: the Hilbert space, the superposition and interference structure, the uncertainty relations, the Schrödinger dynamics, the angular momentum and hydrogen spectrum, and the harmonic oscillator with its coherent states. This single-particle framework describes one closure configuration evolving in a scalar–conformal background. The present paper, QM7, opens the multi-particle sector: the extension of the framework to systems consisting of two or more transport closure configurations, each with its own spatial degrees of freedom, and their mutual interaction through the exchange-sector coupling. The transition from one particle to many is not a quantitative extension of the single-particle framework but a qualitative one: two structures appear in the multi-particle theory that have no single-particle analogue — *entanglement* (the impossibility of factorizing the state of a composite system into independent subsystem states) and *exchange symmetry* (the constraint on multi-particle states imposed by the indistinguishability of identical transport closure configurations) — and both arise from the tensor product structure that is the subject of Secs. 3–6.

QM7 depends on the prior papers in three structurally specific ways. The tensor product construction of Sec. 3 builds directly on the single-particle Hilbert space $\mathcal{H}$ of QM1: the two-particle space $\mathcal{H}^{(2)} = \mathcal{H} \otimes \mathcal{H}$ is constructed as the completion of the algebraic tensor product of $\mathcal{H}$ with itself, inheriting the inner product and completeness properties of $\mathcal{H}$. The coupled harmonic oscillator of Sec. 7 uses the one-dimensional oscillator results of QM6 directly: the decoupled normal modes each inherit the Fock state structure, the ladder operators, and the coherent state theory of QM6, and the coherent states of the normal modes are the natural candidates for classical-limit states of the coupled system. The angular momentum addition of Sec. 8 uses the QM5 angular momentum algebra and spectrum: the total angular momentum $\hat{\mathbf{L}}_{\text{tot}} = \hat{\mathbf{L}}_1 \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes \hat{\mathbf{L}}_2$ satisfies the same SO(3) commutation algebra as the single-particle angular momentum, with the Clebsch-Gordan decomposition of the tensor product of two irreducible representations providing the connection between the product basis $|\,\ell_1, m_1\,\rangle \otimes |\,\ell_2, m_2\,\rangle$ and the total angular momentum basis $|\,J, M\,\rangle$.

The exchange symmetry principle derived in Sec. 5 connects QM7 to the Q-series holonomy quantization at a deeper structural level. In the Q-series, the holonomy quantization condition — the requirement that the transport phase accumulated along a closed path be an integer multiple of $2\pi\Phi_0$ — produced the quantization of energy (radial closure, principal quantum number n) and angular momentum (azimuthal closure, magnetic quantum number m). In QM7, the same holonomy principle is applied to a different class of closed paths: the exchange path, which continuously moves two indistinguishable transport closure configurations from positions (x_1, x_2) to (x_2, x_1). Since the initial and final configurations are identical by indistinguishability, this exchange path is closed in configuration space, and its holonomy must be quantized. The only two possibilities consistent with applying the exchange twice returning to the original state are holonomy $+1$ (bosons) and holonomy -1 (fermions), derived as a theorem in Theorem 5.7. This is the NUVO framework's account of the boson-fermion dichotomy: not a postulate but a consequence of the holonomy principle applied to exchange paths, in direct structural parallel with the energy and angular momentum quantization of the prior series.

QM7 opens the multi-particle sector that extends through QM8 to QM11. QM8 applies the tensor product construction to the spin degree of freedom: for a spin-$\frac{1}{2}$ particle, the full Hilbert space is $\mathcal{H} \otimes \mathbb{C}^2$, the product of the spatial Hilbert space and the two-dimensional spin space, and the spin operators are derived from the double-cover holonomy of QM8. QM9 analyzes the entanglement structure of states in $\mathcal{H}^{(2)}$ that do not factorize as products $\Psi_1 \otimes \Psi_2$, introducing the Schmidt decomposition as the canonical tool for quantifying entanglement, and treating the Bell states and the EPR setup as the primary examples. QM10 uses the center-of-mass and relative coordinate separation introduced in the coupled oscillator of the present paper to analyze two-body scattering: the relative coordinate $\hat{r} = x_1 - x_2$ carries the scattering dynamics while the center-of-mass coordinate $\hat{R} = (x_1 + x_2)/2$ moves freely. The Fock space of Sec. 6 provides the foundational framework for all of these extensions and for the field-theoretic applications that lie beyond the current series.

1.2 Objective of the Present Work

The central objective of the present paper is to extend the single-particle scalar–conformal NUVO framework to multi-particle systems by constructing the tensor product Hilbert space, deriving the exchange symmetry from holonomy, establishing the Fock space for variable particle number, and applying the framework to the canonical two-body model of the coupled harmonic oscillator. Specifically, the paper establishes six claims.

1. The two-particle Hilbert space $\mathcal{H}^{(2)} = \mathcal{H} \otimes \mathcal{H}$ is the completion of the algebraic tensor product of $\mathcal{H}$ with itself, with inner product $\langle \Phi_1 \otimes \Phi_2, \Psi_1 \otimes \Psi_2 \rangle_{\mathcal{H}^{(2)}} = \langle \Phi_1, \Psi_1 \rangle_{\mathcal{H}} \langle \Phi_2, \Psi_2 \rangle_{\mathcal{H}}$ extended by linearity. A complete orthonormal basis for $\mathcal{H}^{(2)}$ is $\{\phi_j \otimes \phi_k\}_{j,k \geq 1}$ for any ONB $\{\phi_j\}$ of $\mathcal{H}$, and $\mathcal{H}^{(2)} \cong L^2(\mathbb{R}^6, \mathbb{C})$ in position space. Single-particle observables of particle 1 and particle 2 commute on $\mathcal{H}^{(2)}$: $[A^{(1)}, B^{(2)}] = 0$.

2. The exchange operator $\hat{P}_{12}$ satisfying $\hat{P}_{12}(\Psi_1 \otimes \Psi_2) = \Psi_2 \otimes \Psi_1$ is a self-adjoint unitary with $\hat{P}_{12}^2 = \hat{\mathbf{1}}_{\mathcal{H}^{(2)}}$ and spectrum $\sigma(\hat{P}_{12}) = \{+1, -1\}$. The two-particle space decomposes orthogonally as $\mathcal{H}^{(2)} = \mathcal{H}_{\text{sym}} \oplus \mathcal{H}_{\text{anti}}$, where $\mathcal{H}_{\text{sym}}$ and $\mathcal{H}_{\text{anti}}$ are the ± 1 eigenspaces of $\hat{P}_{12}$. The symmetrization and antisymmetrization operators $\hat{S}_+ = \frac{1}{2}(\hat{\mathbf{1}} + \hat{P}_{12})$ and $\hat{S}_- = \frac{1}{2}(\hat{\mathbf{1}} - \hat{P}_{12})$ project onto these subspaces.

3. For indistinguishable transport closure configurations, the exchange path in configuration space — the path continuously moving configuration 1 from $\mathbf{x}_1$ to $\mathbf{x}_2$ while configuration 2 moves from $\mathbf{x}_2$ to $\mathbf{x}_1$ — is a closed path by indistinguishability, and its holonomy is a phase $\pi \in \{+1, -1\}$ (from $\pi^2 = 1$). The closure state must therefore satisfy $\hat{P}_{12}\Psi = \pi\Psi$: bosonic configurations ($\pi = +1$) have states in $\mathcal{H}_{\text{sym}}$ and fermionic configurations ($\pi = -1$) have states in $\mathcal{H}_{\text{anti}}$. The Pauli exclusion principle — no two identical fermions in the same single-particle state — follows as a corollary.

4. The Fock space $\mathcal{F} = \bigoplus_{N=0}^{\infty} \mathcal{H}^{(N)}$ is the natural state space for variable particle number, with vacuum $|0\rangle$ spanning $\mathcal{H}^{(0)} = \mathbb{C}$. The bosonic Fock space $\mathcal{F}_+$ and fermionic Fock space $\mathcal{F}_-$ are constructed from the symmetric and antisymmetric N-particle subspaces. For an ONB $\{\phi_j\}$ of $\mathcal{H}$, the creation and annihilation operators on Fock space satisfy the bosonic canonical commutation relations $[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}\hat{\mathbf{1}}$ or the fermionic canonical anticommutation relations $\{\hat{a}_j, \hat{a}_k^\dagger\} = \delta_{jk}\hat{\mathbf{1}}$.

5. The coupled harmonic oscillator $\hat{H}_{\text{coup}} = \hat{H}_{\text{osc}}^{(1)} + \hat{H}_{\text{osc}}^{(2)} + \kappa\, \hat{x}_1 \hat{x}_2$ is decoupled by the normal mode transformation $Q_+ = (x_1 + x_2)/\sqrt{2}$ and $Q_- = (x_1 - x_2)/\sqrt{2}$, giving two independent oscillators with normal mode frequencies $\omega_+ = \sqrt{\omega^2 + \kappa/m}$ and $\omega_- = \sqrt{\omega^2 - \kappa/m}$. The complete spectrum is $E_{n_+, n_-} = (n_+ + \frac{1}{2})\Phi_0\omega_+ + (n_- + \frac{1}{2})\Phi_0\omega_-$ for $n_\pm \in \{0, 1, 2, \ldots\}$, and the ground state is entangled in the original particle coordinates for $\kappa \neq 0$.

6. The tensor product of two angular momentum multiplets, $\mathcal{H}_{\ell_1} \otimes \mathcal{H}_{\ell_2}$, decomposes into irreducible SO(3) representations $\bigoplus_{J=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \mathcal{H}_J$ (the Clebsch-Gordan decomposition), establishing the total dimension count $\sum_J (2J+1) = (2\ell_1+1)(2\ell_2+1)$ and the triangle selection rule. The explicit Clebsch-Gordan coefficients are deferred to the full angular momentum addition treatment.

Claims (1) through (6) are logically ordered. The tensor product construction of claim (1) is the mathematical foundation for all subsequent claims. The exchange operator structure of claim (2) identifies the physically relevant subspaces $\mathcal{H}_{\text{sym}}$ and $\mathcal{H}_{\text{anti}}$ of the two-particle space. The holonomy derivation of claim (3) establishes which subspace physical states of identical particles must occupy. The Fock space of claim (4) extends this to variable particle number. The coupled oscillator of claim (5) is the first concrete multi-particle application, using the QM6 oscillator structure in the two-particle setting. The angular momentum addition preview of claim (6) connects the multi-particle framework to the QM5 angular momentum structure in preparation for QM8.

1.3 What Is Not Assumed

The present work maintains without modification the interpretive discipline of the prior series. Four exclusions are of particular importance for QM7.

The tensor product structure is not postulated as a separate axiom of multi-particle quantum mechanics. In standard formulations, the multi-particle state space is postulated to be the tensor product of single-particle spaces; the choice of tensor product rather than, say, direct sum is an assumption. In the NUVO framework, the tensor product $\mathcal{H} \otimes \mathcal{H}$ is the unique Hilbert space that represents two independent transport closure configurations with independent spatial degrees of freedom, in the sense that product states $\Psi_1 \otimes \Psi_2$ represent configurations that are uncorrelated and have inner products that factor as required by statistical independence. This uniqueness (up to isomorphism) is the content of the universal property of the tensor product, and it makes the tensor product the natural rather than postulated structure.

The exchange symmetry principle is not postulated. The standard quantum-mechanical treatment of identical particles introduces the symmetrization postulate: the state of a system of identical bosons must be symmetric under particle exchange, and the state of identical fermions must be antisymmetric. In the NUVO framework, this is derived as Theorem 5.7 from the holonomy quantization of the exchange path: the exchange path is closed by indistinguishability, its holonomy must be quantized, and the only consistent values are ± 1. The symmetrization postulate is therefore a theorem in the NUVO program, not an independent assumption.

The Pauli exclusion principle is not postulated. It follows in Corollary 5.10 as a two-line consequence of the antisymmetry of fermionic states: if both particles occupy the same single-particle state ϕ, the antisymmetrized state $\hat{S}_-(\phi \otimes \phi) = \frac{1}{2}(\phi \otimes \phi - \phi \otimes \phi) = 0$ vanishes identically.

The spin-statistics connection is not derived in the present paper and is not assumed. The theorem that integer-spin particles are bosons and half-integer-spin particles are fermions — the spin-statistics theorem — requires the relativistic framework: in the scalar–conformal NUVO program, it will be derived in the RQM-series from the structure of the relativistic transport closure system. QM7 establishes that $\pi \in \{+1, -1\}$ and that each particle type has a definite exchange parity, but the identification of specific particle types (electrons as fermions, photons as bosons) with their spin requires the QM8 spin algebra and the relativistic extension. This is recorded explicitly so that the QM7 derivation is not overstated: the boson-fermion dichotomy is established here; the spin-statistics connection is deferred.

1.4 Structure of the Paper

Sec. 2 recalls the single-particle Hilbert space from QM1, the oscillator structure from QM6, the angular momentum structure from QM5, and the Q-series holonomy principle as it applies to exchange paths. Sec. 3 constructs the two-particle Hilbert space $\mathcal{H}^{(2)} = \mathcal{H} \otimes \mathcal{H}$ as the completion of the algebraic tensor product, establishes its inner product, ONB, and isomorphism with $L^2(\mathbb{R}^6)$, and derives the observable algebra on the product space including the commutation $[A^{(1)}, B^{(2)}] = 0$. Sec. 4 introduces the exchange operator $\hat{P}_{12}$, derives its eigenvalues and eigenspaces, and constructs the symmetrization and antisymmetrization projectors $\hat{S}_+$ and $\hat{S}_-$. Sec. 5 derives the exchange symmetry principle for indistinguishable transport closure configurations from the holonomy of the exchange path, establishes the boson-fermion dichotomy as a theorem, and derives the Pauli exclusion principle as a corollary. Sec. 6 constructs the Fock space $\mathcal{F} = \bigoplus_N \mathcal{H}^{(N)}$, defines the bosonic and fermionic Fock spaces and the vacuum state, introduces the creation and annihilation operators on Fock space, and derives the bosonic CCR and fermionic CAR. Sec. 7 applies the two-particle framework to the coupled harmonic oscillator: derives the center-of-mass and relative coordinate separation, performs the normal mode transformation, obtains the normal mode frequencies and complete spectrum, and analyzes the entanglement in the coupled ground state. Sec. 8 derives the total angular momentum algebra for two-particle systems, records the Clebsch-Gordan decomposition structure, and previews the full angular momentum addition theory deferred to the complete treatment. Sec. 9 collects interpretive clarifications and records the scope of the present construction. Sec. 10 summarizes the thirteen principal results, records their programmatic significance, and prepares the transition to QM8.

2 Recalled Structure from Prior Papers

The present section collects the results from the Q-series, QM1, QM5, and QM6 that are directly required for the constructions of Secs. 3–8. Nothing in this section is new. The recalled material falls into four categories: the single-particle Hilbert space infrastructure that is duplicated and tensored in Sec. 3, the oscillator structure that appears in the two copies of Sec. 7, the angular momentum structure used in Sec. 8, and the holonomy principle applied to exchange paths in Sec. 5.

2.1 The Single-Particle Hilbert Space from QM1

The single-particle Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C})$ was established in QM1 with the following properties that are used directly in the tensor product construction.

Inner product and completeness (QM1 Definition 4.1 and Theorem 4.3). For $\Phi, \Psi \in \mathcal{H}$:

$$\langle \Phi, \Psi \rangle_{\mathcal{H}} := \int_{\mathbb{R}^3} \overline{\Phi(\mathbf{x})}\, \Psi(\mathbf{x})\, \mathrm{d}^3 x, \tag{1}$$

and $\mathcal{H}$ is complete: every Cauchy sequence in $\mathcal{H}$-norm converges in $\mathcal{H}$.

Complete orthonormal basis (QM1, from the spectral theorem Theorem 6.1). There exists a complete orthonormal basis (ONB) $\{\phi_j\}_{j\geq 1}$ for $\mathcal{H}$: $\langle \phi_j, \phi_k \rangle_{\mathcal{H}} = \delta_{jk}$ and every $\Psi \in \mathcal{H}$ expands as $\Psi = \sum_j \langle \phi_j, \Psi \rangle_{\mathcal{H}}\, \phi_j$ with the sum converging in $\mathcal{H}$-norm. The resolution of the identity is $\hat{\mathbf{1}}_{\mathcal{H}} = \sum_j |\phi_j\rangle\langle\phi_j|$.

Canonical commutation relation (QM1 Proposition 5.4). On the dense domain $\mathcal{S}(\mathbb{R}^3) \subset \mathcal{H}$:

$$[\hat{x}^j, \hat{p}_k] = \mathrm{i}\Phi_0\, \delta^j{}_k\, \hat{\mathbf{1}}, \qquad [\hat{x}^j, \hat{x}^k] = 0, \qquad [\hat{p}_j, \hat{p}_k] = 0. \tag{2}$$

These will be used in two copies on $\mathcal{H}^{(2)}$: the observables of particle 1 satisfy the CCR among themselves, and similarly for particle 2, while observables of particle 1 and particle 2 commute (established in Sec. 3.4).

Spectral theorem and self-adjointness (QM1 Theorem 6.1). Every self-adjoint operator A on $\mathcal{H}$ has a spectral decomposition $A = \int \lambda\, \mathrm{d}E_\lambda$, and the eigenstates (or generalized eigenstates) of A form a complete system in $\mathcal{H}$. This is used in Sec. 3.3 to establish that the product basis of a pair of self-adjoint operators is complete in $\mathcal{H}^{(2)}$.

Remark 2.1. *The four properties recalled above — inner product, completeness, ONB, and the spectral theorem — are exactly the properties needed for the tensor product construction of Sec. 3. The inner product of $\mathcal{H}$ defines the inner product of $\mathcal{H}^{(2)}$ via Eq.* (11). *The completeness of $\mathcal{H}$ is inherited by $\mathcal{H}^{(2)}$ through the completion of the algebraic tensor product. The ONB $\{\phi_j\}$ generates the product ONB $\{\phi_j \otimes \phi_k\}$ of $\mathcal{H}^{(2)}$. The spectral theorem provides the ONBs needed for the observable algebra on $\mathcal{H}^{(2)}$. No new structure from the prior papers is needed for the tensor product construction; QM1 already contains all the ingredients.*

2.2 The Oscillator Framework from QM6

The harmonic oscillator results of QM6 are used in two distinct ways in the present paper. In Sec. 7, two copies of the one-dimensional oscillator Hamiltonian appear in the coupled system, and the normal mode transformation converts the coupled system back into a pair of independent oscillators. In Sec. 6, the bosonic Fock space ladder operators $\hat{a}_j$ and $\hat{a}_j^\dagger$ are direct generalizations of the single-mode annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ of QM6.

Hamiltonian decomposition (QM6 Theorem 3.2). For a single mode of frequency ω:

$$\hat{H}_{\text{osc}} = \Phi_0\omega\Big(\hat{N} + \tfrac{1}{2}\hat{\mathbf{1}}\Big), \qquad \hat{N} = \hat{a}^\dagger\hat{a}, \qquad [\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}. \tag{3}$$

Fock state structure (QM6 Theorem 4.1 and Proposition 4.3). The spectrum of $\hat{H}_{\text{osc}}$ is $\{(n + \frac{1}{2})\Phi_0\omega : n \geq 0\}$, non-degenerate. The Fock states $|n\rangle = (\hat{a}^\dagger)^n|0\rangle/\sqrt{n!}$ are a complete ONB for the single-mode Hilbert space, with $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

Two-mode extension. For the coupled oscillator of Sec. 7, two independent modes with annihilation operators $\hat{a}_1$ and $\hat{a}_2$ act on different factors of the tensor product $\mathcal{H} \otimes \mathcal{H}$: $\hat{a}_1 = \hat{a} \otimes \hat{\mathbf{1}}$ and $\hat{a}_2 = \hat{\mathbf{1}} \otimes \hat{a}$. The commutation relations are $[\hat{a}_1, \hat{a}_1^\dagger] = \hat{\mathbf{1}}$, $[\hat{a}_2, \hat{a}_2^\dagger] = \hat{\mathbf{1}}$, and $[\hat{a}_1, \hat{a}_2] = [\hat{a}_1, \hat{a}_2^\dagger] = 0$ (operators on different tensor factors commute, by the general result of Proposition 3.8).

Remark 2.2. *The Fock space bosonic CCR $[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}\hat{\mathbf{1}}$ of Theorem 6.5 is the multi-mode generalization of the single-mode relation $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$ of QM6 Lemma 3.1. The δ_{jk} factor arises because modes $j \neq k$ act on different factors of the Fock space (and hence their operators commute by Proposition 3.8), while the same mode $j = k$ gives the QM6 relation. The fermionic anticommutation relation is the new result of QM7 that has no single-mode analogue in QM6.*

2.3 Angular Momentum from QM5

The angular momentum structure of QM5 enters QM7 in Sec. 8, where the total angular momentum of a two-particle system is analyzed. The following results are recalled in the form in which they will be applied.

Angular momentum commutation algebra (QM5 Theorem 3.1). On $\mathcal{S}(\mathbb{R}^3) \subset \mathcal{H}$:

$$[\hat{L}_j, \hat{L}_k] = \mathrm{i}\Phi_0\, \epsilon_{jkl}\, \hat{L}_l. \tag{4}$$

Spectrum of $\hat{L}^2$ and $\hat{L}_3$ (QM5 Theorem 5.2). The joint eigenstates $|\,\ell,\, m\,\rangle$ satisfy:

$$\hat{L}^2|\,\ell,\, m\,\rangle = \ell(\ell+1)\Phi_0^2|\,\ell,\, m\,\rangle, \qquad \hat{L}_3|\,\ell,\, m\,\rangle = m\Phi_0|\,\ell,\, m\,\rangle, \tag{5}$$

for $\ell \in \{0, 1, 2, \dots\}$ and $m \in \{-\ell, \dots, +\ell\}$, with $2\ell + 1$-fold degeneracy in m for each ℓ.

Total angular momentum in the two-particle system. For a two-particle system on $\mathcal{H}^{(2)} = \mathcal{H} \otimes \mathcal{H}$, the total angular momentum operators are

$$\hat{L}_{\text{tot}}^j \coloneqq \hat{L}_j \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes \hat{L}_j, \tag{6}$$

for $j = 1, 2, 3$. These satisfy the same commutation algebra Eq. (4) (established in Sec. 8), so $(\hat{L}_{\text{tot}})^2$ and $\hat{L}_{\text{tot}}^3$ are simultaneously diagonalizable, and the Clebsch-Gordan decomposition of $\mathcal{H}_{\ell_1} \otimes \mathcal{H}_{\ell_2}$ into irreducible SO(3) representations is established in Proposition 8.8.

2.4 The Holonomy Principle Applied to Exchange Paths

The Q-series holonomy quantization principle, recalled here in the form applicable to exchange paths, is the key input to Sec. 5.

The holonomy principle (Q-series, Q1–Q2). The transport phase accumulated along any admissible closed path in the scalar–conformal exchange sector is quantized: on a closed path γ that returns the transport closure configuration to its initial state, the accumulated transport phase satisfies

$$\Delta\phi_\gamma = n \cdot 2\pi\Phi_0, \qquad n \in \mathbb{Z}. \tag{7}$$

This is the principle that quantized the hydrogenic energy levels (radial closed paths, Q-series) and the magnetic quantum number (azimuthal closed paths, QM5 Theorem 5.1).

Application to exchange paths. For two indistinguishable transport closure configurations at positions $\mathbf{x}_1$ and $\mathbf{x}_2$, the exchange path is the continuous motion in configuration space that takes $(\mathbf{x}_1, \mathbf{x}_2) \to (\mathbf{x}_2, \mathbf{x}_1)$. By indistinguishability, the initial configuration $(\mathbf{x}_1, \mathbf{x}_2)$ and the final configuration $(\mathbf{x}_2, \mathbf{x}_1)$ are physically identical, so this exchange path is a closed path in the configuration space $\mathbb{R}^3 \times \mathbb{R}^3/\mathrm{Sym}_2$ (the symmetrized product of two copies of $\mathbb{R}^3$). The holonomy of this closed path must be quantized by Eq. (7); only the values $\mathrm{e}^{\mathrm{i}\cdot 0} = +1$ and $\mathrm{e}^{\mathrm{i}\pi} = -1$ are consistent with the constraint $\pi^2 = 1$ imposed by applying the exchange twice. The derivation is given in Theorem 5.7.

Remark 2.3. *The present application of the holonomy principle to exchange paths is the third distinct use of the same geometric principle in the NUVO program. In the Q-series, the holonomy of radial closure cycles quantized the principal quantum number n. In QM5 Theorem 5.1, the holonomy of azimuthal rotation paths quantized the magnetic quantum number $m \in \mathbb{Z}$. In QM7 Theorem 5.7, the holonomy of exchange paths in configuration space quantizes the exchange parity $\pi \in \{+1, -1\}$. All three quantization results have the same geometric origin: on a closed path in the relevant configuration space, the accumulated transport phase must be a multiple of $2\pi\Phi_0$. The discreteness of the quantum spectrum is, in the NUVO framework, a universal consequence of the holonomy structure of the transport closure system rather than a separately postulated feature of each physical sector. QM8 will add a fourth instance: the holonomy of the double-cover exchange path on $\mathrm{SU}(2)$ quantizes the spin quantum number $j \in \frac{1}{2}\mathbb{Z}$.*

3 The Two-Particle Hilbert Space

The two-particle Hilbert space is the mathematical structure that represents a physical system consisting of two distinct transport closure configurations, each with its own spatial degrees of freedom. The present section constructs this space as the tensor product $\mathcal{H}^{(2)} = \mathcal{H} \otimes \mathcal{H}$, establishes its inner product, complete orthonormal basis, and isomorphism with $L^2(\mathbb{R}^6, \mathbb{C})$, and derives the observable algebra on the product space. The construction is mathematical throughout: no new physical input beyond the single-particle Hilbert space $\mathcal{H}$ of QM1 is required, and the multi-particle structure emerges from the algebraic properties of the tensor product operation.

3.1 The Algebraic Tensor Product

The tensor product $\mathcal{H}^{(2)} = \mathcal{H} \otimes \mathcal{H}$ is built from the single-particle Hilbert space $\mathcal{H}$ in two steps: first the algebraic tensor product (a vector space), then its completion to a Hilbert space.

Definition 3.1 (Algebraic tensor product). *The* algebraic tensor product $\mathcal{H} \odot \mathcal{H}$ *is the vector space over* $\mathbb{C}$ *generated by formal products* $\Psi_1 \odot \Psi_2$ *for* $\Psi_1, \Psi_2 \in \mathcal{H}$*, subject to the bilinearity relations*

$$(c\Psi_1) \odot \Psi_2 = \Psi_1 \odot (c\Psi_2) = c(\Psi_1 \odot \Psi_2), \tag{8}$$

$$(\Psi_1 + \Psi_1') \odot \Psi_2 = \Psi_1 \odot \Psi_2 + \Psi_1' \odot \Psi_2, \tag{9}$$

$$\Psi_1 \odot (\Psi_2 + \Psi_2') = \Psi_1 \odot \Psi_2 + \Psi_1 \odot \Psi_2', \tag{10}$$

for all $c \in \mathbb{C}$ *and* $\Psi_1, \Psi_1', \Psi_2, \Psi_2' \in \mathcal{H}$*. A general element of* $\mathcal{H} \odot \mathcal{H}$ *is a finite linear combination* $\sum_k c_k(\Phi_k \odot \Psi_k)$*.*

Remark 3.2. *The formal product* $\Psi_1 \odot \Psi_2$ *is not the pointwise product of functions: it is a new abstract object representing the pair* (Ψ_1, Ψ_2) *of single-particle states, with the bilinearity relations encoding the physical requirement that scaling or superposing one particle's state is equivalent at the two-particle level. Not every element of* $\mathcal{H} \odot \mathcal{H}$ *is a simple tensor* $\Psi_1 \odot \Psi_2$*; the linear combinations* $\sum_k c_k(\Phi_k \odot \Psi_k)$ *that do not reduce to a single product are the* entangled *states, which have no factorized form. The existence of entangled states — and its physical consequences — is the central new feature of the multi-particle framework.*

3.2 The Two-Particle Hilbert Space

The algebraic tensor product $\mathcal{H} \odot \mathcal{H}$ is equipped with an inner product extending the single-particle inner product Eq. (1), and then completed to a Hilbert space.

Definition 3.3 (Two-particle Hilbert space). *The* two-particle inner product *on* $\mathcal{H} \odot \mathcal{H}$ *is defined on simple tensors by*

$$\langle \Phi_1 \odot \Phi_2, \Psi_1 \odot \Psi_2 \rangle_{\mathcal{H}^{(2)}} := \langle \Phi_1, \Psi_1 \rangle_{\mathcal{H}} \, \langle \Phi_2, \Psi_2 \rangle_{\mathcal{H}}, \tag{11}$$

and extended by sesquilinearity to all of $\mathcal{H} \odot \mathcal{H}$*. The* two-particle Hilbert space *is*

$$\mathcal{H}^{(2)} := \mathcal{H} \otimes \mathcal{H} := \overline{\mathcal{H} \odot \mathcal{H}}^{\|\cdot\|_{\mathcal{H}^{(2)}}}, \tag{12}$$

the completion of $\mathcal{H} \odot \mathcal{H}$ *with respect to the norm* $\|\Psi\|_{\mathcal{H}^{(2)}} = \langle \Psi, \Psi \rangle_{\mathcal{H}^{(2)}}^{1/2}$*. Simple tensors in* $\mathcal{H}^{(2)}$ *are denoted* $\Psi_1 \otimes \Psi_2$*.*

Remark 3.4. *The extension of Eq. (11) from simple tensors to all of* $\mathcal{H} \odot \mathcal{H}$ *by sesquilinearity must be verified to be well-defined and positive definite. Well-definedness follows from the universal property of the algebraic tensor product: the assignment* $(\Phi_1, \Phi_2, \Psi_1, \Psi_2) \mapsto \langle \Phi_1, \Psi_1 \rangle \langle \Phi_2, \Psi_2 \rangle$ *is separately antilinear in* Φ_j *and linear in* Ψ_j*, so it factors through the algebraic tensor product consistently with the bilinearity relations Eqs. (8)–(10). Positive definiteness:* $\langle \Psi, \Psi \rangle_{\mathcal{H}^{(2)}} = 0$ *implies* $\Psi = 0$ *in* $\mathcal{H} \odot \mathcal{H}$*; this follows from the positive definiteness of the single-particle inner products and the definition of the algebraic tensor product [1].*

3.3 Orthonormal Basis and Isomorphism with $L^2(\mathbb{R}^6)$

Proposition 3.5 (ONB and position-space isomorphism of $\mathcal{H}^{(2)}$). *Let* $\{\phi_j\}_{j \geq 1}$ *be a complete orthonormal basis for* $\mathcal{H}$*. Then the family* $\{\phi_j \otimes \phi_k\}_{j,k \geq 1}$ *is a complete orthonormal basis for* $\mathcal{H}^{(2)}$*, satisfying:*

$$\left\langle \phi_j \otimes \phi_k, \phi_{j'} \otimes \phi_{k'} \right\rangle_{\mathcal{H}^{(2)}} = \delta_{jj'} \, \delta_{kk'}, \tag{13}$$

and every $\Psi \in \mathcal{H}^{(2)}$ *expands as*

$$\Psi = \sum_{j,k=1}^{\infty} c_{jk}\, \phi_j \otimes \phi_k, \qquad c_{jk} = \langle \phi_j \otimes \phi_k, \Psi \rangle_{\mathcal{H}^{(2)}}\,, \tag{14}$$

with $\sum_{j,k} |c_{jk}|^2 = \|\Psi\|^2_{\mathcal{H}^{(2)}} < \infty$. *The resolution of the identity on* $\mathcal{H}^{(2)}$ *is*

$$\hat{\mathbf{1}}_{\mathcal{H}^{(2)}} = \sum_{j,k=1}^{\infty} |\phi_j \otimes \phi_k\rangle\langle \phi_j \otimes \phi_k|. \tag{15}$$

Furthermore, $\mathcal{H}^{(2)} \cong L^2(\mathbb{R}^6, \mathbb{C})$*, with the isomorphism given by*

$$(\Psi_1 \otimes \Psi_2)(\mathbf{x}_1, \mathbf{x}_2) = \Psi_1(\mathbf{x}_1)\, \Psi_2(\mathbf{x}_2), \tag{16}$$

for simple tensors, extended by linearity and closure to all of $\mathcal{H}^{(2)}$.

Proof. Orthonormality Eq. (13)*:* By Eq. (11) and the orthonormality of $\{\phi_j\}$:

$$\left\langle \phi_j \otimes \phi_k, \phi_{j'} \otimes \phi_{k'} \right\rangle_{\mathcal{H}^{(2)}} = \left\langle \phi_j, \phi_{j'} \right\rangle_{\mathcal{H}} \left\langle \phi_k, \phi_{k'} \right\rangle_{\mathcal{H}} = \delta_{jj'}\, \delta_{kk'}.$$

Completeness: Every simple tensor $\Psi_1 \otimes \Psi_2$ expands as $\Psi_1 = \sum_j \langle \phi_j, \Psi_1 \rangle\, \phi_j$ and $\Psi_2 = \sum_k \langle \phi_k, \Psi_2 \rangle\, \phi_k$, giving $\Psi_1 \otimes \Psi_2 = \sum_{j,k} \langle \phi_j, \Psi_1 \rangle \langle \phi_k, \Psi_2 \rangle\; \phi_j \otimes \phi_k$ with $\sum_{j,k} |\langle \phi_j, \Psi_1 \rangle \langle \phi_k, \Psi_2 \rangle|^2 = \|\Psi_1\|^2_{\mathcal{H}} \|\Psi_2\|^2_{\mathcal{H}} = \|\Psi_1 \otimes \Psi_2\|^2_{\mathcal{H}^{(2)}} < \infty$. General elements of $\mathcal{H}^{(2)}$ (finite linear combinations of simple tensors and their limits) expand similarly, since the span of $\{\phi_j \otimes \phi_k\}$ is dense in $\mathcal{H}^{(2)}$ by the density of simple tensors [1].

Isomorphism with $L^2(\mathbb{R}^6)$*:* The map Eq. (16) is an isometric isomorphism: it preserves the inner product, since $\langle \Phi_1 \otimes \Phi_2, \Psi_1 \otimes \Psi_2 \rangle_{\mathcal{H}^{(2)}} = \langle \Phi_1, \Psi_1 \rangle \langle \Phi_2, \Psi_2 \rangle = \int_{\mathbb{R}^6} \overline{\Phi_1(\mathbf{x}_1)\Phi_2(\mathbf{x}_2)}\Psi_1(\mathbf{x}_1)\Psi_2(\mathbf{x}_2)\, \mathrm{d}^3x_1\mathrm{d}^3x_2 = \langle \Phi_1 \otimes \Phi_2, \Psi_1 \otimes \Psi_2 \rangle_{L^2(\mathbb{R}^6)}$. The image of the product ONB $\{\phi_j(\mathbf{x}_1)\phi_k(\mathbf{x}_2)\}$ is a complete ONB for $L^2(\mathbb{R}^6)$, establishing surjectivity. □

Remark 3.6. *The isomorphism* $\mathcal{H}^{(2)} \cong L^2(\mathbb{R}^6, \mathbb{C})$ *has a concrete physical interpretation: the two-particle closure state* $\Psi \in \mathcal{H}^{(2)}$ *is a complex-valued square-integrable function* $\Psi(\mathbf{x}_1, \mathbf{x}_2)$ *of the positions of both transport closure configurations simultaneously. The closure density is* $|\Psi(\mathbf{x}_1, \mathbf{x}_2)|^2$*, a function on* $\mathbb{R}^6$*, and by the Born frequency law of QB6 (extended to* $\mathcal{H}^{(2)}$ *by the same argument as QM1 extended it to* $\mathcal{H}$*), this gives the asymptotic joint event frequency at positions* $(\mathbf{x}_1, \mathbf{x}_2)$ *in the detection region. For a simple tensor state* $\Psi_1(\mathbf{x}_1)\Psi_2(\mathbf{x}_2)$*, the joint event frequency factorizes as* $|\Psi_1(\mathbf{x}_1)|^2|\Psi_2(\mathbf{x}_2)|^2$*: the two configurations are statistically independent. For an entangled state that does not factorize, the joint event frequency does not separate, and the two configurations are statistically correlated.*

3.4 The Observable Algebra on the Two-Particle Space

Physical observables in the two-particle system are either single-particle observables (acting on one particle's degree of freedom) or genuine two-particle observables (acting on both simultaneously). The algebra of these observables on $\mathcal{H}^{(2)}$ is derived here.

Definition 3.7 (Single-particle observables on $\mathcal{H}^{(2)}$)**.** *For a self-adjoint operator* A *on* $\mathcal{H}$*, the* extensions to particle 1 and particle 2 *on* $\mathcal{H}^{(2)}$ *are:*

$$A^{(1)} := A \otimes \hat{\mathbf{1}}_{\mathcal{H}}, \qquad A^{(2)} := \hat{\mathbf{1}}_{\mathcal{H}} \otimes A. \tag{17}$$

Their action on simple tensors is:

$$A^{(1)}(\Psi_1 \otimes \Psi_2) = (A\Psi_1) \otimes \Psi_2, \qquad A^{(2)}(\Psi_1 \otimes \Psi_2) = \Psi_1 \otimes (A\Psi_2). \tag{18}$$

Proposition 3.8 (Commutativity of particle-1 and particle-2 observables)**.** *For any self-adjoint operators A and B on $\mathcal{H}$:*

$$[A^{(1)}, B^{(2)}] = 0 \quad \textit{on } \mathcal{H}^{(2)}. \tag{19}$$

In particular, all observables of particle 1 commute with all observables of particle 2.

Proof. On any simple tensor $\Psi_1 \otimes \Psi_2$, using Eq. (18):

$$A^{(1)}B^{(2)}(\Psi_1 \otimes \Psi_2) = A^{(1)}(\Psi_1 \otimes B\Psi_2) = (A\Psi_1) \otimes (B\Psi_2),$$
$$B^{(2)}A^{(1)}(\Psi_1 \otimes \Psi_2) = B^{(2)}((A\Psi_1) \otimes \Psi_2) = (A\Psi_1) \otimes (B\Psi_2).$$

Both give the same result, so $[A^{(1)}, B^{(2)}](\Psi_1 \otimes \Psi_2) = 0$. Since simple tensors are dense in $\mathcal{H}^{(2)}$ and the commutator is bounded (for bounded A and B; for unbounded operators the argument extends to the common domain), the result holds on all of $\mathcal{H}^{(2)}$. □

Remark 3.9. *Proposition 3.8 expresses a fundamental physical principle: observables of two independent transport closure configurations do not interfere with each other. This is the precise sense in which the tensor product structure encodes* physical independence*: particle 1 and particle 2 can be measured simultaneously without any algebraic constraint between their measurements. The contrast with the single-particle case is instructive: within one particle, position and momentum do not commute ($[\hat{x}, \hat{p}] = \mathrm{i}\Phi_0 \hat{\mathbf{1}}$), reflecting the intrinsic quantum uncertainty. Between particles, position of particle 1 and momentum of particle 2 do commute ($[\hat{x}^{(1)}, \hat{p}^{(2)}] = 0$), reflecting the independence of the two closure configurations. This independence can be broken by interactions: if the Hamiltonian contains a term $V(\hat{x}_1, \hat{x}_2)$ coupling the two particles, the time evolution will generate correlations between them, and a state that initially factorizes may evolve into an entangled state.*

Proposition 3.10 (Two-particle Hamiltonian structure)**.** *For two-particle systems with Hamiltonians $\hat{H}^{(1)} = \hat{H} \otimes \hat{\mathbf{1}}$ and $\hat{H}^{(2)} = \hat{\mathbf{1}} \otimes \hat{H}$ and an interaction potential $\hat{V}_{12}$ depending on both particle positions, the two-particle Hamiltonian is:*

$$\hat{H}_{\text{tot}} = \hat{H}^{(1)} + \hat{H}^{(2)} + \hat{V}_{12} \tag{20}$$

on $\mathcal{H}^{(2)}$. For $\hat{V}_{12} = 0$ (non-interacting particles), the spectrum of $\hat{H}_{\text{tot}}$ is the set of all sums $E_n^{(1)} + E_m^{(2)}$ where $E_n^{(1)}$ and $E_m^{(2)}$ are eigenvalues of $\hat{H}$ on $\mathcal{H}$, and the corresponding eigenstates are simple tensors $\phi_n \otimes \phi_m$.

Proof. For $\hat{V}_{12} = 0$: $\hat{H}_{\text{tot}}(\phi_n \otimes \phi_m) = (\hat{H}\phi_n) \otimes \phi_m + \phi_n \otimes (\hat{H}\phi_m) = (E_n^{(1)} + E_m^{(2)})(\phi_n \otimes \phi_m)$. Completeness of $\{\phi_n \otimes \phi_m\}$ in $\mathcal{H}^{(2)}$ (Proposition 3.5) gives the complete spectral decomposition. □

Remark 3.11. *Proposition 3.10 shows that in the non-interacting case the eigenstates are simple (product) tensors: no entanglement is generated by a non-interacting Hamiltonian from an initially product state. The interaction term $\hat{V}_{12}$ breaks this product structure: it couples the two particles and can generate entanglement under the time evolution $\mathrm{e}^{-\mathrm{i}\hat{H}_{\text{tot}}t/\Phi_0}$. The coupled harmonic oscillator of Sec. 7 is the canonical example: the coupling $\kappa\hat{x}_1\hat{x}_2$ is an interaction of exactly this form, and the ground state of the coupled system is entangled for $\kappa \neq 0$. The precise characterization of entanglement — its quantification, the Schmidt decomposition, and the Bell inequality structure — is developed in QM9. The present paper introduces entanglement (in the coupled oscillator analysis) but does not develop its full theory.*

3.5 The N-Particle Hilbert Space

The two-particle construction generalizes straightforwardly to N particles.

Definition 3.12 (N-particle Hilbert space)**.** *The* N-particle Hilbert space *is*

$$\mathcal{H}^{(N)} := \underbrace{\mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{N}, \tag{21}$$

with inner product $\left\langle \bigotimes_{j=1}^{N} \Phi_j, \bigotimes_{j=1}^{N} \Psi_j \right\rangle_{\mathcal{H}^{(N)}} = \prod_{j=1}^{N} \langle \Phi_j, \Psi_j \rangle_{\mathcal{H}}$ *extended by sesquilinearity. The position-space isomorphism is* $\mathcal{H}^{(N)} \cong L^2(\mathbb{R}^{3N}, \mathbb{C})$*, with product states represented as* $\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \prod_{j=1}^{N} \Psi_j(\mathbf{x}_j)$*.*

Remark 3.13. *The N-particle Hilbert space $\mathcal{H}^{(N)} \cong L^2(\mathbb{R}^{3N})$ grows rapidly with N: a complete ONB for $\mathcal{H}^{(N)}$ requires a countably infinite family $\{\phi_{j_1} \otimes \cdots \otimes \phi_{j_N}\}_{j_1,\ldots,j_N \geq 1}$ indexed by N-tuples. For identical particles, the physical subspace is much smaller: $\mathcal{H}^{(N)}_{\text{sym}}$ (bosons, symmetric under all permutations of N particles) or $\mathcal{H}^{(N)}_{\text{anti}}$ (fermions, antisymmetric under all permutations). The Fock space of Sec. 6 provides the natural framework for organizing these physical subspaces across all particle numbers $N = 0, 1, 2, \ldots$ simultaneously.*

4 The Exchange Operator and Symmetry Sectors

The two-particle Hilbert space $\mathcal{H}^{(2)}$ constructed in Sec. 3 contains all square-integrable functions of two particle positions, including states that are neither symmetric nor antisymmetric under the exchange of the two particles. For systems of two identical transport closure configurations, however, not all of $\mathcal{H}^{(2)}$ is physically accessible: the indistinguishability of the configurations restricts the physical states to a specific subspace. The present section introduces the exchange operator $\hat{P}_{12}$ that implements particle exchange on $\mathcal{H}^{(2)}$, derives its eigenvalues and eigenspaces by elementary algebra, and constructs the symmetrization and antisymmetrization projectors that select the physically accessible subspaces. The physical constraint that determines which subspace is occupied by a given particle type is the content of Sec. 5; the present section establishes the mathematical structure that makes that constraint precise.

4.1 Definition and Basic Properties of the Exchange Operator

Definition 4.1 (Exchange operator)**.** *The* exchange operator $\hat{P}_{12}$ *on* $\mathcal{H}^{(2)}$ *is defined on simple tensors by*

$$\hat{P}_{12}\big(\Psi_1 \otimes \Psi_2\big) = \Psi_2 \otimes \Psi_1, \tag{22}$$

and extended by linearity and continuity to all of $\mathcal{H}^{(2)}$*. In position space, Eq.* (22) *reads*

$$(\hat{P}_{12}\Psi)(\mathbf{x}_1, \mathbf{x}_2) = \Psi(\mathbf{x}_2, \mathbf{x}_1), \tag{23}$$

i.e., the exchange operator swaps the position arguments of the two-particle wave function.

Proposition 4.2 (Properties of the exchange operator)**.** *The exchange operator* $\hat{P}_{12}$ *on* $\mathcal{H}^{(2)}$ *is:*

(i) Bounded: $\left\| \hat{P}_{12} \right\|_{\text{op}} = 1$.

(ii) Self-adjoint: $\hat{P}_{12}^{\dagger} = \hat{P}_{12}$.

(iii) Unitary: $\hat{P}_{12}^{\dagger}\hat{P}_{12} = \hat{P}_{12}\hat{P}_{12}^{\dagger} = \hat{\mathbf{1}}_{\mathcal{H}^{(2)}}$.

(iv) Involutory: $\hat{P}_{12}^2 = \hat{\mathbf{1}}_{\mathcal{H}^{(2)}}$.

Proof. (i) Bounded: For any simple tensor, $\left\|\hat{P}_{12}(\Psi_1 \otimes \Psi_2)\right\|^2_{\mathcal{H}^{(2)}} = \|\Psi_2 \otimes \Psi_1\|^2_{\mathcal{H}^{(2)}} = \|\Psi_2\|^2_{\mathcal{H}} \|\Psi_1\|^2_{\mathcal{H}} = \|\Psi_1 \otimes \Psi_2\|^2_{\mathcal{H}^{(2)}}$, so $\hat{P}_{12}$ is an isometry on simple tensors, hence bounded with $\left\|\hat{P}_{12}\right\|_{\text{op}} = 1$ on all of $\mathcal{H}^{(2)}$ by density.

(ii) Self-adjoint: For simple tensors $\Phi_1 \otimes \Phi_2$ and $\Psi_1 \otimes \Psi_2$:

$$\left\langle \Phi_1 \otimes \Phi_2, \hat{P}_{12}(\Psi_1 \otimes \Psi_2)\right\rangle_{\mathcal{H}^{(2)}} = \langle \Phi_1 \otimes \Phi_2, \Psi_2 \otimes \Psi_1\rangle_{\mathcal{H}^{(2)}} = \langle \Phi_1, \Psi_2\rangle_{\mathcal{H}} \langle \Phi_2, \Psi_1\rangle_{\mathcal{H}},$$
$$\left\langle \hat{P}_{12}(\Phi_1 \otimes \Phi_2), \Psi_1 \otimes \Psi_2\right\rangle_{\mathcal{H}^{(2)}} = \langle \Phi_2 \otimes \Phi_1, \Psi_1 \otimes \Psi_2\rangle_{\mathcal{H}^{(2)}} = \langle \Phi_2, \Psi_1\rangle_{\mathcal{H}} \langle \Phi_1, \Psi_2\rangle_{\mathcal{H}}.$$

Both expressions are equal, confirming $\left\langle \Phi, \hat{P}_{12}\Psi\right\rangle = \left\langle \hat{P}_{12}\Phi, \Psi\right\rangle$ on simple tensors, hence on all of $\mathcal{H}^{(2)}$ by sesquilinearity and density.

(iii) Unitary: Follows from self-adjointness and part (iv): $\hat{P}_{12}^{\dagger}\hat{P}_{12} = \hat{P}_{12}^2 = \hat{\mathbf{1}}$.

(iv) Involutory: For any simple tensor: $\hat{P}_{12}^2(\Psi_1 \otimes \Psi_2) = \hat{P}_{12}(\Psi_2 \otimes \Psi_1) = \Psi_1 \otimes \Psi_2$, so $\hat{P}_{12}^2 = \hat{\mathbf{1}}_{\mathcal{H}^{(2)}}$ on simple tensors and hence on all of $\mathcal{H}^{(2)}$. □

Remark 4.3. *The exchange operator $\hat{P}_{12}$ transforms single-particle observables by swapping the particle labels. For any self-adjoint operator A on $\mathcal{H}$:*

$$\hat{P}_{12}\, A^{(1)}\, \hat{P}_{12} = A^{(2)}, \qquad \hat{P}_{12}\, A^{(2)}\, \hat{P}_{12} = A^{(1)}, \tag{24}$$

as verified by computing $\hat{P}_{12}(A \otimes \hat{\mathbf{1}})\hat{P}_{12}(\Psi_1 \otimes \Psi_2) = \hat{P}_{12}(A \otimes \hat{\mathbf{1}})(\Psi_2 \otimes \Psi_1) = \hat{P}_{12}(A\Psi_2 \otimes \Psi_1) = \Psi_1 \otimes A\Psi_2 = (\hat{\mathbf{1}} \otimes A)(\Psi_1 \otimes \Psi_2) = A^{(2)}(\Psi_1 \otimes \Psi_2)$. This relabeling property will be central in Sec. 5: for a system of indistinguishable particles, any physical observable $\hat{O}$ must be symmetric under particle exchange — $[\hat{O}, \hat{P}_{12}] = 0$ — because the labeling of particle 1 versus particle 2 carries no physical meaning.

4.2 Spectrum of the Exchange Operator and the Symmetry Sectors

The involutory property $\hat{P}_{12}^2 = \hat{\mathbf{1}}_{\mathcal{H}^{(2)}}$ severely constrains the spectrum of $\hat{P}_{12}$.

Proposition 4.4 (Spectrum and eigenspace decomposition of $\hat{P}_{12}$)**.** *The spectrum of the exchange operator is $\sigma(\hat{P}_{12}) = \{+1, -1\}$. The corresponding eigenspaces are:*

$$\mathcal{H}_{\text{sym}} := \ker(\hat{P}_{12} - \hat{\mathbf{1}}_{\mathcal{H}^{(2)}}) = \left\{\Psi \in \mathcal{H}^{(2)} \;\middle|\; \hat{P}_{12}\Psi = +\Psi\right\}, \tag{25}$$

$$\mathcal{H}_{\text{anti}} := \ker(\hat{P}_{12} + \hat{\mathbf{1}}_{\mathcal{H}^{(2)}}) = \left\{\Psi \in \mathcal{H}^{(2)} \;\middle|\; \hat{P}_{12}\Psi = -\Psi\right\}, \tag{26}$$

and $\mathcal{H}^{(2)}$ decomposes as the orthogonal direct sum

$$\mathcal{H}^{(2)} = \mathcal{H}_{\text{sym}} \oplus \mathcal{H}_{\text{anti}}. \tag{27}$$

Proof. Spectrum: If $\hat{P}_{12}\Psi = \lambda\Psi$ for some $\Psi \neq 0$, then $\lambda^2\Psi = \hat{P}_{12}^2\Psi = \Psi$, giving $\lambda^2 = 1$, so $\lambda \in \{+1, -1\}$. Both values are achieved: for any $\Psi_1, \Psi_2 \in \mathcal{H}$ with $\Psi_1 \neq \Psi_2$, the states

$$\Psi_+ = \Psi_1 \otimes \Psi_2 + \Psi_2 \otimes \Psi_1 \in \mathcal{H}_{\text{sym}}, \qquad \Psi_- = \Psi_1 \otimes \Psi_2 - \Psi_2 \otimes \Psi_1 \in \mathcal{H}_{\text{anti}},$$

are non-zero eigenstates with eigenvalues $+1$ and -1 respectively.

Orthogonality of eigenspaces: For $\Psi_+ \in \mathcal{H}_{\mathrm{sym}}$ and $\Psi_- \in \mathcal{H}_{\mathrm{anti}}$, using self-adjointness of $\hat{P}_{12}$:

$$\langle \Psi_+, \Psi_- \rangle_{\mathcal{H}^{(2)}} = \left\langle \hat{P}_{12}\Psi_+, \Psi_- \right\rangle_{\mathcal{H}^{(2)}} = \left\langle \Psi_+, \hat{P}_{12}\Psi_- \right\rangle_{\mathcal{H}^{(2)}} = \langle \Psi_+, -\Psi_- \rangle_{\mathcal{H}^{(2)}} = -\langle \Psi_+, \Psi_- \rangle_{\mathcal{H}^{(2)}},$$

so $\langle \Psi_+, \Psi_- \rangle = 0$.

Decomposition $\mathcal{H}^{(2)} = \mathcal{H}_{\mathrm{sym}} \oplus \mathcal{H}_{\mathrm{anti}}$: Any $\Psi \in \mathcal{H}^{(2)}$ decomposes as $\Psi = \frac{1}{2}(\Psi + \hat{P}_{12}\Psi) + \frac{1}{2}(\Psi - \hat{P}_{12}\Psi)$, where $\hat{P}_{12}[\frac{1}{2}(\Psi + \hat{P}_{12}\Psi)] = \frac{1}{2}(\hat{P}_{12}\Psi + \Psi) = \frac{1}{2}(\Psi + \hat{P}_{12}\Psi)$ (the first term is in $\mathcal{H}_{\mathrm{sym}}$) and $\hat{P}_{12}[\frac{1}{2}(\Psi - \hat{P}_{12}\Psi)] = \frac{1}{2}(\hat{P}_{12}\Psi - \Psi) = -\frac{1}{2}(\Psi - \hat{P}_{12}\Psi)$ (the second term is in $\mathcal{H}_{\mathrm{anti}}$). □

Remark 4.5. *Explicit elements of $\mathcal{H}_{\mathrm{sym}}$ and $\mathcal{H}_{\mathrm{anti}}$ for a given ONB $\{\phi_j\}$ of $\mathcal{H}$ are:*

- Symmetric (in $\mathcal{H}_{\mathrm{sym}}$): *For $j = k$: $\phi_j \otimes \phi_j$ (already symmetric). For $j \neq k$: $(\phi_j \otimes \phi_k + \phi_k \otimes \phi_j)/\sqrt{2}$ (normalized symmetric combination).*
- Antisymmetric (in $\mathcal{H}_{\mathrm{anti}}$): *For $j \neq k$: $(\phi_j \otimes \phi_k - \phi_k \otimes \phi_j)/\sqrt{2}$ (normalized antisymmetric combination). For $j = k$: $\phi_j \otimes \phi_j - \phi_j \otimes \phi_j = 0$ (no antisymmetric state with both particles in the same mode).*

The last observation — that there is no non-zero antisymmetric state with both particles in the same single-particle mode — is the Pauli exclusion principle, derived in Corollary 5.10 from the antisymmetry of fermionic states.

4.3 Symmetrization and Antisymmetrization Operators

The orthogonal projectors onto $\mathcal{H}_{\mathrm{sym}}$ and $\mathcal{H}_{\mathrm{anti}}$ are determined by the spectral decomposition of $\hat{P}_{12}$.

Proposition 4.6 (Symmetrization and antisymmetrization operators)**.** *The operators*

$$\hat{S}_+ := \frac{1}{2}(\hat{\mathbf{1}}_{\mathcal{H}^{(2)}} + \hat{P}_{12}), \qquad \hat{S}_- := \frac{1}{2}(\hat{\mathbf{1}}_{\mathcal{H}^{(2)}} - \hat{P}_{12}), \tag{28}$$

are the orthogonal projectors onto $\mathcal{H}_{\mathrm{sym}}$ and $\mathcal{H}_{\mathrm{anti}}$ respectively, satisfying:

$$\hat{S}_+^2 = \hat{S}_+, \quad \hat{S}_-^2 = \hat{S}_-, \tag{29}$$

$$\hat{S}_+^\dagger = \hat{S}_+, \quad \hat{S}_-^\dagger = \hat{S}_-, \tag{30}$$

$$\hat{S}_+ + \hat{S}_- = \hat{\mathbf{1}}_{\mathcal{H}^{(2)}}, \quad \hat{S}_+\hat{S}_- = \hat{S}_-\hat{S}_+ = 0. \tag{31}$$

Their action on product states is:

$$\hat{S}_+(\Psi_1 \otimes \Psi_2) = \frac{1}{2}\big(\Psi_1 \otimes \Psi_2 + \Psi_2 \otimes \Psi_1\big), \tag{32}$$

$$\hat{S}_-(\Psi_1 \otimes \Psi_2) = \frac{1}{2}\big(\Psi_1 \otimes \Psi_2 - \Psi_2 \otimes \Psi_1\big). \tag{33}$$

Proof. Projector properties Eq. (29)*:* $\hat{S}_+^2 = \frac{1}{4}(\hat{\mathbf{1}} + \hat{P}_{12})^2 = \frac{1}{4}(\hat{\mathbf{1}} + 2\hat{P}_{12} + \hat{P}_{12}^2) = \frac{1}{4}(\hat{\mathbf{1}} + 2\hat{P}_{12} + \hat{\mathbf{1}}) = \frac{1}{2}(\hat{\mathbf{1}} + \hat{P}_{12}) = \hat{S}_+$, using $\hat{P}_{12}^2 = \hat{\mathbf{1}}$. Similarly $\hat{S}_-^2 = \hat{S}_-$.

Self-adjointness Eq. (30)*:* From $\hat{P}_{12}^\dagger = \hat{P}_{12}$: $\hat{S}_+^\dagger = \frac{1}{2}(\hat{\mathbf{1}} + \hat{P}_{12})^\dagger = \frac{1}{2}(\hat{\mathbf{1}} + \hat{P}_{12}) = \hat{S}_+$, and similarly for $\hat{S}_-$.

Partition of unity and orthogonality Eq. (31)*:* $\hat{S}_+ + \hat{S}_- = \frac{1}{2}(\hat{\mathbf{1}} + \hat{P}_{12}) + \frac{1}{2}(\hat{\mathbf{1}} - \hat{P}_{12}) = \hat{\mathbf{1}}$. $\hat{S}_+\hat{S}_- = \frac{1}{4}(\hat{\mathbf{1}} + \hat{P}_{12})(\hat{\mathbf{1}} - \hat{P}_{12}) = \frac{1}{4}(\hat{\mathbf{1}} - \hat{P}_{12} + \hat{P}_{12} - \hat{P}_{12}^2) = \frac{1}{4}(\hat{\mathbf{1}} - \hat{\mathbf{1}}) = 0$.

Actions Eqs. (32)–(33)*:* Direct from Definition 4.1 and Eq. (28). □

Remark 4.7. *The symmetrized and antisymmetrized product states of Remark 4.5 are not normalized when* $\Psi_1 \neq \Psi_2$ *(their norm is* $1/\sqrt{2}$ *rather than* 1*). The normalized symmetric and antisymmetric two-particle states built from ONB elements* $\phi_j \neq \phi_k$ *are:*

$$\phi_j \vee \phi_k := \frac{1}{\sqrt{2}}\big(\phi_j \otimes \phi_k + \phi_k \otimes \phi_j\big), \quad j \neq k, \tag{34}$$

$$\phi_j \wedge \phi_k := \frac{1}{\sqrt{2}}\big(\phi_j \otimes \phi_k - \phi_k \otimes \phi_j\big), \quad j \neq k. \tag{35}$$

These are the normalized building blocks of the bosonic and fermionic two-particle Hilbert spaces. The antisymmetric product $\phi_j \wedge \phi_k$ *is the two-particle Slater determinant:*

$$\phi_j \wedge \phi_k = \frac{1}{\sqrt{2}}\begin{vmatrix} \phi_j(\mathbf{x}_1) & \phi_j(\mathbf{x}_2) \\ \phi_k(\mathbf{x}_1) & \phi_k(\mathbf{x}_2) \end{vmatrix}, \tag{36}$$

the standard antisymmetric two-particle wave function of fermionic quantum mechanics, derived here from the antisymmetrization operator rather than postulated.

Remark 4.8. *The symmetrization and antisymmetrization for* $N > 2$ *particles generalizes by replacing* $\hat{P}_{12}$ *(transposition of two elements) with the full permutation group* Sym_N *acting on* $\mathcal{H}^{(N)}$*. The symmetric subspace* $\mathcal{H}^{(N)}_{\mathrm{sym}}$ *consists of states invariant under all* $N!$ *permutations, and the antisymmetric subspace* $\mathcal{H}^{(N)}_{\mathrm{anti}}$ *consists of states that pick up the sign of the permutation (the* signature $\mathrm{sgn}(\sigma)$ *for* $\sigma \in \mathrm{Sym}_N$*). For* $N = 2$*,* $\mathrm{Sym}_2 = \{\mathrm{id}, \hat{P}_{12}\}$ *has only two elements and one non-trivial transposition, giving exactly the* $\hat{S}_+$ *and* $\hat{S}_-$ *of Proposition 4.6. The* N*-particle Slater determinant is the antisymmetric product of* N *single-particle states* $\phi_{j_1}, \ldots, \phi_{j_N}$*, and the* N*-particle permanent is the symmetric product; both are derived from the* N*-particle symmetrization operators. The present paper treats the* $N = 2$ *case in detail and uses the Fock space of Sec. 6 for the general* N *case.*

5 Exchange Symmetry from Holonomy

The exchange operator $\hat{P}_{12}$ and its eigenspaces $\mathcal{H}_{\mathrm{sym}}$ and $\mathcal{H}_{\mathrm{anti}}$ are mathematical structures available for any two-particle Hilbert space $\mathcal{H}^{(2)}$, regardless of whether the two particles are identical. For distinguishable particles — two different species of transport closure configuration — the full space $\mathcal{H}^{(2)} = \mathcal{H}_{\mathrm{sym}} \oplus \mathcal{H}_{\mathrm{anti}}$ is physically accessible, and no exchange symmetry constraint applies. For *indistinguishable* particles — two transport closure configurations of the same type, with identical internal structure — the holonomy quantization principle of the Q-series imposes a constraint: the physical states must lie entirely within $\mathcal{H}_{\mathrm{sym}}$ or entirely within $\mathcal{H}_{\mathrm{anti}}$, and the choice is a fixed property of the particle type. The present section derives this exchange symmetry constraint, establishes the boson-fermion dichotomy as a theorem, and derives the Pauli exclusion principle as its most immediate corollary.

5.1 Indistinguishable Transport Closure Configurations

The concept of indistinguishability requires precise definition in the NUVO transport closure framework.

Definition 5.1 (Indistinguishable transport closure configurations)**.** *Two transport closure configurations are* indistinguishable *if no observable of the scalar–conformal exchange-sector transport* *system can distinguish one from the other: for every self-adjoint observable $\hat{O}$ on $\mathcal{H}^{(2)}$ that is a function of the transport closure variables of the system, the expectation value of $\hat{O}$ in the state Ψ is equal to the expectation value in the exchanged state $\hat{P}_{12}\Psi$:*

$$\langle \hat{O} \rangle_\Psi = \langle \hat{O} \rangle_{\hat{P}_{12}\Psi} \quad \textit{for all physical observables } \hat{O}. \tag{37}$$

Remark 5.2. *The indistinguishability condition Eq.* (37) *is a statement about the transport closure geometry, not about experimental precision. Two configurations are indistinguishable not because they are difficult to distinguish experimentally but because the scalar–conformal transport system contains no physical variable that assigns a label to one configuration versus another. If both configurations have the same type of transport closure loop (same charge, same mass, same internal structure), there is no exchange-sector observable that can tell them apart. The labeling "particle 1" and "particle 2" is a mathematical convenience with no physical content; any physical statement must be invariant under the relabeling.*

Proposition 5.3 (Indistinguishability implies exchange invariance of observables)**.** *If two transport closure configurations are indistinguishable in the sense of Definition 5.1, then every physical observable $\hat{O}$ on $\mathcal{H}^{(2)}$ commutes with the exchange operator:*

$$[\hat{O}, \hat{P}_{12}] = 0. \tag{38}$$

Proof. For any normalized $\Psi \in \mathcal{H}^{(2)}$, the indistinguishability condition Eq. (37) gives $\left\langle \Psi, \hat{O}\Psi \right\rangle_{\mathcal{H}^{(2)}} = \left\langle \hat{P}_{12}\Psi, \hat{O}\,\hat{P}_{12}\Psi \right\rangle_{\mathcal{H}^{(2)}} = \left\langle \Psi, \hat{P}_{12}^\dagger \hat{O}\,\hat{P}_{12}\Psi \right\rangle_{\mathcal{H}^{(2)}}$, using $\hat{P}_{12}^\dagger = \hat{P}_{12}$. Since this holds for all Ψ, the operators $\hat{O}$ and $\hat{P}_{12}^\dagger \hat{O} \hat{P}_{12} = \hat{P}_{12}\hat{O}\hat{P}_{12}$ have identical expectation values in all states, hence are equal as operators: $\hat{O} = \hat{P}_{12}\hat{O}\hat{P}_{12}$. Multiplying on the left by $\hat{P}_{12}$ and using $\hat{P}_{12}^2 = \mathbf{1}$: $\hat{P}_{12}\hat{O} = \hat{O}\hat{P}_{12}$, which is Eq. (38). □

Remark 5.4. *Proposition 5.3 implies that $\hat{P}_{12}$ is a* superselection operator*: every physical observable commutes with it. Consequently, the eigenspaces $\mathcal{H}_{\text{sym}}$ (eigenvalue $+1$) and $\mathcal{H}_{\text{anti}}$ (eigenvalue -1) are* superselection sectors *— subspaces that are mapped to themselves by all physical observables and between which no physical process can create a coherent superposition. A state $\Psi = \Psi_+ + \Psi_-$ with $\Psi_+ \in \mathcal{H}_{\text{sym}}$ and $\Psi_- \in \mathcal{H}_{\text{anti}}$ both non-zero has $\langle \hat{O} \rangle_\Psi = \langle \hat{O} \rangle_{\Psi_+} + \langle \hat{O} \rangle_{\Psi_-}$ with no cross-terms $\left\langle \Psi_+, \hat{O}\Psi_- \right\rangle$ surviving (since $\left\langle \Psi_+, \hat{O}\Psi_- \right\rangle = \left\langle \Psi_+, \hat{P}_{12}\hat{O}\Psi_- \right\rangle = \left\langle \hat{P}_{12}^\dagger \Psi_+, \hat{O}\Psi_- \right\rangle = \left\langle \Psi_+, \hat{O}\Psi_- \right\rangle$ but also $= \left\langle \hat{P}_{12}\Psi_+, \hat{O}\hat{P}_{12}\Psi_- \right\rangle = \left\langle \Psi_+, \hat{O}(-\Psi_-) \right\rangle = -\left\langle \Psi_+, \hat{O}\Psi_- \right\rangle$, forcing the cross term to vanish). This means the symmetric and antisymmetric sectors are completely physically decoupled: no physical process can prepare, measure, or evolve across the boundary between them.*

5.2 The Exchange Path in Configuration Space

The exchange symmetry constraint — whether the physical states of identical particles lie in $\mathcal{H}_{\text{sym}}$ or $\mathcal{H}_{\text{anti}}$ — is determined by the holonomy of a specific path in the configuration space of the two-particle system.

Definition 5.5 (Exchange path)**.** *For two indistinguishable transport closure configurations at positions $\mathbf{x}_1$ and $\mathbf{x}_2$, an* exchange path *is a continuous path in configuration space $\mathbb{R}^3 \times \mathbb{R}^3$ that takes the configuration $(\mathbf{x}_1, \mathbf{x}_2)$ continuously to the configuration $(\mathbf{x}_2, \mathbf{x}_1)$.*

Remark 5.6. *For distinguishable particles, the exchange path connects two distinct points in configuration space:* $(\mathbf{x}_1, \mathbf{x}_2)$ *is physically different from* $(\mathbf{x}_2, \mathbf{x}_1)$*, and the exchange path is an open arc in* $\mathbb{R}^3 \times \mathbb{R}^3$*. For indistinguishable particles, the configuration* $(\mathbf{x}_1, \mathbf{x}_2)$ *and the configuration* $(\mathbf{x}_2, \mathbf{x}_1)$ *are physically identical (by Definition 5.1), so the exchange path connects a configuration to itself. It is therefore a* closed path *in the physical configuration space, which for identical particles is not* $\mathbb{R}^3 \times \mathbb{R}^3$ *but the* symmetrized product $(\mathbb{R}^3 \times \mathbb{R}^3)/\mathrm{Sym}_2$*, the quotient that identifies* $(\mathbf{x}_1, \mathbf{x}_2)$ *with* $(\mathbf{x}_2, \mathbf{x}_1)$*. The fundamental group of this symmetrized configuration space is* $\pi_1((\mathbb{R}^3 \times \mathbb{R}^3)/\mathrm{Sym}_2) = \mathbb{Z}_2$*, reflecting the two topologically distinct types of exchange paths: those that can be continuously deformed to the identity (trivial holonomy* $+1$*) and those that cannot (holonomy* -1*).*

5.3 The Exchange Symmetry Theorem

Theorem 5.7 (Exchange symmetry from holonomy quantization)**.** *For a system of two indistinguishable transport closure configurations, the closure state must satisfy*

$$\hat{P}_{12}\Psi = \pi\,\Psi, \qquad \pi \in \{+1, -1\}, \tag{39}$$

where π *is the* exchange parity *of the particle type. States with* $\pi = +1$ *(*bosonic configurations*, lying in* $\mathcal{H}_{\mathrm{sym}}$*) are symmetric under exchange; states with* $\pi = -1$ *(*fermionic configurations*, lying in* $\mathcal{H}_{\mathrm{anti}}$*) are antisymmetric. The exchange parity is derived from the holonomy of the exchange path in the physical configuration space* $(\mathbb{R}^3 \times \mathbb{R}^3)/\mathrm{Sym}_2$*: applying the exchange operation twice returns to the initial state, requiring*

$$\pi^2 = 1, \tag{40}$$

which selects $\pi \in \{+1, -1\}$*.*

Proof. Step 1: Physical states must be exchange eigenstates. By Proposition 5.3, $\hat{P}_{12}$ commutes with all physical observables. Since $\hat{P}_{12}$ is self-adjoint with spectrum $\{+1, -1\}$ (Proposition 4.4), the physical Hilbert space decomposes as $\mathcal{H}^{(2)} = \mathcal{H}_{\mathrm{sym}} \oplus \mathcal{H}_{\mathrm{anti}}$. A state $\Psi = \Psi_+ + \Psi_-$ with $\Psi_+ \in \mathcal{H}_{\mathrm{sym}}$ and $\Psi_- \in \mathcal{H}_{\mathrm{anti}}$ both non-zero would lie in neither eigenspace. By the superselection argument of Remark 5.4, the cross-terms $\left\langle \Psi_+, \hat{O}\Psi_- \right\rangle$ vanish for all physical $\hat{O}$, so the components Ψ_+ and Ψ_- evolve and are measured independently — they represent distinct physical states that happen to be described by the same mathematical vector. For a pure state of identical particles, the physical state occupies exactly one superselection sector.

Step 2: The exchange parity from the holonomy condition. Consider the transport closure state Ψ of the two-particle system as a function on the physical configuration space $(\mathbb{R}^3 \times \mathbb{R}^3)/\mathrm{Sym}_2$. The exchange path of Definition 5.5 is a closed loop γ in this space (by Remark 5.6). The holonomy of γ is the phase π accumulated by Ψ as γ is traversed: $\Psi(\mathbf{x}_2, \mathbf{x}_1) = \pi\,\Psi(\mathbf{x}_1, \mathbf{x}_2)$. By the Q-series holonomy quantization principle Eq. (7), the accumulated phase along any closed transport closure path must be quantized. Traversing the exchange loop twice returns to the initial configuration: $(\hat{P}_{12})^2\Psi = \Psi$, giving $\pi^2\Psi = \Psi$, hence $\pi^2 = 1$. The only solutions are $\pi = +1$ and $\pi = -1$.

Step 3: π *is a fixed invariant of the particle type.* Since π is a topological invariant of the exchange path in $(\mathbb{R}^3 \times \mathbb{R}^3)/\mathrm{Sym}_2$ — it cannot change under continuous deformation of the path — it is a fixed property of the particle type, the same for all states of a given species of indistinguishable transport closure configuration. □

Remark 5.8. *Theorem 5.7 establishes that the exchange parity* $\pi \in \{+1, -1\}$ *is a fixed invariant of the particle type. It does not, however, determine which specific particle types have* $\pi = +1$ *(bosons) and which have* $\pi = -1$ *(fermions). In the standard quantum-mechanical framework,* *this is answered by the spin-statistics theorem: particles with integer spin are bosons and particles with half-integer spin are fermions. In the NUVO program, this connection requires the relativistic framework: the full spin-statistics theorem is a consequence of the relativistic transport closure structure and will be derived in the RQM-series. What QM7 establishes is the structural dichotomy — every identical particle system has a definite exchange parity* ± 1 *— and the consequences of each choice (symmetry and bosonic statistics for* $\pi = +1$*; antisymmetry, fermionic statistics, and Pauli exclusion for* $\pi = -1$*) are derived in the remainder of the present section.*

Remark 5.9. *The restriction* $\pi \in \{+1, -1\}$ *in three spatial dimensions is a consequence of the topology of the symmetrized configuration space* $(\mathbb{R}^3 \times \mathbb{R}^3)/\mathrm{Sym}_2$*: its fundamental group is* $\pi_1 = \mathbb{Z}_2$*, which has only two elements, corresponding to the two possible holonomies* $+1$ *and* -1*. In two spatial dimensions, the configuration space* $(\mathbb{R}^2 \times \mathbb{R}^2)/\mathrm{Sym}_2$ *has fundamental group* $\pi_1 = \mathbb{Z}$ *(the braid group), which is infinite, and the holonomy can take any value* $\mathrm{e}^{\mathrm{i}\theta}$ *for* $\theta \in [0, 2\pi)$*. Particles in two dimensions with* $\theta \neq 0, \pi$ *are called* anyons *(any statistics). The scalar–conformal NUVO program operates in three spatial dimensions, so anyons do not arise; the full anyon structure requires a two-dimensional transport closure system and is outside the scope of the present paper.*

5.4 The Pauli Exclusion Principle

The Pauli exclusion principle — the impossibility of two identical fermions occupying the same single-particle state — follows immediately from the antisymmetry of fermionic states.

Corollary 5.10 (Pauli exclusion principle)**.** *For two indistinguishable fermionic transport closure configurations* $(\pi = -1)$*, the two-particle closure state* $\Psi \in \mathcal{H}_{\mathrm{anti}}$ *satisfies* $\Psi = 0$ *whenever both configurations occupy the same single-particle state* $\phi \in \mathcal{H}$*. Equivalently:*

$$\hat{S}_-(\phi \otimes \phi) = 0 \quad \textit{for all } \phi \in \mathcal{H}. \tag{41}$$

Proof. By Eq. (33):

$$\hat{S}_-(\phi \otimes \phi) = \frac{1}{2}(\phi \otimes \phi - \phi \otimes \phi) = 0.$$

□

Remark 5.11. *The Pauli exclusion principle, derived in two lines from the antisymmetry of fermionic states, is a structural theorem in the NUVO framework rather than a separate postulate. Its physical content in transport closure language is: if two indistinguishable fermionic transport closure configurations were to occupy the same spatial closure mode* ϕ*, the antisymmetrized state would be identically zero — not a state of small amplitude, but a zero vector, i.e., no such physical state exists. The exclusion is therefore not a rule imposed on the system but a consequence of the topology of the exchange path: the holonomy* $\pi = -1$ *produces antisymmetric states, and antisymmetric states with both particles in the same mode simply do not exist in the Hilbert space.*

Remark 5.12. *The Pauli exclusion principle has profound consequences for the structure of matter that propagate throughout the QM-series. For two fermionic configurations in distinct modes* $\phi_j \neq \phi_k$*, the antisymmetric two-particle state* $\phi_j \wedge \phi_k = (\phi_j \otimes \phi_k - \phi_k \otimes \phi_j)/\sqrt{2}$ *(the Slater determinant Eq.* (36)*) is non-zero. The structure of atomic energy levels, the periodic table, the stability of matter, and the Fermi-Dirac statistics of many-fermion systems all follow from the exclusion principle combined with the single-particle energy levels of the relevant Hamiltonian. In the QM-series, the most immediate application is the electronic structure of atoms (QM8: spin-orbit coupling and the aufbau principle) and the fermionic Fock space of Sec. 6.*

6 Fock Space and Second Quantization

The two-particle Hilbert space $\mathcal{H}^{(2)}$ describes systems with exactly two transport closure configurations. Physical systems in which the number of configurations is not fixed — systems that can emit or absorb configurations, or systems that are most naturally described in terms of occupation numbers rather than labeled particle positions — require a state space that accommodates *all* particle numbers simultaneously. The Fock space $\mathcal{F} = \bigoplus_{N=0}^{\infty} \mathcal{H}^{(N)}$ is this structure: a direct sum of zero-particle, one-particle, two-particle, and higher spaces, with a vacuum state spanning the zero-particle sector and creation and annihilation operators that move between sectors by adding or removing one configuration at a time. The present section constructs the Fock space, introduces the creation and annihilation operators and derives their commutation or anticommutation relations, and establishes the number operator and the structure of multi-mode Fock states.

6.1 The Fock Space Construction

Definition 6.1 (Fock space)**.** *The* Fock space *over $\mathcal{H}$ is the Hilbert space direct sum*

$$\mathcal{F} := \bigoplus_{N=0}^{\infty} \mathcal{H}^{(N)}, \tag{42}$$

where $\mathcal{H}^{(0)} := \mathbb{C}$ (the vacuum sector*) and $\mathcal{H}^{(N)} := \bigotimes_{j=1}^{N} \mathcal{H}$ for $N \geq 1$. An element of $\mathcal{F}$ is a sequence $\Psi = (\Psi^{(0)}, \Psi^{(1)}, \Psi^{(2)}, \ldots)$ with $\Psi^{(N)} \in \mathcal{H}^{(N)}$ for each N and $\sum_{N=0}^{\infty} \left\|\Psi^{(N)}\right\|^2_{\mathcal{H}^{(N)}} < \infty$. The inner product on $\mathcal{F}$ is*

$$\langle \Phi, \Psi \rangle_{\mathcal{F}} = \sum_{N=0}^{\infty} \left\langle \Phi^{(N)}, \Psi^{(N)} \right\rangle_{\mathcal{H}^{(N)}}. \tag{43}$$

The vacuum state *$|0\rangle$ is the element with $\Psi^{(0)} = 1 \in \mathbb{C}$ and $\Psi^{(N)} = 0$ for $N \geq 1$, normalized as $\langle 0, 0 \rangle_{\mathcal{F}} = 1$.*

Definition 6.2 (Bosonic and fermionic Fock spaces)**.** *The* bosonic Fock space *is*

$$\mathcal{F}_+ := \bigoplus_{N=0}^{\infty} \mathcal{H}^{(N)}_{\text{sym}}, \tag{44}$$

where $\mathcal{H}^{(0)}_{\text{sym}} = \mathbb{C}$, $\mathcal{H}^{(1)}_{\text{sym}} = \mathcal{H}$, and $\mathcal{H}^{(N)}_{\text{sym}}$ is the fully symmetrized N-particle subspace for $N \geq 2$. The fermionic Fock space *is*

$$\mathcal{F}_- := \bigoplus_{N=0}^{\infty} \mathcal{H}^{(N)}_{\text{anti}}, \tag{45}$$

where $\mathcal{H}^{(0)}_{\text{anti}} = \mathbb{C}$, $\mathcal{H}^{(1)}_{\text{anti}} = \mathcal{H}$, and $\mathcal{H}^{(N)}_{\text{anti}}$ is the fully antisymmetrized N-particle subspace for $N \geq 2$. By Theorem 5.7, physical bosonic systems have states in $\mathcal{F}_+$ and physical fermionic systems have states in $\mathcal{F}_-$.

Remark 6.3. *The full Fock space $\mathcal{F}$ carries a natural* particle number operator

$$\hat{N}_{\text{tot}} \Psi := (0 \cdot \Psi^{(0)},\, 1 \cdot \Psi^{(1)},\, 2 \cdot \Psi^{(2)},\, \ldots), \tag{46}$$

which multiplies the N-particle component by N. The eigenspaces of $\hat{N}_{\text{tot}}$ are exactly the individual particle-number sectors $\mathcal{H}^{(N)}$. A state in a definite N-particle sector is an eigenstate of $\hat{N}_{\text{tot}}$ with eigenvalue N. A superposition of states from different sectors has indefinite particle number; the expectation value $\langle \hat{N}_{\text{tot}} \rangle$ gives the mean particle number. The Fock space accommodates both definite-number and indefinite-number states within a single Hilbert space.

6.2 Creation and Annihilation Operators on Fock Space

For a fixed single-particle mode $\phi \in \mathcal{H}$, the creation and annihilation operators add and remove one configuration in state ϕ.

Definition 6.4 (Creation and annihilation operators on Fock space)**.** *Let $\phi \in \mathcal{H}$ be a normalized single-particle state. The* creation operator $\hat{a}^\dagger(\phi)$ *on $\mathcal{F}_+$ (bosons) or $\mathcal{F}_-$ (fermions) is defined on the N-particle sector by*

$$\hat{a}^\dagger(\phi)\,\Psi^{(N)} := \sqrt{N+1}\;\hat{S}_\pm\big(\phi \otimes \Psi^{(N)}\big), \tag{47}$$

where $\hat{S}_+$ is used for bosons and $\hat{S}_-$ for fermions, and the $\sqrt{N+1}$ normalization ensures that the creation operator on the Fock space is the adjoint of the annihilation operator. The annihilation operator $\hat{a}(\phi) = [\hat{a}^\dagger(\phi)]^\dagger$ *removes one configuration in state ϕ from the N-particle sector, mapping to the $(N-1)$-particle sector. In terms of an ONB $\{\phi_j\}$ of $\mathcal{H}$, the mode operators are $\hat{a}_j := \hat{a}(\phi_j)$ and $\hat{a}_j^\dagger := \hat{a}^\dagger(\phi_j)$.*

Theorem 6.5 (Bosonic CCR and fermionic CAR on Fock space)**.** *For an ONB $\{\phi_j\}_{j\geq 1}$ of $\mathcal{H}$, the mode creation and annihilation operators satisfy:*

(i) Bosonic canonical commutation relations (CCR):

$$[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}\,\hat{\mathbf{1}}_{\mathcal{F}_+}, \tag{48}$$

$$[\hat{a}_j, \hat{a}_k] = [\hat{a}_j^\dagger, \hat{a}_k^\dagger] = 0. \tag{49}$$

(ii) Fermionic canonical anticommutation relations (CAR):

$$\{\hat{a}_j, \hat{a}_k^\dagger\} = \delta_{jk}\,\hat{\mathbf{1}}_{\mathcal{F}_-}, \tag{50}$$

$$\{\hat{a}_j, \hat{a}_k\} = \{\hat{a}_j^\dagger, \hat{a}_k^\dagger\} = 0, \tag{51}$$

where $\{A, B\} := AB + BA$ is the anticommutator.

In both cases, the vacuum is annihilated: $\hat{a}_j|0\rangle = 0$ for all j.

Proof. Bosonic CCR. The bosonic mode operators $\hat{a}_j$ act on the symmetrized N-particle sector $\mathcal{H}^{(N)}_{\mathrm{sym}}$ via Eq. (47) with $\hat{S}_+$. For the same mode $j = k$, the action on the vacuum gives $\hat{a}_j^\dagger|0\rangle = \phi_j$ (a one-particle state) and $\hat{a}_j\phi_j = |0\rangle$, so $[\hat{a}_j, \hat{a}_j^\dagger]|0\rangle = |0\rangle$. For the general Fock state $|n_1, n_2, \ldots\rangle$ with occupation numbers n_j particles in mode j: $\hat{a}_j^\dagger\hat{a}_j|\ldots, n_j, \ldots\rangle = n_j|\ldots, n_j, \ldots\rangle$ and $\hat{a}_j\hat{a}_j^\dagger|\ldots, n_j, \ldots\rangle = (n_j+1)|\ldots, n_j, \ldots\rangle$, giving $[\hat{a}_j, \hat{a}_j^\dagger]|\ldots, n_j, \ldots\rangle = |\ldots, n_j, \ldots\rangle$ for all occupation number states. Since occupation number states span $\mathcal{F}_+$, the CCR $[\hat{a}_j, \hat{a}_j^\dagger] = \hat{\mathbf{1}}$ holds. For $j \neq k$, modes act on different single-particle factors; by the independence of operators on different tensor factors (Proposition 3.8), $[\hat{a}_j, \hat{a}_k^\dagger] = 0$. The vanishing commutators Eq. (49) follow from symmetry of the bosonic product and the anti-Hermitian nature of $[\hat{a}_j, \hat{a}_k] = -[\hat{a}_k, \hat{a}_j]$, which combined with the bosonic CCR gives zero.

Fermionic CAR. For the same mode $j = k$ in the fermionic case, the antisymmetry of $\mathcal{H}^{(N)}_{\mathrm{anti}}$ and the Pauli exclusion principle (Corollary 5.10) give $\hat{a}_j^\dagger\hat{a}_j^\dagger = 0$ (two particles cannot be created in the same mode). The anticommutator $\{\hat{a}_j, \hat{a}_j^\dagger\} = \hat{a}_j\hat{a}_j^\dagger + \hat{a}_j^\dagger\hat{a}_j$ acts on a state $|\ldots, n_j, \ldots\rangle$ with $n_j \in \{0, 1\}$ (since $n_j \geq 2$ is forbidden by Pauli): for $n_j = 0$: $\hat{a}_j\hat{a}_j^\dagger|\ldots, 0, \ldots\rangle = |\ldots, 0, \ldots\rangle$ and $\hat{a}_j^\dagger\hat{a}_j|\ldots, 0, \ldots\rangle = 0$, sum $= |\ldots, 0, \ldots\rangle$; for $n_j = 1$: $\hat{a}_j\hat{a}_j^\dagger|\ldots, 1, \ldots\rangle = 0$ and $\hat{a}_j^\dagger\hat{a}_j|\ldots, 1, \ldots\rangle = |\ldots, 1, \ldots\rangle$, sum $= |\ldots, 1, \ldots\rangle$. In both cases $\{\hat{a}_j, \hat{a}_j^\dagger\}|\ldots, n_j, \ldots\rangle = |\ldots, n_j, \ldots\rangle$, confirming

Eq. (50) for $j = k$. For $j \neq k$: the antisymmetry of $\mathcal{H}^{(N)}_{\text{anti}}$ under mode exchange combined with the definition of $\hat{S}_-$ gives $\{\hat{a}_j, \hat{a}_k^\dagger\} = 0$ (modes in different positions anticommute under the antisymmetrization). The anticommutators $\{\hat{a}_j, \hat{a}_k\} = 0$ follow from $(\hat{a}_j)^2 = 0$ (for $j = k$, by Pauli) and the antisymmetry (for $j \neq k$). □

Remark 6.6. *The single-mode bosonic CCR $[\hat{a}_j, \hat{a}_j^\dagger] = \hat{\mathbf{1}}$ of Theorem 6.5 is precisely the single-mode commutation relation $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$ of QM6 Lemma 3.1, with the mode index j specifying which single-particle state is being occupied. The QM6 harmonic oscillator is therefore the single-mode bosonic Fock space with mode frequency $\omega_j = \omega$: the Fock states $|n\rangle$ of QM6 are the occupation number states $|n_j = n\rangle$ of the j-th mode, and the QM6 Hamiltonian $\hat{H}_{\text{osc}} = \Phi_0\omega(\hat{N} + \frac{1}{2}\hat{\mathbf{1}})$ is the single-mode restriction of the many-mode Hamiltonian $\hat{H} = \sum_j \Phi_0\omega_j(\hat{n}_j + \frac{1}{2}\hat{\mathbf{1}})$. The Fock space of the present section is therefore the natural multi-mode generalization of the QM6 single-mode oscillator, not a new structure.*

6.3 Occupation Number States and the Fock Basis

The complete orthonormal basis for $\mathcal{F}_+$ and $\mathcal{F}_-$ is provided by the occupation number states.

Definition 6.7 (Occupation number states)**.** *For an ONB $\{\phi_j\}_{j\geq 1}$ of $\mathcal{H}$, the* bosonic occupation number state *with n_j particles in mode j for each j is*

$$|n_1, n_2, n_3, \ldots\rangle := \prod_{j=1}^{\infty} \frac{(\hat{a}_j^\dagger)^{n_j}}{\sqrt{n_j!}} \, |0\rangle, \qquad n_j \in \{0, 1, 2, \ldots\}, \tag{52}$$

where the product is understood to act in order and only finitely many n_j are non-zero. The fermionic occupation number state *is*

$$|n_1, n_2, n_3, \ldots\rangle := \prod_{j=1}^{\infty} (\hat{a}_j^\dagger)^{n_j} \, |0\rangle, \qquad n_j \in \{0, 1\}, \tag{53}$$

where the Pauli exclusion principle (Corollary 5.10) restricts each occupation number to $n_j \in \{0, 1\}$ in the fermionic case.

Proposition 6.8 (Fock basis and ladder actions)**.** *The occupation number states form a complete orthonormal basis for $\mathcal{F}_+$ and $\mathcal{F}_-$ respectively. The ladder operators act on them as:*

(i) Bosonic:

$$\hat{a}_j^\dagger |\ldots, n_j, \ldots\rangle = \sqrt{n_j + 1} \, |\ldots, n_j + 1, \ldots\rangle, \tag{54}$$

$$\hat{a}_j |\ldots, n_j, \ldots\rangle = \sqrt{n_j} \, |\ldots, n_j - 1, \ldots\rangle. \tag{55}$$

(ii) Fermionic:

$$\hat{a}_j^\dagger |\ldots, n_j, \ldots\rangle = \begin{cases} (-1)^{\sum_{k<j} n_k} \, |\ldots, 1_j, \ldots\rangle & \textit{if } n_j = 0, \\ 0 & \textit{if } n_j = 1, \end{cases} \tag{56}$$

$$\hat{a}_j |\ldots, n_j, \ldots\rangle = \begin{cases} (-1)^{\sum_{k<j} n_k} \, |\ldots, 0_j, \ldots\rangle & \textit{if } n_j = 1, \\ 0 & \textit{if } n_j = 0. \end{cases} \tag{57}$$

The mode number operators $\hat{n}_j := \hat{a}_j^\dagger \hat{a}_j$ satisfy $\hat{n}_j |\ldots, n_j, \ldots\rangle = n_j |\ldots, n_j, \ldots\rangle$ in both cases.

Proof. Bosonic: The matrix elements Eqs. (54) and (55) are the multi-mode generalization of QM6 Proposition 4.3, derived by the same normalization argument: $\left\|\hat{a}_j^\dagger|\ldots,n_j,\ldots\rangle\right\|^2 = \left\langle\ldots,n_j,\ldots,\hat{a}_j\hat{a}_j^\dagger|\ldots,n_j,\ldots\right\rangle = \left\langle\ldots,n_j,\ldots,(\hat{n}_j+\hat{\mathbf{1}})|\ldots,n_j,\ldots\right\rangle = n_j+1$. Different modes $k \neq j$ are unaffected by $\hat{a}_j^\dagger$.

Fermionic: The sign factor $(-1)^{\sum_{k<j} n_k}$ in Eqs. (56) and (57) arises from the antisymmetry of the fermionic Fock space: to add a particle in mode j, it must be moved past all the particles in modes $k < j$ that are already occupied, each transposition generating a sign -1. The factor $(-1)^{\sum_{k<j} n_k}$ counts the total sign from commuting the new particle through all lower-mode occupants. The mode number operator result follows directly from the definition $\hat{n}_j = \hat{a}_j^\dagger\hat{a}_j$ and the ladder actions. Orthonormality and completeness of the occupation number states follow from the completeness of $\{\phi_j\}$ in $\mathcal{H}$ and the construction of $\mathcal{F}_+$ and $\mathcal{F}_-$ as the completions of the respective algebraic structures. □

Remark 6.9. *The sign factor $(-1)^{\sum_{k<j} n_k}$ in the fermionic ladder actions Eqs.* (56) *and* (57) *is known as the Jordan-Wigner factor. It is the precise expression of the antisymmetry of the fermionic Fock space at the level of the creation and annihilation operators: it ensures that the operators satisfy the CAR $\{\hat{a}_j, \hat{a}_k^\dagger\} = \delta_{jk}$ with the correct signs. The Jordan-Wigner factor depends on the ordering of the modes and is therefore conventional; different mode orderings give equivalent descriptions. For the physical applications in QM8-QM11, the mode ordering is fixed by the energy eigenvalue ordering, and the Jordan-Wigner factors are computed with respect to that ordering.*

6.4 The Connection Between Fock Space and the N-Particle Theory

Proposition 6.10 (Fock space as extension of fixed-N theory)**.** *The N-particle sector $\mathcal{H}_{\text{sym}}^{(N)}$ (bosons) or $\mathcal{H}_{\text{anti}}^{(N)}$ (fermions) sits inside the Fock space as the eigenspace of $\hat{N}_{\text{tot}}$ with eigenvalue N. The bosonic occupation number state $|n_1, n_2, \ldots\rangle$ with $\sum_j n_j = N$ corresponds to the symmetrized N-particle state in $\mathcal{H}_{\text{sym}}^{(N)}$ with n_j particles in each mode ϕ_j. Explicitly, for $N = 2$ with modes $j \neq k$:*

$$\hat{a}_j^\dagger\hat{a}_k^\dagger|0\rangle = \sqrt{2}\,(\phi_j \vee \phi_k), \tag{58}$$

$$\hat{a}_j^\dagger\hat{a}_k^\dagger|0\rangle = \sqrt{2}\,(\phi_j \wedge \phi_k) \quad \textit{(fermions)}, \tag{59}$$

where $\phi_j \vee \phi_k$ and $\phi_j \wedge \phi_k$ are the normalized two-particle states of Eqs. (34)–(35)*.*

Proof. For bosons, $\hat{a}_j^\dagger\hat{a}_k^\dagger|0\rangle = \hat{a}_j^\dagger\phi_k = \sqrt{2}\,\hat{S}_+(\phi_j\otimes\phi_k) = \sqrt{2}\,(\phi_j\otimes\phi_k+\phi_k\otimes\phi_j)/\sqrt{2}\cdot\sqrt{2} = \sqrt{2}\,\phi_j\vee\phi_k$, where the $\sqrt{2}$ factor comes from the creation operator normalization in Eq. (47). The fermionic case is identical with $\hat{S}_+$ replaced by $\hat{S}_-$ and a sign convention fixed by the mode ordering. □

Remark 6.11. *Proposition 6.10 confirms that the Fock space formalism is not a replacement for the fixed-N tensor product theory but its extension. For systems with a fixed, conserved particle number N, the physics lies entirely within the N-particle sector $\mathcal{H}^{(N)}$, and the Fock space formalism reduces to the fixed-N theory of Secs. 3–5. The Fock space becomes essential when the particle number is not fixed: for the coupled oscillator of Sec. 7 (where the normal modes have definite energy but can be in any Fock state $|n_+, n_-\rangle$ with varying total excitation number), for photon emission and absorption in QM10 (where the radiation field has variable photon number), and for the quantum field theoretic extensions of the RQM-series. The key new content of the Fock space formalism — the CCR and CAR of Theorem 6.5 — provides the algebraic framework for all of these applications.*

7 The Coupled Harmonic Oscillator

The coupled harmonic oscillator is the canonical two-body model of the QM-series: it is the simplest system in which the tensor product structure of Sec. 3, the Fock space algebra of Sec. 6, and the single-particle oscillator theory of QM6 all interact in a single derivation. The system consists of two particles of equal mass m, each bound to its own harmonic potential with frequency ω, and coupled by a bilinear interaction $\kappa\hat{x}_1\hat{x}_2$. The coupling is exactly of the form described in Remark 3.11: it is a genuine two-particle observable that breaks the product structure of the non-interacting Hamiltonian and generates entanglement in the ground state. The strategy for solving the coupled system is the same strategy used in QM5 for the hydrogen atom: a coordinate transformation that simplifies the potential. There, the transformation from laboratory coordinates to center-of-mass and relative coordinates reduced a two-body Coulomb problem to a one-body problem in the reduced mass. Here, the normal mode transformation — a linear canonical transformation — decouples the coupled oscillator into two independent oscillators, each governed by the QM6 theory with a modified frequency.

7.1 The Coupled Oscillator Hamiltonian

Definition 7.1 (Coupled harmonic oscillator Hamiltonian)**.** *The* coupled harmonic oscillator Hamiltonian *for two particles of mass m and natural frequency ω, with bilinear coupling constant κ, is*

$$\hat{H}_{\text{coup}} := \frac{\hat{p}_1^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_1^2 + \frac{\hat{p}_2^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_2^2 + \kappa\,\hat{x}_1\hat{x}_2, \tag{60}$$

acting on $\mathcal{H}^{(2)} = L^2(\mathbb{R}^2, \mathbb{C})$, where $\hat{x}_j$ and $\hat{p}_j$ are the position and momentum operators for particle j, extended to $\mathcal{H}^{(2)}$ via Definition 3.7. The coupling satisfies $|\kappa| < m\omega^2$, ensuring that the normal mode frequencies are real.

Remark 7.2. *The coupled oscillator potential $V(x_1, x_2) = \frac{1}{2}m\omega^2x_1^2 + \frac{1}{2}m\omega^2x_2^2 + \kappa x_1x_2$ is a quadratic form in (x_1, x_2), positive definite for $|\kappa| < m\omega^2$ (since the matrix $\begin{pmatrix} m\omega^2 & \kappa/2 \\ \kappa/2 & m\omega^2 \end{pmatrix}$ is positive definite under this condition). A positive definite quadratic potential is a Kato-class potential on $\mathbb{R}^2$ (QM4 Definition 3.2), so $\hat{H}_{\text{coup}}$ is self-adjoint on the Sobolev domain $H^2(\mathbb{R}^2)$ by QM4 Theorem 4.2. Stone's theorem (QM4 Theorem 3.1) gives the strongly continuous unitary time evolution $U(t) = \mathrm{e}^{-\mathrm{i}\hat{H}_{\text{coup}}t/\Phi_0}$, well-defined on $\mathcal{H}^{(2)}$ for all $t \in \mathbb{R}$.*

7.2 The Normal Mode Transformation

The decoupling is achieved by passing to normal mode coordinates, which diagonalize the potential energy matrix of the coupled system.

Definition 7.3 (Normal mode coordinates)**.** *The* normal mode coordinates *and their conjugate momenta are defined by the orthogonal transformation:*

$$Q_+ := \frac{x_1 + x_2}{\sqrt{2}}, \quad P_+ := \frac{p_1 + p_2}{\sqrt{2}}, \tag{61}$$

$$Q_- := \frac{x_1 - x_2}{\sqrt{2}}, \quad P_- := \frac{p_1 - p_2}{\sqrt{2}}, \tag{62}$$

with the inverse:

$$x_1 = \frac{Q_+ + Q_-}{\sqrt{2}}, \quad x_2 = \frac{Q_+ - Q_-}{\sqrt{2}}, \quad p_1 = \frac{P_+ + P_-}{\sqrt{2}}, \quad p_2 = \frac{P_+ - P_-}{\sqrt{2}}. \tag{63}$$

Lemma 7.4 (Normal mode CCR)**.** *The normal mode operators satisfy the canonical commutation relations:*

$$[Q_+, P_+] = \mathrm{i}\Phi_0\,\hat{\mathbf{1}}_{\mathcal{H}^{(2)}}, \quad [Q_-, P_-] = \mathrm{i}\Phi_0\,\hat{\mathbf{1}}_{\mathcal{H}^{(2)}}, \tag{64}$$

$$[Q_+, P_-] = [Q_-, P_+] = 0, \tag{65}$$

and all other commutators among $\{Q_+, Q_-, P_+, P_-\}$ *vanish.*

Proof. The normal mode operators are orthogonal linear combinations of the original position and momentum operators. Using Eq. (2) for the original CCR and Proposition 3.8 for the cross-particle commutators:

$$[Q_+, P_+] = \frac{1}{2}[x_1+x_2, p_1+p_2] = \frac{1}{2}\big([x_1,p_1]+[x_2,p_2]+[x_1,p_2]+[x_2,p_1]\big) = \frac{1}{2}(\mathrm{i}\Phi_0+\mathrm{i}\Phi_0+0+0) = \mathrm{i}\Phi_0.$$

Similarly for $[Q_-, P_-]$. The cross-commutators:

$$[Q_+, P_-] = \frac{1}{2}[x_1 + x_2, p_1 - p_2] = \frac{1}{2}(\mathrm{i}\Phi_0 - \mathrm{i}\Phi_0 + 0 - 0) = 0.$$

□

Remark 7.5. *Lemma 7.4 establishes that the normal mode transformation Eqs.* (61)–(62) *is a* canonical transformation*: it maps a canonical pair* (x_j, p_j) *to a new canonical pair* (Q_+, P_+) *and* (Q_-, P_-)*, preserving the commutation structure. The two normal mode pairs are independent in the algebraic sense: operators of the + mode commute with operators of the − mode, by Eq.* (65)*. This independence is the algebraic expression of the physical decoupling: after the transformation, the + mode and the − mode do not interact.*

7.3 Hamiltonian Decoupling and Normal Mode Frequencies

Proposition 7.6 (Decoupling of the coupled oscillator)**.** *In normal mode coordinates, the coupled oscillator Hamiltonian Eq.* (60) *separates as*

$$\hat{H}_{\text{coup}} = \hat{H}_+ + \hat{H}_-, \tag{66}$$

where

$$\hat{H}_+ := \frac{P_+^2}{2m} + \frac{1}{2}m\omega_+^2 Q_+^2, \tag{67}$$

$$\hat{H}_- := \frac{P_-^2}{2m} + \frac{1}{2}m\omega_-^2 Q_-^2, \tag{68}$$

with normal mode frequencies

$$\omega_+ := \sqrt{\omega^2 + \frac{\kappa}{m}}, \qquad \omega_- := \sqrt{\omega^2 - \frac{\kappa}{m}}. \tag{69}$$

Both frequencies are real for $|\kappa| < m\omega^2$*.*

Proof. Substitute Eq. (63) into Eq. (60).

Kinetic energy: $\frac{p_1^2+p_2^2}{2m} = \frac{1}{2m} \cdot \frac{(P_++P_-)^2+(P_+-P_-)^2}{2} = \frac{P_+^2+P_-^2}{2m}$.

Potential energy:

$$\frac{m\omega^2}{2}(x_1^2 + x_2^2) = \frac{m\omega^2}{2} \cdot \frac{(Q_+ + Q_-)^2 + (Q_+ - Q_-)^2}{2} = \frac{m\omega^2}{2}(Q_+^2 + Q_-^2).$$

For the coupling term, using Eq. (63):

$$\kappa\, x_1 x_2 = \kappa \cdot \frac{(Q_+ + Q_-)(Q_+ - Q_-)}{2} = \frac{\kappa}{2}(Q_+^2 - Q_-^2).$$

Combining the potential terms:

$$\begin{aligned}
&\frac{m\omega^2}{2}(Q_+^2 + Q_-^2) + \frac{\kappa}{2}(Q_+^2 - Q_-^2) \\
&= \frac{1}{2}\Big(m\omega^2 + \kappa\Big)Q_+^2 + \frac{1}{2}\Big(m\omega^2 - \kappa\Big)Q_-^2 \\
&= \frac{1}{2}m\omega_+^2 Q_+^2 + \frac{1}{2}m\omega_-^2 Q_-^2,
\end{aligned}$$

where $\omega_+^2 = \omega^2 + \kappa/m$ and $\omega_-^2 = \omega^2 - \kappa/m$. Assembling: $\hat{H}_{\text{coup}} = (P_+^2/2m + \frac{1}{2}m\omega_+^2 Q_+^2) + (P_-^2/2m + \frac{1}{2}m\omega_-^2 Q_-^2) = \hat{H}_+ + \hat{H}_-$, confirming Eq. (66). Positivity of ω_+^2 and ω_-^2 for $|\kappa| < m\omega^2$ is immediate from the definition. □

Remark 7.7. *The two terms $\hat{H}_+$ and $\hat{H}_-$ commute with each other — $[\hat{H}_+, \hat{H}_-] = 0$ by Lemma 7.4 — since $\hat{H}_+$ involves only Q_+ and P_+ while $\hat{H}_-$ involves only Q_- and P_-. The + mode describes in-phase oscillation of the two particles (x_1 and x_2 move together, Q_+ large, $Q_- \approx 0$) with frequency $\omega_+ > \omega$: the coupling stiffens the in-phase mode. The − mode describes out-of-phase oscillation (x_1 and x_2 move oppositely, Q_- large, $Q_+ \approx 0$) with frequency $\omega_- < \omega$: the coupling softens the out-of-phase mode. The limiting case $\kappa = 0$ recovers $\omega_+ = \omega_- = \omega$ (two decoupled oscillators with the same frequency), as expected. The case $\kappa = m\omega^2$ gives $\omega_- = 0$: the out-of-phase mode has zero frequency, corresponding to a flat potential in the relative coordinate and a free relative motion (no restoring force).*

7.4 Complete Spectrum and Eigenstates

Theorem 7.8 (Complete spectrum of the coupled oscillator)**.** *The coupled harmonic oscillator $\hat{H}_{\text{coup}}$ has the complete discrete spectrum*

$$E_{n_+,n_-} = \Big(n_+ + \tfrac{1}{2}\Big)\Phi_0\omega_+ + \Big(n_- + \tfrac{1}{2}\Big)\Phi_0\omega_-, \qquad n_\pm \in \{0, 1, 2, \ldots\}, \tag{70}$$

with corresponding eigenstates

$$\Psi_{n_+,n_-}(Q_+, Q_-) = \Psi_{n_+}^{(+)}(Q_+)\,\Psi_{n_-}^{(-)}(Q_-), \tag{71}$$

where $\Psi_{n_+}^{(+)}$ and $\Psi_{n_-}^{(-)}$ are the QM6 Hermite-Gaussian eigenstates of frequency ω_+ and ω_- respectively (QM6 Theorem 5.2). The ground state energy is

$$E_{0,0} = \frac{1}{2}\Phi_0\omega_+ + \frac{1}{2}\Phi_0\omega_- = \frac{\Phi_0}{2}(\omega_+ + \omega_-), \tag{72}$$

which reduces to $\Phi_0\omega$ when $\kappa = 0$ (two decoupled oscillators each contributing $\Phi_0\omega/2$).

Proof. By Proposition 7.6, $\hat{H}_{\text{coup}} = \hat{H}_+ + \hat{H}_-$ with $[\hat{H}_+, \hat{H}_-] = 0$. Each $\hat{H}_\pm$ is a single-mode harmonic oscillator Hamiltonian with frequency ω_+ or ω_-, acting on the normal mode coordinates (Q_+, P_+) and (Q_-, P_-) respectively, each satisfying the CCR Eq. (64). By QM6 Theorem 4.1 applied to each mode independently: $\sigma(\hat{H}_\pm) = \{(n_\pm + \frac{1}{2})\Phi_0\omega_+{}'\}$ where $\omega_+{}'$ is ω_+ for $\hat{H}_+$ and ω_- for $\hat{H}_-$. Since $\hat{H}_+$ and $\hat{H}_-$ act on independent factors (in the sense of Lemma 7.4 and Proposition 3.8), the eigenvalues of $\hat{H}_{\rm coup} = \hat{H}_+ + \hat{H}_-$ are all sums $E_{n_+} + E_{n_-}$, and the eigenstates are products $\Psi^{(+)}_{n_+} \otimes \Psi^{(-)}_{n_-}$ (Proposition 3.10 applied to $\hat{H}_+$ and $\hat{H}_-$). Completeness of the product eigenstates in $\mathcal{H}^{(2)}$ follows from the completeness of each set $\{\Psi^{(\pm)}_{n_\pm}\}$ in $L^2(\mathbb{R})$ (QM6 Proposition 5.3). □

Remark 7.9. *In terms of normal mode ladder operators defined via QM6 Definition 3.1 applied to each mode:*

$$\hat{b}_+ := \frac{m\omega_+ Q_+ + \mathrm{i}P_+}{\sqrt{2m\omega_+\Phi_0}}, \qquad \hat{b}_- := \frac{m\omega_- Q_- + \mathrm{i}P_-}{\sqrt{2m\omega_-\Phi_0}}, \tag{73}$$

with adjoints $\hat{b}_+^\dagger = (\hat{b}_+)^\dagger$ and $\hat{b}_-^\dagger = (\hat{b}_-)^\dagger$, the decoupled Hamiltonian takes the form

$$\hat{H}_{\rm coup} = \Phi_0\omega_+\left(\hat{b}_+^\dagger\hat{b}_+ + \tfrac{1}{2}\hat{\mathbf{1}}\right) + \Phi_0\omega_-\left(\hat{b}_-^\dagger\hat{b}_- + \tfrac{1}{2}\hat{\mathbf{1}}\right), \tag{74}$$

by QM6 Theorem 3.2 applied to each mode. The Fock state $|n_+, n_-\rangle = (\hat{b}_+^\dagger)^{n_+}(\hat{b}_-^\dagger)^{n_-}|0\rangle/\sqrt{n_+!\,n_-!}$ is the eigenstate with energy Eq. (70). The ladder operators satisfy $[\hat{b}_+, \hat{b}_+^\dagger] = \hat{\mathbf{1}}$, $[\hat{b}_-, \hat{b}_-^\dagger] = \hat{\mathbf{1}}$, and $[\hat{b}_+, \hat{b}_-] = [\hat{b}_+, \hat{b}_-^\dagger] = 0$.

7.5 Entanglement in the Ground State

The ground state $\Psi_{0,0}$ of the coupled oscillator is a Gaussian in the normal mode coordinates (Q_+, Q_-) and hence in the original coordinates (x_1, x_2). Whether it is entangled depends on whether it factorizes as a product in the original particle coordinates.

Proposition 7.10 (Ground state entanglement for $\kappa \neq 0$)**.** *The coupled oscillator ground state in original coordinates is:*

$$\Psi_{0,0}(x_1, x_2) = \mathcal{N}\exp\left(-\frac{m\omega_+}{2\Phi_0}\frac{(x_1+x_2)^2}{2} - \frac{m\omega_-}{2\Phi_0}\frac{(x_1-x_2)^2}{2}\right), \tag{75}$$

where $\mathcal{N} = (m^2\omega_+\omega_-/\pi^2\Phi_0^2)^{1/4}$ is the normalization constant. For $\kappa \neq 0$ (equivalently $\omega_+ \neq \omega_-$), this state does not factorize as $\Psi_1(x_1)\Psi_2(x_2)$ and is therefore entangled. *For $\kappa = 0$ (equivalently $\omega_+ = \omega_- = \omega$), the ground state factorizes as $\Psi_0^\omega(x_1)\Psi_0^\omega(x_2)$ and is a product state.*

Proof. The ground state in normal mode coordinates is the product Gaussian:

$$\begin{aligned}\Psi_{0,0}(Q_+, Q_-) &= \Psi_0^{(+)}(Q_+)\,\Psi_0^{(-)}(Q_-)\\ &= \frac{1}{(\pi\ell_{0+}^2)^{1/4}}\mathrm{e}^{-Q_+^2/(2\ell_{0+}^2)} \cdot \frac{1}{(\pi\ell_{0-}^2)^{1/4}}\mathrm{e}^{-Q_-^2/(2\ell_{0-}^2)},\end{aligned}$$

where $\ell_{0\pm} = \sqrt{\Phi_0/(m\omega_{+'})}$. Substituting $Q_+ = (x_1+x_2)/\sqrt{2}$ and $Q_- = (x_1-x_2)/\sqrt{2}$:

$$\begin{aligned}\frac{Q_+^2}{2\ell_{0+}^2} + \frac{Q_-^2}{2\ell_{0-}^2} &= \frac{(x_1+x_2)^2}{4\ell_{0+}^2} + \frac{(x_1-x_2)^2}{4\ell_{0-}^2}\\ &= \frac{m\omega_+}{4\Phi_0}(x_1+x_2)^2 + \frac{m\omega_-}{4\Phi_0}(x_1-x_2)^2,\end{aligned}$$

giving Eq. (75).

Entanglement for $\kappa \neq 0$: Expanding Eq. (75):

$$\begin{aligned}&\frac{m\omega_+}{4\Phi_0}(x_1+x_2)^2 + \frac{m\omega_-}{4\Phi_0}(x_1-x_2)^2\\ &= \frac{m(\omega_+ + \omega_-)}{4\Phi_0}(x_1^2 + x_2^2) + \frac{m(\omega_+ - \omega_-)}{2\Phi_0}x_1x_2.\end{aligned}$$

The cross term $(m(\omega_+ - \omega_-)/2\Phi_0)x_1x_2$ vanishes if and only if $\omega_+ = \omega_-$, i.e., $\kappa = 0$. When the cross term is present, the exponent does not separate as $f(x_1) + g(x_2)$, so $\Psi_{0,0}(x_1, x_2)$ does not factorize as $\Psi_1(x_1)\Psi_2(x_2)$: the ground state is entangled. For $\kappa = 0$: $\omega_+ = \omega_- = \omega$, the cross term vanishes, and $\Psi_{0,0}(x_1, x_2) = \Psi_0^\omega(x_1)\Psi_0^\omega(x_2)$: the ground state is a product. □

Remark 7.11. *Proposition 7.10 is the first concrete demonstration of entanglement in the QM-series. The coupled oscillator ground state Eq.* (75) *is a two-mode Gaussian state that is entangled for any non-zero coupling $\kappa \neq 0$, no matter how small. An infinitesimally small coupling generates a non-factorizable ground state: entanglement is not a threshold phenomenon but arises for arbitrarily weak interactions. The degree of entanglement, measured by the Schmidt number or the von Neumann entropy of the reduced density matrix (to be defined in QM9), grows monotonically with $|\kappa|$ from zero (product state at $\kappa = 0$) to maximal entanglement as $|\kappa| \to m\omega^2$ (where $\omega_- \to 0$ and the relative mode becomes delocalized). The full entanglement theory — Schmidt decomposition, entanglement entropy, separability criteria — is developed in QM9 using the tensor product framework of Sec. 3 as its foundation.*

Remark 7.12. *Proposition 7.6 can alternatively be formulated in terms of center-of-mass and relative coordinates:*

$$\hat{R} = \frac{x_1 + x_2}{2}, \quad P = p_1 + p_2, \quad \hat{r} = x_1 - x_2, \quad p = \frac{p_1 - p_2}{2}, \tag{76}$$

with total mass $M = 2m$ and reduced mass $\mu = m/2$. The Hamiltonian separates as $\hat{H}_{\text{coup}} = \hat{H}_{\text{CM}} + \hat{H}_{\text{rel}}$ where $\hat{H}_{\text{CM}} = P^2/(2M) + \frac{1}{2}M\omega^2\hat{R}^2$ is a harmonic oscillator of mass M and frequency ω (the coupling does not affect the center-of-mass), and $\hat{H}_{\text{rel}} = p^2/(2\mu) + \frac{1}{2}\mu(2\omega^2 + 2\kappa/m)\hat{r}^2/2$ is a harmonic oscillator of mass μ and frequency $\tilde{\omega} = \sqrt{\omega^2 + 2\kappa/m}$ (the coupling shifts the relative frequency). The normal mode coordinates $Q_+ = \sqrt{2}\hat{R}$ and $Q_- = \hat{r}/\sqrt{2}$ are related to the CM/relative coordinates by a scaling, confirming consistency between the two formulations. This CM/relative separation is the direct two-body analogue of the reduction from the two-body hydrogen atom to the one-body problem in the reduced mass established in QM4-QM5, and it is the prototype for the two-body scattering analysis of QM10.

8 Angular Momentum Addition: Preview

The angular momentum of a two-particle system is the sum of the angular momenta of its constituent particles. In the single-particle setting of QM5, the angular momentum operators $\hat{L}_j$ generate rotations of the single-particle configuration space $\mathbb{R}^3$, and the holonomy quantization of the azimuthal closure path selects integer values for the magnetic quantum number. In the two-particle setting of the present paper, the rotation of the full configuration space $\mathbb{R}^3 \times \mathbb{R}^3$ is generated by the *total* angular momentum, which is the sum of the single-particle generators acting on each factor of the tensor product. The present section establishes the algebra of the total angular momentum operators, identifies the total angular momentum squared and its third component as the natural quantum numbers for the two-particle system, and records the structure of the Clebsch-Gordan decomposition — the decomposition of the tensor product of two angular momentum multiplets into irreducible SO(3) representations. The full derivation of the Clebsch-Gordan coefficients requires a systematic treatment of the SO(3) representation theory that is beyond the scope of the present paper; the structure theorem is established here and the explicit coefficients are deferred.

8.1 The Total Angular Momentum Operators

Definition 8.1 (Total angular momentum of a two-particle system)**.** *For a two-particle system on* $\mathcal{H}^{(2)} = \mathcal{H} \otimes \mathcal{H}$*, the* total angular momentum operators *are*

$$\hat{L}^j_{\text{tot}} \coloneqq \hat{L}_j \otimes \hat{\mathbf{1}}_{\mathcal{H}} + \hat{\mathbf{1}}_{\mathcal{H}} \otimes \hat{L}_j = \hat{L}^{(1)}_j + \hat{L}^{(2)}_j, \quad j = 1, 2, 3, \tag{77}$$

where $\hat{L}^{(1)}_j = \hat{L}_j \otimes \hat{\mathbf{1}}$ *and* $\hat{L}^{(2)}_j = \hat{\mathbf{1}} \otimes \hat{L}_j$ *are the single-particle extensions of Definition 3.7. The* total angular momentum squared *is*

$$\hat{L}^2_{\text{tot}} \coloneqq \sum_{j=1}^{3} (\hat{L}^j_{\text{tot}})^2. \tag{78}$$

Theorem 8.2 (Total angular momentum algebra)**.** *The total angular momentum operators satisfy the* SO(3) *commutation algebra:*

$$[\hat{L}^j_{\text{tot}}, \hat{L}^k_{\text{tot}}] = \mathrm{i}\Phi_0 \, \epsilon_{jkl} \, \hat{L}^l_{\text{tot}}, \tag{79}$$

and consequently $[\hat{L}^2_{\text{tot}}, \hat{L}^j_{\text{tot}}] = 0$ *for all* j*.*

Proof. Expand the commutator using Eq. (77):

$$\begin{aligned}
[\hat{L}^j_{\text{tot}}, \hat{L}^k_{\text{tot}}] &= [\hat{L}^{(1)}_j + \hat{L}^{(2)}_j, \hat{L}^{(1)}_k + \hat{L}^{(2)}_k] \\
&= [\hat{L}^{(1)}_j, \hat{L}^{(1)}_k] + [\hat{L}^{(2)}_j, \hat{L}^{(2)}_k] + [\hat{L}^{(1)}_j, \hat{L}^{(2)}_k] + [\hat{L}^{(2)}_j, \hat{L}^{(1)}_k].
\end{aligned}$$

The cross-particle commutators vanish by Proposition 3.8: $[\hat{L}^{(1)}_j, \hat{L}^{(2)}_k] = [\hat{L}^{(2)}_j, \hat{L}^{(1)}_k] = 0$. The single-particle commutators follow from the QM5 algebra Eq. (4) extended to $\mathcal{H}^{(2)}$:

$$\begin{aligned}
[\hat{L}^{(1)}_j, \hat{L}^{(1)}_k] &= \mathrm{i}\Phi_0 \, \epsilon_{jkl} \, \hat{L}^{(1)}_l, \\
[\hat{L}^{(2)}_j, \hat{L}^{(2)}_k] &= \mathrm{i}\Phi_0 \, \epsilon_{jkl} \, \hat{L}^{(2)}_l.
\end{aligned}$$

Summing:

$$[\hat{L}^j_{\text{tot}}, \hat{L}^k_{\text{tot}}] = \mathrm{i}\Phi_0 \, \epsilon_{jkl} (\hat{L}^{(1)}_l + \hat{L}^{(2)}_l) = \mathrm{i}\Phi_0 \, \epsilon_{jkl} \, \hat{L}^l_{\text{tot}},$$

which is Eq. (79). The commutativity $[\hat{L}^2_{\text{tot}}, \hat{L}^j_{\text{tot}}] = 0$ follows from the same algebraic identity as in QM5 Theorem 3.2 applied to the algebra Eq. (79). □

Remark 8.3. *Theorem 8.2 shows that the total angular momentum satisfies the* same SO(3) *algebra as the single-particle angular momentum of QM5. This means all the QM5 spectral theory applies to* $\hat{L}_{\text{tot}}$*: the eigenvalues of* $\hat{L}^2_{\text{tot}}$ *are* $J(J+1)\Phi_0^2$ *and those of* $\hat{L}^3_{\text{tot}}$ *are* $M\Phi_0$*, for some range of* J *and* M *values. The new question is: what are the allowed values of* J *for a system with single-particle angular momenta* ℓ_1 *and* ℓ_2*? The answer is the Clebsch-Gordan decomposition of Proposition 8.8:* J *ranges from* $|\ell_1 - \ell_2|$ *to* $\ell_1 + \ell_2$ *in integer steps.*

8.2 Expansion of the Total Angular Momentum Squared

The structure of $\hat{L}^2_{\text{tot}}$ reveals the two-particle nature of the problem: it contains a genuine two-particle interaction term that is not a tensor product of single-particle operators.

Proposition 8.4 (Expansion of $\hat{L}^2_{\text{tot}}$)**.** *The total angular momentum squared on $\mathcal{H}^{(2)}$ expands as:*

$$\hat{L}^2_{\text{tot}} = \hat{L}^{2(1)} + \hat{L}^{2(2)} + 2\sum_{j=1}^{3} \hat{L}_j^{(1)}\hat{L}_j^{(2)}, \tag{80}$$

where $\hat{L}^{2(1)} = \hat{L}^2 \otimes \hat{\mathbf{1}}$ and $\hat{L}^{2(2)} = \hat{\mathbf{1}} \otimes \hat{L}^2$ are the single-particle angular momentum squared operators, and the spin-orbit coupling term $2\sum_j \hat{L}_j^{(1)}\hat{L}_j^{(2)} = 2\hat{\mathbf{L}}_1 \cdot \hat{\mathbf{L}}_2$ *is a genuine two-particle operator.*

Proof. Expand $\hat{L}^2_{\text{tot}} = \sum_j (\hat{L}^j_{\text{tot}})^2$ using Eq. (77):

$$(\hat{L}^j_{\text{tot}})^2 = (\hat{L}_j^{(1)} + \hat{L}_j^{(2)})^2 = (\hat{L}_j^{(1)})^2 + \hat{L}_j^{(1)}\hat{L}_j^{(2)} + \hat{L}_j^{(2)}\hat{L}_j^{(1)} + (\hat{L}_j^{(2)})^2.$$

Using $[\hat{L}_j^{(1)}, \hat{L}_j^{(2)}] = 0$ (Proposition 3.8), the two cross-terms give $2\hat{L}_j^{(1)}\hat{L}_j^{(2)}$. Summing over j:

$$\hat{L}^2_{\text{tot}} = \sum_j (\hat{L}_j^{(1)})^2 + \sum_j (\hat{L}_j^{(2)})^2 + 2\sum_j \hat{L}_j^{(1)}\hat{L}_j^{(2)} = \hat{L}^{2(1)} + \hat{L}^{2(2)} + 2\hat{\mathbf{L}}_1 \cdot \hat{\mathbf{L}}_2. \qquad \square$$

Remark 8.5. *The operator $\hat{\mathbf{L}}_1 \cdot \hat{\mathbf{L}}_2 = \sum_j \hat{L}_j^{(1)}\hat{L}_j^{(2)}$ in Eq.* (80) *is a sum of products of single-particle angular momentum components, one from each particle. It is* not *of the form $A \otimes B$ for any single pair of operators, since it involves a sum of three such terms. This operator is therefore a genuinely irreducible two-particle operator: it cannot be expressed as a finite sum of simple tensor products $A \otimes B$. Its eigenvalues in the coupled basis $|\, J, M\,\rangle$ are:*

$$\hat{\mathbf{L}}_1 \cdot \hat{\mathbf{L}}_2 \,|\, J, M\,\rangle = \frac{\Phi_0^2}{2}\big[J(J+1) - \ell_1(\ell_1+1) - \ell_2(\ell_2+1)\big]|\, J, M\,\rangle, \tag{81}$$

obtained from Eq. (80) *by solving for the cross term: $2\hat{\mathbf{L}}_1 \cdot \hat{\mathbf{L}}_2 = \hat{L}^2_{\text{tot}} - \hat{L}^{2(1)} - \hat{L}^{2(2)}$. The eigenvalues $J(J+1)\Phi_0^2$ of $\hat{L}^2_{\text{tot}}$ determined by the Clebsch-Gordan decomposition then give the eigenvalues Eq.* (81) *directly. This is the key simplification of the coupled basis: in the uncoupled product basis $|\, \ell_1, m_1\,\rangle \otimes |\, \ell_2, m_2\,\rangle$, $\hat{\mathbf{L}}_1 \cdot \hat{\mathbf{L}}_2$ is not diagonal; in the coupled basis $|\, J, M\,\rangle$ it is.*

8.3 The Coupled and Uncoupled Bases

For the angular momentum addition problem, two natural orthonormal bases exist on the tensor product space $\mathcal{H}_{\ell_1} \otimes \mathcal{H}_{\ell_2}$.

Definition 8.6 (Uncoupled and coupled bases)**.** *The* uncoupled basis *is $\{|\, \ell_1, m_1\,\rangle \otimes |\, \ell_2, m_2\,\rangle\}$ for $m_1 \in \{-\ell_1, \ldots, +\ell_1\}$ and $m_2 \in \{-\ell_2, \ldots, +\ell_2\}$, consisting of* $(2\ell_1+1)(2\ell_2+1)$ *simultaneous eigenstates of $\hat{L}^{2(1)}$, $\hat{L}_3^{(1)}$, $\hat{L}^{2(2)}$, and $\hat{L}_3^{(2)}$.*

The coupled basis *is $\{|\, J, M\,\rangle\}$ for $J \in \{|\ell_1 - \ell_2|, \ldots, \ell_1 + \ell_2\}$ and $M \in \{-J, \ldots, +J\}$, consisting of simultaneous eigenstates of $\hat{L}^2_{\text{tot}}$, $\hat{L}^3_{\text{tot}}$, $\hat{L}^{2(1)}$, and $\hat{L}^{2(2)}$:*

$$\hat{L}^2_{\text{tot}}|\, J, M\,\rangle = J(J+1)\Phi_0^2|\, J, M\,\rangle, \tag{82}$$

$$\hat{L}^3_{\text{tot}}|\, J, M\,\rangle = M\Phi_0|\, J, M\,\rangle, \tag{83}$$

$$\hat{L}^{2(1)}|\, J, M\,\rangle = \ell_1(\ell_1+1)\Phi_0^2|\, J, M\,\rangle, \tag{84}$$

$$\hat{L}^{2(2)}|\, J, M\,\rangle = \ell_2(\ell_2+1)\Phi_0^2|\, J, M\,\rangle. \tag{85}$$

Both bases are complete orthonormal bases for the same $(2\ell_1+1)(2\ell_2+1)$*-dimensional space.*

Remark 8.7. *The total magnetic quantum number $M = m_1 + m_2$ is conserved: $\hat{L}^3_{\text{tot}}(|\,\ell_1,\, m_1\,\rangle \otimes |\,\ell_2,\, m_2\,\rangle) = (m_1+m_2)\Phi_0(|\,\ell_1,\, m_1\,\rangle \otimes |\,\ell_2,\, m_2\,\rangle)$. The constraint $M = m_1 + m_2$ holds in both bases and is the key organizing principle of the Clebsch-Gordan decomposition: for fixed M, only states with $m_1 + m_2 = M$ contribute to the coupled state $|\,J,\, M\,\rangle$. The maximum value of M is $m_1^{\max} + m_2^{\max} = \ell_1 + \ell_2$, which corresponds to $J_{\max} = \ell_1 + \ell_2$: the state with $M = J = \ell_1 + \ell_2$ is unique and equals $|\,\ell_1,\, \ell_1\,\rangle \otimes |\,\ell_2,\, \ell_2\,\rangle$. The $J_{\min} = |\ell_1 - \ell_2|$ lower bound is established by the dimension count of Proposition 8.8.*

8.4 The Clebsch-Gordan Decomposition

Proposition 8.8 (Clebsch-Gordan decomposition: structure)**.** *The tensor product of the angular momentum ℓ_1-multiplet $\mathcal{H}_{\ell_1}$ (dimension $2\ell_1+1$) and the ℓ_2-multiplet $\mathcal{H}_{\ell_2}$ (dimension $2\ell_2+1$) decomposes as a direct sum of irreducible* SO(3) *representations:*

$$\mathcal{H}_{\ell_1} \otimes \mathcal{H}_{\ell_2} \cong \bigoplus_{J=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \mathcal{H}_J, \tag{86}$$

where J ranges in integer steps from $|\ell_1 - \ell_2|$ to $\ell_1 + \ell_2$, each multiplet $\mathcal{H}_J$ has dimension $2J+1$, and the total dimension is preserved:

$$\sum_{J=|\ell_1-\ell_2|}^{\ell_1+\ell_2} (2J+1) = (2\ell_1+1)(2\ell_2+1). \tag{87}$$

The isomorphism is implemented by the Clebsch-Gordan coefficients $\langle \ell_1, m_1; \ell_2, m_2 | J, M\rangle$, defined by

$$|\,J,\, M\,\rangle = \sum_{\substack{m_1=-\ell_1 \\ m_2=M-m_1}}^{\ell_1} \langle \ell_1, m_1; \ell_2, M-m_1 | J, M\rangle \; |\,\ell_1,\, m_1\,\rangle \otimes |\,\ell_2,\, M-m_1\,\rangle, \tag{88}$$

with the constraint $m_2 = M - m_1$ expressing conservation of $M = m_1 + m_2$.

Proof. Dimension count Eq. (87)*:* The sum $\sum_{J=|\ell_1-\ell_2|}^{\ell_1+\ell_2}(2J+1)$ contains $\ell_1+\ell_2-|\ell_1-\ell_2|+1 = 2\min(\ell_1,\ell_2)+1$ terms. Taking $\ell_1 \geq \ell_2$ without loss of generality, the sum becomes $\sum_{J=\ell_1-\ell_2}^{\ell_1+\ell_2}(2J+1)$, which evaluates by the arithmetic series formula:

$$\begin{aligned}\sum_{J=\ell_1-\ell_2}^{\ell_1+\ell_2}(2J+1) &= \sum_{J=0}^{\ell_1+\ell_2}(2J+1) - \sum_{J=0}^{\ell_1-\ell_2-1}(2J+1) \\ &= (\ell_1+\ell_2+1)^2 - (\ell_1-\ell_2)^2 = (2\ell_1+1)(2\ell_2+1),\end{aligned}$$

using $\sum_{J=0}^{K}(2J+1) = (K+1)^2$ and $(A+1)^2 - B^2 = (A+1+B)(A+1-B) = (2\ell_1+1)(2\ell_2+1)$ with $A = \ell_1+\ell_2$ and $B = \ell_1 - \ell_2$.

Range of J and the triangle rule: The maximum M eigenvalue of $\hat{L}^3_{\text{tot}}$ in the product space is $(\ell_1+\ell_2)\Phi_0$, achieved uniquely by $|\,\ell_1,\, \ell_1\,\rangle \otimes |\,\ell_2,\, \ell_2\,\rangle$. By Theorem 8.2 the total angular momentum algebra is the SO(3) algebra, so $J_{\max} = \ell_1+\ell_2$ and the multiplet $\mathcal{H}_{J_{\max}}$ occupies $2(\ell_1+\ell_2)+1$ dimensions. The remaining states in the product basis are distributed among smaller multiplets $J < J_{\max}$, with the minimum J value determined by the total dimension constraint Eq. (87): after exhausting all multiplets from $J_{\max}$ down to $J_{\min} = |\ell_1-\ell_2|$, the dimension count is exactly satisfied. The explicit construction of the Clebsch-Gordan coefficients $\langle \ell_1, m_1; \ell_2, m_2 | J, M\rangle$ via the ladder operator technique (applying $\hat{L}^-_{\text{tot}} = \hat{L}^{(1)}_- + \hat{L}^{(2)}_-$ to the highest-weight state and using orthogonality) is deferred to the full treatment. □

Remark 8.9. *For the lowest non-trivial cases, the Clebsch-Gordan decomposition takes explicit forms. For $\ell_1 = \ell_2 = \frac{1}{2}$ (relevant for spin in QM8): $\mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2} \cong \mathcal{H}_1 \oplus \mathcal{H}_0$, a triplet ($J = 1$, three states) and a singlet ($J = 0$, one state), total $4 = 2 \cdot 2$ dimensions. For $\ell_1 = 1$, $\ell_2 = \frac{1}{2}$ (the spin-orbit coupling case): $\mathcal{H}_1 \otimes \mathcal{H}_{1/2} \cong \mathcal{H}_{3/2} \oplus \mathcal{H}_{1/2}$, a quartet and a doublet, total $6 = 3 \cdot 2$ dimensions. For $\ell_1 = \ell_2 = 1$: $\mathcal{H}_1 \otimes \mathcal{H}_1 \cong \mathcal{H}_2 \oplus \mathcal{H}_1 \oplus \mathcal{H}_0$, a quintuplet, triplet, and singlet, total $9 = 3 \cdot 3$ dimensions. The first case ($\frac{1}{2} \otimes \frac{1}{2}$) is the primary input to QM8, where the spin degrees of freedom of two particles are combined.*

Remark 8.10. *The present paper establishes the structure of the Clebsch-Gordan decomposition (Proposition 8.8): the range of J, the dimension count, the organization into the coupled basis, and the form of the expansion Eq.* (88). *The explicit values of the Clebsch-Gordan coefficients $\langle \ell_1, m_1; \ell_2, m_2 | J, M \rangle$ — computed via the ladder operator method, tabulated for low quantum numbers, and expressed in closed form via the Racah formula — are deferred to the full treatment in QM8, where the spin-$\frac{1}{2}$ case is the primary application. The deferral is appropriate to the scope of the present paper: QM7 opens the multi-particle sector and introduces the tensor product framework; the angular momentum addition problem is a structural consequence of applying that framework to the* SO(3) *angular momentum sector already established in QM5, and its full development belongs in the context where it is first concretely needed — the spin theory of QM8.*

9 Interpretive Clarifications and Scope

The present section collects the interpretive constraints governing the multi-particle analysis of the preceding sections and records the precise boundary between what the present paper establishes and what is deferred to subsequent papers. Three items are addressed: the status of the tensor product as a derived rather than postulated structure, the scope of the exchange symmetry derivation relative to the spin-statistics theorem, and the complete inventory of what QM7 establishes and does not establish.

9.1 The Tensor Product Without Additional Postulates

In the standard formulation of multi-particle quantum mechanics, the statement that the state space of a composite system is the tensor product of the state spaces of its parts is a separate postulate — the *composition postulate* — added to the single-particle axioms. The NUVO framework does not introduce this postulate. Instead, the tensor product $\mathcal{H}^{(2)} = \mathcal{H} \otimes \mathcal{H}$ is the structure that emerges from the requirement that two independent transport closure configurations be described by a single Hilbert space that (a) contains the single-particle states of each configuration as subsystem states, (b) assigns statistically independent joint distributions to uncorrelated configurations, and (c) is consistent with the inner product and completeness of QM1. The algebraic content of these three requirements is precisely the universal property of the tensor product: the tensor product is the unique (up to isomorphism) Hilbert space satisfying all three simultaneously.

The practical consequence of this derivation is the commutation relation $[A^{(1)}, B^{(2)}] = 0$ of Proposition 3.8, which holds not because it is postulated but because operators on independent tensor factors cannot fail to commute. The physical independence of the two transport closure configurations is expressed algebraically as the independence of their operators, and the tensor product is the unique algebraic structure that enforces this independence while respecting linearity and the inner product.

Entangled states — elements of $\mathcal{H}^{(2)}$ that do not factorize as $\Psi_1 \otimes \Psi_2$ — are therefore not a separate postulate either: they are the states in $\mathcal{H}^{(2)}$ that are not simple tensors, and they exist because the Hilbert space $\mathcal{H}^{(2)}$ is the completion of the algebraic tensor product, which is larger than the set of simple tensors. The existence of entanglement is a mathematical consequence of the tensor product structure, not an additional physical assumption.

9.2 Exchange Symmetry Without the Spin-Statistics Theorem

Theorem 5.7 establishes the following chain of results:

(i) For indistinguishable transport closure configurations, the exchange operator $\hat{P}_{12}$ commutes with all physical observables (Proposition 5.3).

(ii) The eigenspaces $\mathcal{H}_{\text{sym}}$ and $\mathcal{H}_{\text{anti}}$ are superselection sectors (Remark 5.4).

(iii) Physical states of identical configurations must lie in one or the other sector.

(iv) The holonomy of the exchange path in $(\mathbb{R}^3 \times \mathbb{R}^3)/\text{Sym}_2$ is quantized to $\pi \in \{+1, -1\}$ by the constraint $\pi^2 = 1$.

(v) The Pauli exclusion principle follows from the antisymmetry of fermionic states.

This chain is complete and self-contained within QM7.

What QM7 does *not* establish is the connection between the exchange parity π and the particle's spin. The spin-statistics theorem — that particles with integer spin have $\pi = +1$ (bosons) and particles with half-integer spin have $\pi = -1$ (fermions) — is a result of relativistic quantum field theory. Its proof requires the analyticity of the n-point functions in the relativistic framework, the TCP symmetry of local field theories, and the structure of the relativistic transport closure system that is developed in the RQM-series of the NUVO program. QM7 correctly derives the dichotomy $\pi \in \{+1, -1\}$ and all its consequences (Pauli exclusion, bosonic and fermionic statistics, Fock space CCR and CAR); it does not claim to connect π to spin without the relativistic framework. This interpretive boundary is recorded explicitly because the spin-statistics theorem is sometimes stated in non-relativistic quantum mechanics as if it can be derived without relativistic input; the NUVO program takes the position that the relativistic origin of the theorem is essential and defers its derivation accordingly.

9.3 Scope of the Present Construction

The present paper establishes the following results, which are available as inputs to subsequent QM-series papers.

Tensor product Hilbert space structure: The two-particle Hilbert space $\mathcal{H}^{(2)} = \mathcal{H} \otimes \mathcal{H}$ with inner product (Definition 3.3), the product ONB $\{\phi_j \otimes \phi_k\}$ and isomorphism $\mathcal{H}^{(2)} \cong L^2(\mathbb{R}^6)$ (Proposition 3.5), the observable algebra $[A^{(1)}, B^{(2)}] = 0$ (Proposition 3.8), the non-interacting two-particle spectrum as sums of single-particle eigenvalues (Proposition 3.10), and the N-particle generalization $\mathcal{H}^{(N)} \cong L^2(\mathbb{R}^{3N})$ (Definition 3.12).

Exchange operator and symmetry sectors: The exchange operator $\hat{P}_{12}$ with properties self-adjoint, unitary, and $\hat{P}_{12}^2 = \hat{\mathbf{1}}$ (Proposition 4.2), the eigenspace decomposition $\mathcal{H}^{(2)} = \mathcal{H}_{\text{sym}} \oplus \mathcal{H}_{\text{anti}}$ (Proposition 4.4), the symmetrization and antisymmetrization projectors $\hat{S}_+$ and $\hat{S}_-$ with the Slater determinant as the two-particle fermionic state (Proposition 4.6 and Remark 4.7).

Exchange symmetry from holonomy: Indistinguishability implies $[\hat{O}, \hat{P}_{12}] = 0$ for all physical $\hat{O}$ (Proposition 5.3); $\mathcal{H}_{\text{sym}}$ and $\mathcal{H}_{\text{anti}}$ are superselection sectors (Remark 5.4); the exchange parity $\pi \in \{+1, -1\}$ is derived from the holonomy of the exchange path (Theorem 5.7); the Pauli exclusion principle (Corollary 5.10); anyons excluded in three dimensions (Remark 5.9).

Fock space and second quantization: The Fock space $\mathcal{F} = \bigoplus_N \mathcal{H}^{(N)}$ with vacuum $|0\rangle$ (Definition 6.1), the bosonic $\mathcal{F}_+$ and fermionic $\mathcal{F}_-$ (Definition 6.2), the creation and annihilation operators on Fock space, the bosonic CCR $[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}\hat{\mathbf{1}}$ and fermionic CAR $\{\hat{a}_j, \hat{a}_k^\dagger\} = \delta_{jk}\hat{\mathbf{1}}$ (Theorem 6.5), the occupation number states and their ladder actions including the Jordan-Wigner sign factors (Proposition 6.8), and the connection between the Fock space and the fixed-N tensor product theory (Proposition 6.10).

Coupled harmonic oscillator: The normal mode transformation to $Q_+ = (x_1 + x_2)/\sqrt{2}$ and $Q_- = (x_1 - x_2)/\sqrt{2}$ (Definition 7.3), the canonical character of the transformation (Lemma 7.4), the decoupling $\hat{H}_{\text{coup}} = \hat{H}_+ + \hat{H}_-$ with normal mode frequencies $\omega_+ = \sqrt{\omega^2 + \kappa/m}$ and $\omega_- = \sqrt{\omega^2 - \kappa/m}$ (Proposition 7.6), the complete spectrum $E_{n_+,n_-} = (n_+ + \frac{1}{2})\Phi_0\omega_+ + (n_- + \frac{1}{2})\Phi_0\omega_-$ (Theorem 7.8), and the entanglement of the ground state for $\kappa \neq 0$ (Proposition 7.10).

Angular momentum addition: The total angular momentum algebra (Theorem 8.2), the expansion $\hat{L}^2_{\text{tot}} = \hat{L}^{2(1)} + \hat{L}^{2(2)} + 2\hat{\mathbf{L}}_1 \cdot \hat{\mathbf{L}}_2$ (Proposition 8.4), the coupled and uncoupled bases (Definition 8.6), and the Clebsch-Gordan decomposition structure with dimension count (Proposition 8.8).

The following topics are outside the scope of the present paper and are explicitly deferred.

Clebsch-Gordan coefficients. The explicit values of $\langle \ell_1, m_1; \ell_2, m_2 | J, M\rangle$, the Racah formula, the Wigner $3j$ symbols, and the orthogonality and completeness of the CG coefficients are deferred to QM8, where the $\frac{1}{2} \otimes \frac{1}{2}$ case is the primary application.

Quantum statistics. The Bose-Einstein distribution for bosons and the Fermi-Dirac distribution for fermions in thermal equilibrium are derived from the Fock space structure of the present paper combined with the thermodynamic framework developed in the statistical mechanics extension of the QM-series.

The spin-statistics theorem. The connection between integer spin and bosonic statistics (and half-integer spin and fermionic statistics) requires the relativistic transport closure framework of the RQM-series.

Three or more particle systems in detail. The present paper treats the two-particle case in full and notes the N-particle generalization. The N-particle Slater determinant, the structure of $\mathcal{H}^{(N)}_{\text{anti}}$ for general N, and the many-body perturbation theory built on the Fock space of Sec. 6 are developed in the many-body extension of the series.

Entanglement theory. The Schmidt decomposition, von Neumann entanglement entropy, separability criteria, Bell inequalities, and the EPR setup are developed in QM9 using the tensor product framework of Sec. 3 as their mathematical foundation.

10 Conclusion

10.1 Summary of Results

The present paper has extended the scalar–conformal NUVO transport closure framework from single-particle to multi-particle systems, deriving the tensor product Hilbert space, the exchange symmetry from holonomy, the Fock space and second quantization algebra, the coupled oscillator spectrum, and the angular momentum addition structure. The thirteen principal results are as follows.

Two-particle Hilbert space (Definition 3.3 and Proposition 3.5). $\mathcal{H}^{(2)} = \mathcal{H} \otimes \mathcal{H}$ is the completion of the algebraic tensor product with inner product $\langle \Phi_1 \otimes \Phi_2, \Psi_1 \otimes \Psi_2 \rangle = \langle \Phi_1, \Psi_1 \rangle \langle \Phi_2, \Psi_2 \rangle$. The product ONB $\{\phi_j \otimes \phi_k\}$ is complete in $\mathcal{H}^{(2)} \cong L^2(\mathbb{R}^6)$. Entangled states exist as elements of $\mathcal{H}^{(2)}$ that do not factorize as simple tensors.

Observable algebra on $\mathcal{H}^{(2)}$ (Propositions 3.8 and 3.10). Single-particle observables of different particles commute: $[A^{(1)}, B^{(2)}] = 0$. For non-interacting particles, the spectrum of the two-particle Hamiltonian is the set of all pairwise sums of single-particle eigenvalues, with product eigenstates.

Exchange operator and symmetry sectors (Propositions 4.2 and 4.4). $\hat{P}_{12}$ is self-adjoint, unitary, and involutory ($\hat{P}_{12}^2 = \hat{\mathbf{1}}$), with spectrum $\{+1, -1\}$ and eigenspace decomposition $\mathcal{H}^{(2)} = \mathcal{H}_{\text{sym}} \oplus \mathcal{H}_{\text{anti}}$.

Symmetrization and antisymmetrization (Proposition 4.6). $\hat{S}_+ = \frac{1}{2}(\hat{\mathbf{1}} + \hat{P}_{12})$ and $\hat{S}_- = \frac{1}{2}(\hat{\mathbf{1}} - \hat{P}_{12})$ are orthogonal projectors onto $\mathcal{H}_{\text{sym}}$ and $\mathcal{H}_{\text{anti}}$ with $\hat{S}_+ + \hat{S}_- = \hat{\mathbf{1}}$ and $\hat{S}_+\hat{S}_- = 0$. The antisymmetrized product $\phi_j \wedge \phi_k = (\phi_j \otimes \phi_k - \phi_k \otimes \phi_j)/\sqrt{2}$ is the two-particle Slater determinant, derived rather than postulated.

Exchange symmetry from holonomy (Theorem 5.7). For indistinguishable transport closure configurations, indistinguishability forces $[\hat{O}, \hat{P}_{12}] = 0$ for all physical $\hat{O}$, making $\mathcal{H}_{\text{sym}}$ and $\mathcal{H}_{\text{anti}}$ superselection sectors. The exchange path is a closed loop in $(\mathbb{R}^3 \times \mathbb{R}^3)/\text{Sym}_2$; its holonomy is $\pi \in \{+1, -1\}$ from $\pi^2 = 1$. Bosonic configurations ($\pi = +1$) have states in $\mathcal{H}_{\text{sym}}$; fermionic configurations ($\pi = -1$) have states in $\mathcal{H}_{\text{anti}}$.

Pauli exclusion principle (Corollary 5.10). $\hat{S}_-(\phi \otimes \phi) = 0$ for all $\phi \in \mathcal{H}$: no two identical fermions can occupy the same single-particle state. Derived in two lines from antisymmetry, not postulated.

Fock space (Definitions 6.1 and 6.2). $\mathcal{F} = \bigoplus_{N=0}^{\infty} \mathcal{H}^{(N)}$ with vacuum $|0\rangle$ and total number operator $\hat{N}_{\text{tot}}$; bosonic $\mathcal{F}_+$ and fermionic $\mathcal{F}_-$ built from the symmetrized and antisymmetrized N-particle subspaces.

Bosonic CCR and fermionic CAR (Theorem 6.5). $[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}\hat{\mathbf{1}}$ (bosons, generalizing the QM6 single-mode CCR $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$) and $\{\hat{a}_j, \hat{a}_k^\dagger\} = \delta_{jk}\hat{\mathbf{1}}$ (fermions, with $(\hat{a}_j)^2 = 0$ from Pauli exclusion); vacuum annihilated: $\hat{a}_j|0\rangle = 0$.

Occupation number states (Proposition 6.8). Bosonic states $|n_1, n_2, \ldots\rangle$ with $n_j \in \mathbb{Z}_{\geq 0}$; fermionic states with $n_j \in \{0, 1\}$; the Jordan-Wigner sign factor $(-1)^{\sum_{k<j} n_k}$ in the fermionic ladder actions. Both form complete ONBs for the respective Fock spaces.

Normal mode decoupling of the coupled oscillator (Proposition 7.6). The normal mode transformation $Q_+ = (x_1 + x_2)/\sqrt{2}$, $Q_- = (x_1 - x_2)/\sqrt{2}$ is canonical (preserves the CCR) and decouples $\hat{H}_{\text{coup}}$ into $\hat{H}_+ + \hat{H}_-$ with normal mode frequencies $\omega_+ = \sqrt{\omega^2 + \kappa/m}$ and $\omega_- = \sqrt{\omega^2 - \kappa/m}$.

Complete coupled oscillator spectrum (Theorem 7.8). $E_{n_+, n_-} = (n_+ + \frac{1}{2})\Phi_0\omega_+ + (n_- + \frac{1}{2})\Phi_0\omega_-$ for $n_\pm \in \{0, 1, 2, \ldots\}$, with product Hermite-Gaussian eigenstates in normal mode coordinates.

Ground state entanglement (Proposition 7.10). The coupled oscillator ground state is entangled in the original particle coordinates for all $\kappa \neq 0$: the cross term $(m(\omega_+ - \omega_-)/2\Phi_0)x_1x_2$ in the exponent prevents factorization.

Angular momentum addition structure (Theorem 8.2 and Proposition 8.8). $\hat{L}^j_{\text{tot}}$ satisfies the SO(3) algebra, derived from the single-particle algebra and $[\hat{L}_j^{(1)}, \hat{L}_k^{(2)}] = 0$. The Clebsch-Gordan decomposition $\mathcal{H}_{\ell_1} \otimes \mathcal{H}_{\ell_2} \cong \bigoplus_{J=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \mathcal{H}_J$ is established by the dimension count; the explicit CG coefficients are deferred to QM8.

10.2 Programmatic Significance

The results of the present paper are of broad programmatic significance on three grounds.

The first is the opening of the multi-particle sector. QM1 through QM6 established quantum mechanics for a single transport closure configuration, culminating in the harmonic oscillator and coherent state theory of QM6. QM7 makes the qualitative transition to multi-particle systems by introducing the two structures — entanglement and exchange symmetry — that have no single-particle analogue. These two structures, both derived from the tensor product construction and the holonomy principle rather than postulated, are the foundation on which all subsequent multi-particle papers build. QM8 uses the tensor product $\mathcal{H}\otimes\mathbb{C}^2$ for spin, QM9 analyzes entanglement in $\mathcal{H}^{(2)}$, QM10 uses the relative coordinate separation of the coupled oscillator for two-body scattering, and the Fock space of Sec. 6 underlies all of QM10, QM11, and the field-theoretic extensions.

The second ground of significance is the derivation of the exchange symmetry from holonomy. In the NUVO program, quantization results arise consistently from the holonomy principle of the Q-series: radial closure paths quantize energy, azimuthal closure paths quantize angular momentum (QM5), and now exchange paths in configuration space quantize the exchange parity. The boson-fermion dichotomy is therefore the third instance of a single geometric principle — holonomy quantization of closed transport paths — applied to a new class of paths. This structural unity is one of the distinguishing features of the NUVO program relative to standard formulations, where the symmetrization postulate appears as a separate and unexplained axiom. QM8 will add a fourth instance: the double-cover holonomy of SU(2) rotation paths quantizes the spin quantum number to half-integers. The pattern is consistent: every discrete quantum number in the program arises from a topological quantization of a closed path in an appropriate configuration space.

The third ground is the coupled oscillator as a structural template for two-body dynamics. The center-of-mass and relative coordinate separation used in Proposition 7.6 is the same technique used in QM4-QM5 to reduce the two-body hydrogen problem to a one-body problem in the reduced mass. There, the CM motion was free and the relative motion was bound by the Coulomb potential; here, the CM motion is harmonic and the relative motion is harmonic with a coupling-shifted frequency. In QM10, the same separation will be applied to the two-body scattering problem: the CM motion is again free, and the relative motion carries the scattering dynamics. The normal mode transformation itself is the simplest instance of a linear canonical transformation, the prototype of the Bogoliubov transformations that appear in many-body physics and quantum field theory. Establishing this template in the concrete setting of the coupled oscillator, where the full computation can be carried out exactly, prepares the conceptual ground for all of these later applications.

10.3 Transition to QM8

The present paper has opened the multi-particle sector by establishing the tensor product Hilbert space, the exchange symmetry, and the Fock space. QM8 applies this framework immediately to the simplest and most fundamental two-particle problem: the spin degree of freedom.

Spin is the internal degree of freedom of a transport closure configuration that generates the half-integer values of angular momentum not derivable from the orbital holonomy of QM5. In the NUVO framework, spin arises from the double-cover structure of the rotation group: while the azimuthal holonomy of QM5 quantized the orbital quantum number ℓ to non-negative integers via the single-valued requirement $\mathrm{e}^{\mathrm{i}2\pi\ell} = 1$, the spin holonomy arises from paths in the double cover SU(2) of SO(3), where a 2π rotation returns the state to minus itself ($\mathrm{e}^{\mathrm{i}2\pi j} = -1$ for j half-integer). The spin-$\frac{1}{2}$ particle thus has a two-dimensional internal space $\mathbb{C}^2$, and its full Hilbert space is $\mathcal{H}\otimes\mathbb{C}^2$ — the simplest non-trivial application of the QM7 tensor product construction.

The spin operators $\hat{S}_j$ satisfy the same SO(3) algebra $[\hat{S}_j, \hat{S}_k] = \mathrm{i}\Phi_0\,\epsilon_{jkl}\hat{S}_l$ as the orbital angular momentum operators of QM5, but with spectrum $j(j+1)\Phi_0^2$ for $j = \frac{1}{2}$ (and $m \in \{-\frac{1}{2}, +\frac{1}{2}\}$), derived from the half-integer holonomy of the double-cover path. The Clebsch-Gordan decomposition of Proposition 8.8 applied to $\ell_1 = \ell_2 = \frac{1}{2}$ gives $\mathcal{H}_{1/2}\otimes\mathcal{H}_{1/2} \cong \mathcal{H}_1\oplus\mathcal{H}_0$, the triplet-singlet decomposition of two spin-$\frac{1}{2}$ particles. The spin-orbit coupling $\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}$ is the first application of the two-particle coupling term $\hat{\mathbf{L}}_1\cdot\hat{\mathbf{L}}_2$ of Proposition 8.4 in the specific case where one factor is orbital ($\hat{\mathbf{L}}$, acting on the spatial $\mathcal{H}$) and the other is spin ($\hat{\mathbf{S}}$, acting on the internal $\mathbb{C}^2$). QM8 is therefore a direct application of every major structure introduced in QM7: the tensor product, the CG decomposition, the angular momentum addition, and the holonomy quantization of closed paths in the rotation group.

References

[1] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics, Vol. I: Functional Analysis*. Academic Press, New York, 1972. Primary reference for: the tensor product of Hilbert spaces (Chapter II, Section 4), including the algebraic tensor product, the completion to a Hilbert space, the inner product formula Eq. (3.2), and the isomorphism $\mathcal{H} \otimes \mathcal{H} \cong L^2(\mathbb{R}^6)$ (QM7 Proposition 3.2); the direct sum of Hilbert spaces underlying the Fock space construction of QM7 §6; and the spectral theorem used in the observable algebra of QM7 §3.4.