QM8 — Spin: The Double-Cover Holonomy, Pauli Algebra, and Spin-Orbit Coupling

in Scalar–Conformal NUVO Systems *Preprint, Version 1.0*[*]

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Notation and Conventions

- $\mathcal{M}$ denotes the spacetime manifold.
- η denotes the reference Lorentzian metric (typically Minkowski in a global chart).
- g denotes the physical metric.
- The scalar field $\Lambda : \mathcal{M} \to \mathbb{R}_{>0}$ is the NUVO modulation field.
- The physical metric is scalar–conformal:

$$g_{\mu\nu} = \Lambda^2 \, \eta_{\mu\nu}.$$

- $\Lambda_0 > 0$ denotes the baseline scalar availability level supported by the intrinsic delivery structure of the underlying field. In the absence of localized structural occupation the scalar field satisfies $\Lambda(x) = \Lambda_0$.
- The dimensionless scalar diagnostic is

$$\lambda(x) := \frac{\Lambda(x)}{\Lambda_0}.$$

- The scalar field represents the *locally available structural capacity* of the underlying delivery field. Localized structures may reduce this availability through occupation or transport, but the intrinsic delivery baseline Λ_0 remains fixed.
- Greek indices $\mu, \nu, \ldots$ range over spacetime coordinates $0, 1, 2, 3$.
- We use the Einstein summation convention unless explicitly stated otherwise.

Remark 0.1. *Unless otherwise stated, the background signature is* $(-, +, +, +)$.

[*]Bibliography is provisional. Cross-references to companion NUVO-series papers (M-, SR-, Q-, QB-, QM-series) will be updated with Zenodo DOIs in subsequent versions.

Program scope.

Abstract

The angular momentum spectrum established in QM5 contains only integer values of the quantum number $\ell \in \{0, 1, 2, \ldots\}$, derived from the requirement that the azimuthal transport closure state be single-valued under a full 2π rotation. The present paper derives the half-integer angular momentum spectrum from the double-cover holonomy of the rotation group: a 4π rotation returns to the identity in SU(2) while a 2π rotation does not, generating a second family of representations with $s \in \{\frac{1}{2}, \frac{3}{2}, \ldots\}$ that is inaccessible from the single-cover orbital holonomy of QM5. This half-integer family is the *spin* degree of freedom.

For a spin-$\frac{1}{2}$ particle, the spin Hilbert space is $\mathbb{C}^2 = \mathbb{C}^2$, and the full single-particle Hilbert space is $\mathcal{H}_{\text{full}} = \mathcal{H} \otimes \mathbb{C}^2$, the simplest non-trivial application of the QM7 tensor product construction. The spin-$\frac{1}{2}$ operators $\hat{S}_1$, $\hat{S}_2$, $\hat{S}_3$ satisfy the SU(2) algebra $[\hat{S}_1, \hat{S}_2] = \mathrm{i}\Phi_0 \hat{S}_3$ (and cyclic permutations) and are represented by the Pauli matrices: $\hat{\mathbf{S}} = \frac{\Phi_0}{2}\boldsymbol{\sigma}$. The spectrum of $\hat{S}^2$ is $\frac{3}{4}\Phi_0^2$ and of $\hat{S}_3$ is $\pm\frac{1}{2}\Phi_0$, with eigenstates the spin-up and spin-down spinors $|\uparrow\rangle$ and $|\downarrow\rangle$.

The interaction of the spin magnetic moment with an external magnetic field produces the Zeeman effect: the spin-up and spin-down states split in energy by $\Delta E = \Phi_0 \omega_L$ where $\omega_L = g\mu_B B/\Phi_0$ is the Larmor frequency. The time evolution of a spin-$\frac{1}{2}$ particle in a magnetic field is derived from the Zeeman Hamiltonian and shown to describe Larmor precession.

The spin-orbit coupling $\hat{H}_{\text{SO}} = \xi\, \hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$, where $\xi = \xi(r)$ is the radial coupling function, is the primary physical consequence of combining spin and orbital degrees of freedom. Applied to the hydrogen atom of QM5, the spin-orbit coupling removes the degeneracy of the $n\ell$-levels: states with the same n and ℓ but different total angular momentum $j = \ell \pm \frac{1}{2}$ are split by the fine structure energy $E^{\text{FS}}_{n\ell \propto} \alpha^4/n^3$.

The Clebsch-Gordan decomposition of $\mathcal{H}_\ell \otimes \mathbb{C}^2$ into total angular momentum sectors is derived in full: for orbital quantum number $\ell \geq 1$, $\mathcal{H}_\ell \otimes \mathbb{C}^2 \cong \mathcal{H}_{\ell+1/2} \oplus \mathcal{H}_{\ell-1/2}$, and for $\ell = 0$, $\mathcal{H}_0 \otimes \mathbb{C}^2 \cong \mathcal{H}_{1/2}$. The explicit Clebsch-Gordan coefficients for the $\ell \otimes \frac{1}{2}$ case are derived, completing the derivation deferred in QM7 Proposition 8.1.

No new postulates are introduced. All results follow from the QM7 tensor product structure, the QM5 angular momentum algebra, and the double-cover holonomy of SU(2).

1 Introduction

1.1 Position Within the QM-Series

The angular momentum structure derived in QM5 rests on a single geometric condition: the azimuthal transport closure state must return to itself under a full 2π rotation of the configuration. This single-valuedness condition, formalized as the holonomy of the azimuthal transport closure path in the rotation group SO(3), selects integer magnetic quantum numbers $m \in \mathbb{Z}$ and thereby restricts the orbital quantum number to $\ell \in \{0, 1, 2, \ldots\}$. The resulting orbital angular momentum structure — the spherical harmonics, the $\ell(\ell+1)\Phi_0^2$ spectrum, the $(2\ell+1)$-fold degeneracy — describes the rotational degrees of freedom of a single transport closure configuration moving in three-dimensional space. The question left open in QM5 was whether the angular momentum algebra $[\hat{L}_j, \hat{L}_k] = \mathrm{i}\Phi_0\, \epsilon_{jkl} \hat{L}_l$ admits representations beyond the orbital (integer) family. The present paper, QM8, answers this question in the affirmative: the same algebra admits a second family of representations — the half-integer representations with $s \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}$ — that are not accessible from the single-cover holonomy of QM5 but emerge from the double-cover holonomy of SU(2). These are the *spin* representations, and they constitute the internal angular momentum degree of freedom of a transport closure configuration.

The key distinction between the orbital and spin cases lies in the topology of the rotation group. The orbital holonomy of QM5 is set in SO(3), whose fundamental group is $\pi_1(\text{SO}(3)) = \mathbb{Z}_2$: a 2π

rotation is a non-contractible loop, and the single-valuedness condition on the closure state forces the holonomy to be $+1$ (ruling out the factor -1 that would arise for half-integer winding). The double-cover SU(2), by contrast, is simply connected ($\pi_1(\mathrm{SU}(2)) = 0$): every loop is contractible. The minimal contractible loop in SU(2) corresponds to a 4π rotation, and the holonomy quantization applied to this loop selects $m_s \in \frac{1}{2}\mathbb{Z}$ — either integer or half-integer magnetic quantum numbers. The integer case recovers the orbital representations of QM5; the half-integer case is new, with the defining property that the closure state acquires a factor -1 under a 2π rotation (returning to $+1$ only under a 4π rotation). This spinor behavior is derived here as a theorem from the double-cover holonomy, not postulated as a new axiom.

QM8 is the fourth and final instance of the holonomy quantization principle in the non-relativistic QM-series. In the Q-series, the holonomy of radial transport closure cycles quantized the principal quantum number $n \in \mathbb{Z}_{>0}$. In QM5, the holonomy of azimuthal rotation paths quantized the orbital quantum number $\ell \in \mathbb{Z}_{\geq 0}$. In QM7, the holonomy of exchange paths in the symmetrized configuration space $(\mathbb{R}^3 \times \mathbb{R}^3)/\mathrm{Sym}_2$ quantized the exchange parity $\pi \in \{+1, -1\}$. In the present paper, the holonomy of rotation paths in the double cover SU(2) quantizes the spin quantum number $s \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. The pattern is consistent: every discrete quantum number in the NUVO program arises from the topological quantization of a closed transport path in an appropriate configuration space. The spin quantum number is not a separate postulate appended to the theory but the double-cover generalization of the orbital holonomy, required by the richer topology of the rotation group. QM11 will add a fifth instance in the relativistic sector: the connection between the spin quantum number and the exchange parity — the spin-statistics theorem — arises from the holonomy structure of the relativistic transport closure system in a way that is not accessible within the non-relativistic framework.

QM8 depends on the prior series in three structurally specific ways. The angular momentum algebra of QM5 is the input to the spin derivation of Sec. 3: the same SO(3) commutation relations $[\hat{\mathbf{S}}_j, \hat{\mathbf{S}}_k] = \mathrm{i}\Phi_0\, \epsilon_{jkl}\hat{\mathbf{S}}_l$ are satisfied by the spin operators, and the same ladder argument of QM5 Sec. 4 derives the spectrum $j(j+1)\Phi_0^2$ once the double-cover holonomy has extended the allowed values of j to include half-integers. The tensor product construction of QM7 is the structural input to the full spin-$\frac{1}{2}$ Hilbert space $\mathcal{H}_{\mathrm{full}} = \mathcal{H} \otimes \mathbb{C}^2$ of Sec. 5: the spatial and spin degrees of freedom are independent subsystems whose operators commute on the product space by QM7 Proposition 3.3. The Clebsch-Gordan decomposition previewed in QM7 Proposition 8.1 is completed in Sec. 8 for the primary case $\mathcal{H}_\ell \otimes \mathbb{C}^2$: the explicit coefficients of the $\ell \otimes \frac{1}{2}$ decomposition are derived and applied immediately to the spin-orbit coupling and hydrogen fine structure of Sec. 7.

1.2 Objective of the Present Work

The central objective of the present paper is to derive the spin degree of freedom and its physical consequences from the double-cover holonomy of SU(2), within the scalar–conformal NUVO transport closure framework. Specifically, the paper establishes six claims.

1. The holonomy quantization principle applied to rotation paths in SU(2) selects magnetic quantum numbers $m_s \in \frac{1}{2}\mathbb{Z}$. For half-integer m_s, a 2π rotation of the closure state produces a factor -1 (spinor behavior), and a 4π rotation returns the state to itself. Combined with the ladder argument of QM5 applied to the SU(2) algebra $[\hat{\mathbf{S}}_j, \hat{\mathbf{S}}_k] = \mathrm{i}\Phi_0\, \epsilon_{jkl}\hat{\mathbf{S}}_l$, the spin spectrum is $\hat{S}^2|s, m_s\rangle = s(s+1)\Phi_0^2|s, m_s\rangle$ and $\hat{S}_3|s, m_s\rangle = m_s\Phi_0|s, m_s\rangle$ for $s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ and $m_s \in \{-s, \ldots, +s\}$.

2. For spin $s = \frac{1}{2}$, the spin Hilbert space is $\mathbb{C}^2 = \mathbb{C}^2$ and the spin operators are $\hat{\mathbf{S}}_j = (\Phi_0/2)\boldsymbol{\sigma}_j$, where the Pauli matrices σ_1, σ_2, σ_3 are the unique (up to unitary equivalence) 2×2 Hermitian

traceless matrices satisfying the SU(2) algebra. The Pauli matrices satisfy $\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k = \delta_{jk}\sigma_0 + \mathrm{i}\epsilon_{jkl}\boldsymbol{\sigma}_l$, which encodes simultaneously the SU(2) commutation algebra and the Clifford algebra $\{\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_k\} = 2\delta_{jk}\sigma_0$.

3. The full single-particle Hilbert space for a spin-$\frac{1}{2}$ particle is $\mathcal{H}_{\text{full}} = \mathcal{H} \otimes \mathbb{C}^2 = L^2(\mathbb{R}^3, \mathbb{C}^2)$, with elements represented as two-component spinors $\chi(\mathbf{x}) = (\Psi_\uparrow(\mathbf{x}), \Psi_\downarrow(\mathbf{x}))^\mathsf{T}$ for $\Psi_\uparrow, \Psi_\downarrow \in \mathcal{H}$. Spatial observables and spin observables commute on $\mathcal{H}_{\text{full}}$ (QM7 Proposition 3.3 applied to $\mathcal{H}$ and $\mathbb{C}^2$). The Pauli equation — the spin-$\frac{1}{2}$ Schrödinger equation on $\mathcal{H}_{\text{full}}$ — follows from the QM4 dynamical framework applied to $\mathcal{H}_{\text{full}}$.

4. The Zeeman Hamiltonian $\hat{H}_Z = (g\mu_B/\Phi_0)B\hat{S}_3$ for a spin-$\frac{1}{2}$ particle in a uniform field $\mathbf{B} = B\hat{z}$ has eigenvalues $E_\pm = \pm g\mu_B B/2$, giving energy splitting $\Delta E = g\mu_B B = \Phi_0\omega_L$ where $\omega_L = g\mu_B B/\Phi_0$ is the Larmor frequency. The time evolution of a general spin state under $\hat{H}_Z$ is Larmor precession: the transverse spin components rotate at ω_L while $\langle\hat{S}_3\rangle$ is conserved.

5. The spin-orbit coupling $\hat{H}_{\text{SO}} = \xi(r)\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$, acting on $\mathcal{H}_{\text{full}}$, has eigenvalues in the total angular momentum basis $|j, m_j\rangle$ given by $(\Phi_0^2/2)[j(j+1) - \ell(\ell+1) - \frac{3}{4}]$ via the identity $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} = (\hat{J}^2 - \hat{L}^2 - \hat{S}^2)/2$. Applied to the hydrogen atom, the spin-orbit coupling splits each $n\ell$-level of QM5 into two levels with $j = \ell + \frac{1}{2}$ and $j = \ell - \frac{1}{2}$, separated by a fine structure energy proportional to $\alpha^2|E_n|/n$ where α is the fine structure constant.

6. The Clebsch-Gordan decomposition $\mathcal{H}_\ell \otimes \mathbb{C}^2 \cong \mathcal{H}_{\ell+1/2} \oplus \mathcal{H}_{\ell-1/2}$ (for $\ell \geq 1$) is proved by the ladder operator method, with the explicit CG coefficients derived: $\langle \ell, m_j \mp \frac{1}{2}; \frac{1}{2}, \pm\frac{1}{2}|j, m_j\rangle = \pm\sqrt{(\ell \pm m_j + \frac{1}{2})/(2\ell+1)}$ for $j = \ell + \frac{1}{2}$ and corresponding coefficients for $j = \ell - \frac{1}{2}$. This completes the derivation deferred in QM7 Proposition 8.1 for the primary physical case.

Claims (1) through (6) are logically ordered. The double-cover holonomy of claim (1) establishes the half-integer spectrum; the Pauli representation of claim (2) is the explicit realization of the spin-$\frac{1}{2}$ case; the full Hilbert space of claim (3) is the setting for all physical applications; the Zeeman effect of claim (4) is the simplest application (spin alone, no orbital coupling); the spin-orbit coupling of claim (5) is the first application combining spin and orbital degrees of freedom; and the CG decomposition of claim (6) is the structural result that makes the spin-orbit coupling analysis precise.

1.3 What Is Not Assumed

The present work maintains without modification the interpretive discipline of the prior series. Five exclusions are of particular importance for QM8.

Spin is not postulated as an additional degree of freedom appended to the single-particle framework. In many treatments of quantum mechanics, the spin of a particle is introduced either by empirical observation (the Stern-Gerlach experiment) or by the Dirac equation as a relativistic necessity. In the NUVO framework, spin is derived in the present paper from the double-cover holonomy of SU(2): the same geometric principle that gives integer angular momentum in QM5 gives half-integer angular momentum when the full double-cover structure of the rotation group is taken into account. The empirical appearance of half-integer angular momentum in experiments is a consequence of this geometric structure, not a separate input to the theory.

The g-factor $g = 2$ for the electron spin magnetic moment is not derived in the present paper. The value $g = 2$ is the leading prediction of the Dirac equation — the relativistic first-principles theory of the spin-$\frac{1}{2}$ particle — which is derived in QM11 as the relativistic transport closure extension of the present spin-$\frac{1}{2}$ framework. QM8 introduces the Zeeman coupling $\hat{H}_Z = (g\mu_B/\Phi_0)B\hat{S}_3$ with g as an external parameter, establishes the Zeeman splitting and Larmor precession for general g, and records that $g = 2$ is the value produced by the Dirac equation. The radiative corrections to g ($g = 2 + \alpha/\pi + \cdots$ from quantum electrodynamics) are beyond the scope of the QM-series.

The spin-statistics theorem is not derived in the present paper. As established in QM7 Sec. **??**, the theorem that half-integer spin particles are fermions and integer spin particles are bosons requires the relativistic framework and will be derived in the RQM-series. QM8 derives the spin structure and the spin-$\frac{1}{2}$ Hilbert space without making any claim about the exchange symmetry of multi-particle spin-$\frac{1}{2}$ states. In particular, QM8 establishes that the spin-$\frac{1}{2}$ particle has $\mathcal{H}_{\text{full}} = \mathcal{H} \otimes \mathbb{C}^2$ as its single-particle state space, but does not assert whether a many-particle system of such configurations is bosonic or fermionic.

Higher spin representations ($s = \frac{3}{2}, 2, \frac{5}{2}, \ldots$) are not developed in detail. The double-cover holonomy of Theorem 3.3 and the ladder spectrum of Theorem 3.5 apply to all $s \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, but the explicit matrix representations of the spin operators for $s > \frac{1}{2}$, their tensor product decompositions, and their physical applications (spin-1 photons, spin-$\frac{3}{2}$ baryons) are deferred. The general CG decomposition for $\ell_1 \otimes \ell_2$ with arbitrary ℓ_2 is similarly deferred; the present paper completes the $\ell \otimes \frac{1}{2}$ case required for the hydrogen fine structure and atomic physics applications of QM8-QM10.

The relativistic spin-orbit coupling is not derived from first principles in the present paper. The non-relativistic spin-orbit Hamiltonian $\hat{H}_{\text{SO}} = \xi(r)\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$ is introduced in Sec. 7 with the radial function $\xi(r)$ given by its non-relativistic reduction; the full derivation of $\xi(r)$ from the Dirac equation, including the Thomas precession factor of $\frac{1}{2}$, is deferred to QM11. The energy eigenvalues of $\hat{H}_{\text{SO}}$ in the total angular momentum basis and the resulting fine structure splitting are derived in full from the non-relativistic Hamiltonian.

1.4 Structure of the Paper

Sec. 2 recalls the angular momentum algebra and orbital holonomy from QM5, the tensor product construction from QM7, and the SU(2) double cover of SO(3) as the group-theoretic setting for the spin derivation. Sec. 3 derives the double-cover holonomy condition, establishes that half-integer magnetic quantum numbers are consistent with a 4π contractible loop in SU(2), and derives the complete spin spectrum by applying the ladder argument of QM5 to the extended set of admissible quantum numbers. Sec. 4 introduces the Pauli matrices as the unique 2×2 matrix representation of the spin-$\frac{1}{2}$ generators, derives the complete Pauli algebra including the product formula, the anticommutation and commutation relations, and the trace and completeness properties, and identifies the spin eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$. Sec. 5 constructs the full single-particle Hilbert space $\mathcal{H}_{\text{full}} = \mathcal{H} \otimes \mathbb{C}^2$, introduces the two-component spinor wave function as its position-space representation, records the observable algebra on $\mathcal{H}_{\text{full}}$, and derives the Pauli equation from the QM4 dynamical framework. Sec. 6 derives the Zeeman Hamiltonian and its eigenvalues for a uniform external magnetic field, establishes the Larmor precession dynamics from the Heisenberg equation of motion, and records the Bloch sphere representation of spin state evolution. Sec. 7 introduces the spin-orbit coupling operator $\hat{H}_{\text{SO}} = \xi(r)\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$ on $\mathcal{H}_{\text{full}}$, evaluates its eigenvalues in the total angular momentum basis using the identity $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} = (\hat{J}^2 - \hat{L}^2 - \hat{S}^2)/2$, and applies the result to derive the hydrogen fine structure energy splitting. Sec. 8 completes the Clebsch-Gordan derivation deferred in QM7: the decomposition $\mathcal{H}_\ell \otimes \mathbb{C}^2 \cong \mathcal{H}_{\ell+1/2} \oplus \mathcal{H}_{\ell-1/2}$ is proved by the ladder operator method and the explicit CG coefficients are derived and recorded. Sec. 9 records the place of QM8 in the holonomy quantization sequence, the interpretive boundary between what is derived and what is deferred, and the scope of the present construction. Sec. 10 summarizes the twelve principal results, records the programmatic significance of the spin derivation and the CG completion, and prepares the transition to QM9.

2 Recalled Structure from Prior Papers

The present section collects the results from QM5, QM7, and the standard theory of the rotation group that are directly required for the derivations of Secs. 3–8. Nothing in this section is new. The recalled material falls into three categories: the angular momentum algebra and orbital holonomy condition from QM5 whose extension to the double cover produces the spin spectrum, the tensor product structure from QM7 whose application to $\mathcal{H} \otimes \mathbb{C}^2$ produces the full spin-$\frac{1}{2}$ Hilbert space, and the group-theoretic relationship between SO(3) and SU(2) that is the geometric setting for the spin derivation.

2.1 The Angular Momentum Algebra and Orbital Holonomy from QM5

The following results from QM5 are used directly in Secs. 3 and 8.

Angular momentum commutation algebra (QM5 Theorem 3.1). On $\mathcal{S}(\mathbb{R}^3) \subset \mathcal{H}$:

$$[\hat{L}_j, \hat{L}_k] = \mathrm{i}\Phi_0\, \epsilon_{jkl}\, \hat{L}_l. \tag{1}$$

The spin operators of QM8 satisfy the same algebra with $\hat{L}_j$ replaced by $\hat{\mathbf{S}}_j$ on the spin Hilbert space $\mathbb{C}^2$; the derivations of Sec. 3 use the ladder argument of QM5 applied verbatim to the SU(2) algebra.

Ladder operator matrix elements (QM5 Proposition 5.3). For the raising and lowering operators $\hat{L}_+ = \hat{L}_1 + \mathrm{i}\hat{L}_2$ and $\hat{L}_- = \hat{L}_1 - \mathrm{i}\hat{L}_2$:

$$\hat{L}_+|\, \ell,\, m\,\rangle = \sqrt{\ell(\ell+1) - m(m+1)}\; \Phi_0\, |\, \ell,\, m+1\,\rangle, \tag{2}$$

and the analogous relation for $\hat{L}_-$. These matrix elements are used in Sec. 8 to derive the CG coefficients for $\ell \otimes \frac{1}{2}$: the lowering operator $\hat{J}_- = \hat{L}_- \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes \hat{S}_-$ applied to the highest-weight state of the total angular momentum multiplet generates all states of that multiplet via Eq. (2) on each factor.

The orbital holonomy condition (QM5 Theorem 5.1 and Sec. 5.2). The orbital angular momentum quantum numbers are restricted to $\ell \in \{0, 1, 2, \ldots\}$ and $m \in \{-\ell, \ldots, +\ell\}$ by the requirement that the azimuthal transport closure state be single-valued under a 2π rotation:

$$\mathrm{e}^{\mathrm{i}2\pi m} = +1 \quad \Rightarrow \quad m \in \mathbb{Z}. \tag{3}$$

This condition eliminates half-integer values of m from the orbital sector because a 2π rotation is a closed loop in SO(3), and the single-valuedness of the closure state on SO(3) requires integer winding numbers. The extension to SU(2) in Sec. 3 replaces the 2π loop condition with the 4π loop condition of the double cover, admitting $m \in \frac{1}{2}\mathbb{Z}$.

Spherical harmonics as the orbital eigenstates (QM5 Theorem 6.2). The joint eigenstates of $\hat{L}^2$ and $\hat{L}_3$ on $L^2(S^2)$ are the spherical harmonics $Y_\ell^m(\theta, \varphi)$, recalled from Eq. (**??**). These enter QM8 in the CG decomposition of Sec. 8 as the angular factor of the spatial component of the total angular momentum eigenstates.

Remark 2.1. *The QM5-to-QM8 program arc is the most direct derivation chain in the QM-series. QM5 establishes the angular momentum algebra and, from it, the orbital spectrum by the ladder* *argument. QM8 takes the same algebra, relaxes the holonomy condition from 2π to 4π, and applies the same ladder argument to obtain the extended spectrum. The two papers therefore have an almost identical formal structure in Secs. 3, with the single difference that the orbital case terminates the half-integer branch at the holonomy step (Eq. (3) forces $m \in \mathbb{Z}$) while the spin case admits it. The algebraic results — ladder matrix elements, the spectrum $j(j+1)\Phi_0^2$, and the non-degeneracy of each multiplet — all carry over verbatim.*

2.2 The Tensor Product Structure from QM7

The following results from QM7 are used in Secs. 5–8.

The tensor product Hilbert space (QM7 Definition 3.1 and Proposition 3.2). For two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$ is the completion of the algebraic tensor product with inner product $\langle \Phi_A \otimes \Phi_B, \Psi_A \otimes \Psi_B \rangle = \langle \Phi_A, \Psi_A \rangle_{\mathcal{H}_A} \langle \Phi_B, \Psi_B \rangle_{\mathcal{H}_B}$. Applied to $\mathcal{H}_A = \mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C})$ and $\mathcal{H}_B = \mathbb{C}^2 = \mathbb{C}^2$:

$$\mathcal{H}_{\text{full}} = \mathcal{H} \otimes \mathbb{C}^2 \cong L^2(\mathbb{R}^3, \mathbb{C}^2), \tag{4}$$

with product states $\Psi(\mathbf{x}) \otimes |m_s\rangle$ represented as two-component spinors $(\Psi_\uparrow(\mathbf{x}), \Psi_\downarrow(\mathbf{x}))^\mathsf{T}$.

Commutation of spatial and spin observables (QM7 Proposition 3.3). For any spatial observable A on $\mathcal{H}$ and any spin observable B on $\mathbb{C}^2$:

$$[A \otimes \hat{\mathbf{1}}_{\mathbb{C}^2},\ \hat{\mathbf{1}}_{\mathcal{H}} \otimes B] = 0 \quad \text{on } \mathcal{H}_{\text{full}}. \tag{5}$$

In particular, $[\hat{L}_j \otimes \hat{\mathbf{1}}, \hat{\mathbf{1}} \otimes \hat{\mathbf{S}}_k] = 0$: orbital and spin angular momentum components commute on $\mathcal{H}_{\text{full}}$. This is the key identity used in the spin-orbit coupling analysis of Sec. 7: the total angular momentum $\hat{J}_j = \hat{L}_j \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes \hat{\mathbf{S}}_j$ satisfies the same SO(3) algebra as its components (as derived in QM7 Theorem 8.1) precisely because the cross commutators vanish.

Clebsch-Gordan decomposition structure (QM7 Proposition 8.1). The tensor product of two angular momentum multiplets decomposes as $\mathcal{H}_{\ell_1} \otimes \mathcal{H}_{\ell_2} \cong \bigoplus_{J=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \mathcal{H}_J$, with the dimension count Eq. (**??**). For the specific case $\ell_1 = \ell$ and $\ell_2 = \frac{1}{2}$, the decomposition stated in QM7 Remark 8.3 is

$$\mathcal{H}_\ell \otimes \mathbb{C}^2 \cong \mathcal{H}_{\ell+1/2} \oplus \mathcal{H}_{\ell-1/2} \quad (\ell \geq 1), \tag{6}$$

whose proof was deferred in QM7 and is completed in Sec. 8 of the present paper. The total angular momentum algebra on $\mathcal{H}_{\text{full}}$ was established in QM7 Theorem 8.1 for orbital-orbital coupling; the same result applies verbatim to orbital-spin coupling since the derivation used only the commutation relations Eq. (5) and the individual SO(3) algebras.

Remark 2.2. *The QM7 tensor product framework is used in QM8 in the specific context where one factor is infinite-dimensional ($\mathcal{H}$, the spatial Hilbert space) and the other is finite-dimensional ($\mathbb{C}^2 = \mathbb{C}^2$, the spin space). This is structurally simpler than the QM7 coupled oscillator setting (where both factors were copies of the same $\mathcal{H}$) because the finite dimensionality of $\mathbb{C}^2$ makes the spectral theory of spin operators elementary: a self-adjoint operator on $\mathbb{C}^2$ is a 2×2 Hermitian matrix, and its spectrum is obtained by solving the characteristic polynomial, which is at most quadratic. The functional analytic subtleties (domains of self-adjointness, spectral measures, deficiency indices) that apply to unbounded operators on $\mathcal{H}$ do not arise for bounded operators on $\mathbb{C}^2 = \mathbb{C}^2$.*

2.3 The SU(2) Double Cover of SO(3)

The group-theoretic relationship between SO(3) and SU(2) is the geometric setting for the spin derivation of Sec. 3. The following facts from the standard theory of Lie groups and their representations are recorded as background, without derivation.

The rotation groups SO(3) *and* SU(2)*:* $\mathrm{SO}(3) = \{R \in M_3(\mathbb{R}) : R^\mathsf{T} R = I_3, \det R = +1\}$ is the group of orientation-preserving rotations of $\mathbb{R}^3$. $\mathrm{SU}(2) = \{U \in M_2(\mathbb{C}) : U^\dagger U = I_2, \det U = +1\}$ is the group of 2×2 special unitary matrices. As topological spaces: $\mathrm{SO}(3) \cong \mathbb{RP}^3$ (real projective three-space) and $\mathrm{SU}(2) \cong S^3$ (the three-sphere).

The covering homomorphism: There is a surjective Lie group homomorphism $\pi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ defined by

$$\pi(U)\mathbf{v} := \text{the rotation of } \mathbf{v} \in \mathbb{R}^3 \text{ corresponding to } U \in \mathrm{SU}(2), \tag{7}$$

realized explicitly via $\pi(U)\mathbf{v} = U(\mathbf{v} \cdot \boldsymbol{\sigma})U^\dagger$ (where $\mathbf{v} \cdot \boldsymbol{\sigma} = v_j \boldsymbol{\sigma}_j$). The kernel of π is $\ker \pi = \{+I_2, -I_2\} \cong \mathbb{Z}_2$, so π is 2-to-1: every rotation $R \in \mathrm{SO}(3)$ has exactly two pre-images $\pm U \in \mathrm{SU}(2)$.

Fundamental groups and contractible loops: The topological difference between SO(3) and SU(2) is captured by their fundamental groups:

$$\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2, \qquad \pi_1(\mathrm{SU}(2)) = 0. \tag{8}$$

In SO(3), the path $\theta \mapsto R_{\hat{z}}(\theta)$ for $\theta \in [0, 2\pi]$ (a full rotation about the z-axis) is a non-contractible loop: it represents the non-trivial element of $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$. The path $\theta \mapsto R_{\hat{z}}(\theta)$ for $\theta \in [0, 4\pi]$ (a double rotation) is contractible in SO(3). In SU(2), since $\pi_1(\mathrm{SU}(2)) = 0$, *every* loop is contractible; in particular, the lift of the 2π rotation to SU(2) — which is the path from $+I_2$ to $-I_2$ — is an open arc, not a closed loop. The minimal closed loop in SU(2) above the 2π rotation of SO(3) is the 4π rotation, whose lift in SU(2) goes $+I_2 \to -I_2 \to +I_2$.

Representations: A representation of SO(3) is automatically a representation of SU(2) (by composing with π). A representation $\rho : \mathrm{SU}(2) \to \mathrm{GL}(V)$ descends to a representation of SO(3) if and only if $\rho(-I_2) = \mathrm{id}_V$, i.e., the central element $-I_2$ acts trivially. For the spin-j representation: $\rho_j(-I_2) = (-1)^{2j}\mathrm{id}$, so:

$$\rho_j(-I_2) = \begin{cases} +\mathrm{id} & \text{if } j \in \{0, 1, 2, \ldots\} \quad \text{(representation of SO(3))}, \\ -\mathrm{id} & \text{if } j \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\} \quad \text{(representation of SU(2) only)}. \end{cases} \tag{9}$$

The integer-j representations are the orbital representations of QM5; the half-integer-j representations are the spin representations that are the subject of the present paper.

Remark 2.3. *The representation criterion Eq.* (9) *is the group-theoretic statement of the double-cover holonomy condition derived in Sec. 3. In holonomy language: the closed* 2π *rotation loop in* SO(3) *lifts to the open arc* $+I_2 \to -I_2$ *in* SU(2)*, which is not a closed loop. A state transforming under the spin-*j *representation picks up the factor* $\rho_j(-I_2) = (-1)^{2j}$ *when the system completes a* 2π *rotation. For integer* j*: factor* $+1$ *(consistent with the QM5 single-cover holonomy condition Eq.* (3)*). For half-integer* j*: factor* -1 *(spinor behavior, consistent with the* 4π *double-cover holonomy condition of Theorem 3.3). The two descriptions — group-theoretic (representation criterion) and geometric (holonomy quantization) — are equivalent and will be used interchangeably in Sec. 3.*

Remark 2.4. *It is instructive to observe how the four holonomy quantizations of the NUVO program relate to group topology. The radial holonomy (Q-series) involves a simply connected configuration space; the quantization comes from the* 2π *periodicity of the phase. The azimuthal holonomy* *of QM5 involves the circle S^1 (azimuthal angle), whose fundamental group is $\pi_1(S^1) = \mathbb{Z}$; integer winding numbers give integer m. The exchange holonomy of QM7 involves $(\mathbb{R}^3 \times \mathbb{R}^3)/\mathrm{Sym}_2$, whose fundamental group is $\pi_1 = \mathbb{Z}_2$; the two elements give $\pi = \pm 1$. The double-cover holonomy of QM8 involves $\mathrm{SO}(3) \cong \mathbb{RP}^3$, whose fundamental group is $\pi_1 = \mathbb{Z}_2$; the two elements give the factor ± 1 under 2π rotation, selecting j integer or half-integer. The correspondence between the fundamental group and the quantization structure is the unifying geometric principle of the NUVO program's approach to quantum numbers.*

3 The Double-Cover Holonomy and the Half-Integer Spectrum

The orbital angular momentum quantum numbers of QM5 are integers because the holonomy quantization of the azimuthal rotation path requires $\mathrm{e}^{\mathrm{i}2\pi m} = +1$. The present section derives the spin quantum numbers by applying the holonomy principle to the larger group SU(2), in which the minimal contractible loop corresponds to a 4π rotation. This change in the loop condition — from 2π to 4π — extends the admissible magnetic quantum numbers from $\mathbb{Z}$ to $\frac{1}{2}\mathbb{Z}$, while retaining the integer case as a special subset. The half-integer family that the 4π condition admits is the spin sector; its spectrum is derived by the same ladder argument used in QM5, applied to the SU(2) algebra.

3.1 The Rotation Path and Its Holonomy in SU(2)

The transport closure state in the exchange sector is a section of a complex line bundle over the configuration space of the system. When the configuration is rotated, the closure state transforms under the rotation, and the holonomy of the rotation path is the phase factor accumulated by the state as the system traverses the path. For a state of orbital angular momentum ℓ and magnetic quantum number m, the azimuthal rotation path $\varphi \mapsto \mathrm{e}^{\mathrm{i}\varphi \hat{L}_3/\Phi_0}$ for $\varphi \in [0, 2\pi]$ accumulates a holonomy $\mathrm{e}^{\mathrm{i}2\pi m}$. The single-valuedness condition $\mathrm{e}^{\mathrm{i}2\pi m} = +1$ selects $m \in \mathbb{Z}$ (QM5 Theorem 5.1).

For the spin degree of freedom, the configuration space is not the laboratory coordinate space $\mathbb{R}^3$ but the rotation group itself: the spin state transforms under rotations of the physical frame, not under translations of the particle position. The relevant group is SU(2) rather than SO(3), and the holonomy principle must be applied to paths in SU(2).

Definition 3.1 (Rotation path in SU(2))**.** *The* rotation path *about the z-axis through angle φ is the path in* SU(2)*:*

$$\varphi \mapsto U(\varphi) := \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\varphi/2} \end{pmatrix} = \exp\left(\frac{\mathrm{i}\varphi}{2}\sigma_3\right), \qquad \varphi \in [0, \varphi_{\max}]. \tag{10}$$

For $\varphi_{\max} = 2\pi$: $U(2\pi) = -I_2$, so the 2π path is an open arc *in* SU(2) *from $+I_2$ to $-I_2$. For $\varphi_{\max} = 4\pi$: $U(4\pi) = +I_2$, so the 4π path is a* closed loop *in* SU(2) *based at $+I_2$.*

Remark 3.2. *The distinction between the 2π and 4π paths is the central geometric fact of spin physics. In* SO(3)*, the 2π rotation returns to the identity: the path $\varphi \mapsto R_{\hat{z}}(\varphi)$ for $\varphi \in [0, 2\pi]$ is a closed loop in* SO(3)*. The holonomy condition on this loop gives $\mathrm{e}^{\mathrm{i}2\pi m} = 1$, which forces $m \in \mathbb{Z}$.*

In SU(2)*, the 2π rotation lifts to the path $U(\varphi)$ for $\varphi \in [0, 2\pi]$, which is an* open arc *from $+I_2$ to $-I_2$: it does not close. The closed loop in* SU(2) *that covers the 2π* SO(3) *rotation must be traversed twice to close, giving the 4π path. The holonomy condition on the 4π closed loop gives $\mathrm{e}^{\mathrm{i}4\pi m_s} = 1$, which forces $m_s \in \frac{1}{2}\mathbb{Z}$.*

Theorem 3.3 (Double-cover holonomy and the half-integer spectrum)**.** *The holonomy quantization principle applied to rotation paths in* SU(2) *selects spin magnetic quantum numbers* $m_s \in \frac{1}{2}\mathbb{Z}$. *Specifically:*

(i) *The closed loop in* SU(2) *corresponding to a* 4π *rotation accumulates holonomy* $\mathrm{e}^{\mathrm{i}4\pi m_s} = +1$*, which is satisfied for all* $m_s \in \frac{1}{2}\mathbb{Z}$*.*

(ii) *The open arc in* SU(2) *corresponding to a* 2π *rotation accumulates holonomy* $\mathrm{e}^{\mathrm{i}2\pi m_s}$*, which equals* $+1$ *for* $m_s \in \mathbb{Z}$ *(*integer case*, orbital representations) and* -1 *for* $m_s \in \mathbb{Z} + \frac{1}{2}$ *(*half-integer case*, spinor behavior).*

(iii) *States in the half-integer case acquire a factor* -1 *under a* 2π *rotation and return to themselves only under a* 4π *rotation; they are* spinors*.*

Proof. Part (i): Let m_s be the eigenvalue of the z-component generator $\hat{S}_3/\Phi_0$ for the spin state under consideration. Under the rotation path Eq. (10), a spin eigenstate with eigenvalue m_s transforms as:

$$|m_s\rangle \;\longmapsto\; \mathrm{e}^{\mathrm{i}\varphi \hat{S}_3/\Phi_0}|m_s\rangle = \mathrm{e}^{\mathrm{i}\varphi m_s}|m_s\rangle.$$

For the 4π closed loop ($\varphi_{\max} = 4\pi$), the holonomy is $\mathrm{e}^{\mathrm{i}4\pi m_s}$. The quantization condition — that the state returns to itself at the end of a closed loop (Q-series holonomy principle, Eq. (**??**)) — requires $\mathrm{e}^{\mathrm{i}4\pi m_s} = +1$, giving $4\pi m_s \in 2\pi\mathbb{Z}$, hence $m_s \in \frac{1}{2}\mathbb{Z}$.

Part (ii): For the 2π arc ($\varphi_{\max} = 2\pi$), the accumulated phase factor is $\mathrm{e}^{\mathrm{i}2\pi m_s}$. For $m_s \in \mathbb{Z}$: $\mathrm{e}^{\mathrm{i}2\pi m_s} = +1$ (the state is unchanged by a 2π rotation, consistent with the orbital holonomy condition Eq. (3)). For $m_s \in \mathbb{Z} + \frac{1}{2}$: $\mathrm{e}^{\mathrm{i}2\pi m_s} = \mathrm{e}^{\mathrm{i}\pi} = -1$ (the state acquires a factor -1).

Part (iii): States with half-integer m_s acquire factor -1 at $\varphi = 2\pi$ (from part (ii)) and factor $\mathrm{e}^{\mathrm{i}4\pi m_s} = +1$ at $\varphi = 4\pi$ (from part (i)). They are therefore spinors in the precise sense: single-valued on the 4π cover but double-valued on the 2π base. □

Remark 3.4. *The factor of* -1 *under* 2π *rotation is not unphysical. While overall phases are unobservable in a single state,* relative *phases between components of a superposition are observable. For a spin-*$\frac{1}{2}$ *particle in the superposition* $|\chi\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$*, a* 2π *rotation sends* $|\uparrow\rangle \to -|\uparrow\rangle$ *and* $|\downarrow\rangle \to -|\downarrow\rangle$ *(both acquire* -1*), leaving the superposition unchanged:* $|\chi\rangle \to -|\chi\rangle$*, which is the same ray in* $\mathbb{C}^2$*. However, for a state entangled between spin and spatial degrees of freedom — such as a neutron passing through a magnetic field region that rotates it by* 2π *on one path and* 0 *on another — the relative phase* -1 *between the two path amplitudes is observable as an interference effect. This phase shift has been directly measured in neutron interferometry experiments and confirms the* 4π *periodicity of spinor states.*

3.2 The Spin Spectrum from the Ladder Argument

With the double-cover holonomy having established $m_s \in \frac{1}{2}\mathbb{Z}$, the complete spin spectrum is derived by the same ladder argument used in QM5 for the orbital case. The argument is self-contained, referring to the SU(2) algebra directly.

Theorem 3.5 (Complete spin spectrum)**.** *For the* SU(2) *commutation algebra* $[\hat{\mathbf{S}}_j, \hat{\mathbf{S}}_k] = \mathrm{i}\Phi_0\, \epsilon_{jkl}\hat{\mathbf{S}}_l$*, the spectrum of* $\hat{S}^2$ *and* $\hat{S}_3$ *on an irreducible representation of spin* s *is:*

$$\hat{S}^2\,|s, m_s\rangle = s(s+1)\Phi_0^2\,|s, m_s\rangle, \tag{11}$$

$$\hat{S}_3\,|s, m_s\rangle = m_s\Phi_0\,|s, m_s\rangle, \tag{12}$$

for $s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ and $m_s \in \{-s, -s+1, \ldots, +s\}$, a total of $2s+1$ states per irreducible representation. The raising and lowering operators $\hat{S}_+ = \hat{S}_1 + \mathrm{i}\hat{S}_2$ and $\hat{S}_- = \hat{S}_1 - \mathrm{i}\hat{S}_2$ act as:

$$\hat{S}_+ \,|s, m_s\rangle = \sqrt{s(s+1) - m_s(m_s+1)}\;\Phi_0\,|s, m_s+1\rangle, \tag{13}$$

$$\hat{S}_- \,|s, m_s\rangle = \sqrt{s(s+1) - m_s(m_s-1)}\;\Phi_0\,|s, m_s-1\rangle. \tag{14}$$

Proof. The proof applies the QM5 ladder argument (QM5 Secs. **??**–**??**) verbatim to the SU(2) algebra, with the single modification that the allowed range of m_s is extended to $\frac{1}{2}\mathbb{Z}$ by Theorem 3.3.

Step 1: The raising and lowering action. Define $\hat{S}_+ = \hat{S}_1 + \mathrm{i}\hat{S}_2$ and $\hat{S}_- = \hat{S}_1 - \mathrm{i}\hat{S}_2$. From $[\hat{\mathbf{S}}_j, \hat{\mathbf{S}}_k] = \mathrm{i}\Phi_0\epsilon_{jkl}\hat{\mathbf{S}}_l$:

$$[\hat{S}_3, \hat{S}_+] = +\Phi_0\,\hat{S}_+, \qquad [\hat{S}_3, \hat{S}_-] = -\Phi_0\,\hat{S}_-, \qquad [\hat{S}_+, \hat{S}_-] = 2\Phi_0\,\hat{S}_3.$$

If $\hat{S}_3|s, m_s\rangle = m_s\Phi_0|s, m_s\rangle$, then $\hat{S}_3(\hat{S}_+|s, m_s\rangle) = (m_s+1)\Phi_0(\hat{S}_+|s, m_s\rangle)$, so $\hat{S}_+$ raises m_s by 1. Similarly, $\hat{S}_-$ lowers m_s by 1.

Step 2: The Casimir operator. The operator $\hat{S}^2 = \hat{S}_1^2 + \hat{S}_2^2 + \hat{S}_3^2$ commutes with all $\hat{\mathbf{S}}_j$: $[\hat{S}^2, \hat{\mathbf{S}}_j] = 0$. In an irreducible representation, $\hat{S}^2 = c \cdot \hat{\mathbf{1}}$ for some constant $c \geq 0$ (Schur's lemma). Writing $\hat{S}^2$ in terms of the ladder operators: $\hat{S}^2 = \hat{S}_-\hat{S}_+ + \hat{S}_3^2 + \Phi_0\hat{S}_3$, the action on $|s, m_s\rangle$ gives $c = s(s+1)\Phi_0^2$ once the bounds on m_s are established in Step 3.

Step 3: Non-negativity and termination. The matrix element $\left\|\hat{S}_+|s, m_s\rangle\right\|^2 = \langle s, m_s|\hat{S}_-\hat{S}_+|s, m_s\rangle = \langle s, m_s|(\hat{S}^2 - \hat{S}_3^2 - \Phi_0\hat{S}_3)|s, m_s\rangle = [c - m_s(m_s+1)\Phi_0^2] \geq 0$, bounding $m_s \leq s$ where s is the positive root of $c = s(s+1)\Phi_0^2$. Similarly $\left\|\hat{S}_-|s, m_s\rangle\right\|^2 = [c - m_s(m_s-1)\Phi_0^2] \geq 0$ bounds $m_s \geq -s$. Termination: $\hat{S}_+|s, s\rangle = 0$ (the state at the top of the multiplet) and $\hat{S}_-|s, -s\rangle = 0$ (at the bottom).

Step 4: Quantization of s. The ladder from $-s$ to $+s$ in unit steps requires $2s \in \mathbb{Z}_{\geq 0}$, i.e., $s \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. By Theorem 3.3, $m_s \in \frac{1}{2}\mathbb{Z}$, so half-integer s is admitted. Integer s gives the orbital representations; half-integer s gives the spin representations.

Step 5: The matrix elements. From $\left\|\hat{S}_+|s, m_s\rangle\right\|^2 = [s(s+1) - m_s(m_s+1)]\Phi_0^2 \geq 0$, choosing the positive real square root gives Eq. (13). The lowering matrix element Eq. (14) follows by the adjoint relation $\hat{S}_- = (\hat{S}_+)^\dagger$. □

Remark 3.6. *The spectrum established in Theorem 3.5 contains two families, distinguished by the holonomy of the rotation path:*

Family	*Spin quantum number*	*2π rotation factor*
Orbital (QM5)	$s \in \{0, 1, 2, \ldots\}$	$+1$
Spin (QM8)	$s \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}$	-1

The orbital family is the one derived in QM5: the holonomy of the 2π rotation path in $\mathrm{SO}(3)$ forces the transport closure state to be unchanged under a 2π rotation, selecting integer m_s. The spin family is new: the holonomy of the 4π rotation path in $\mathrm{SU}(2)$ allows $m_s \in \mathbb{Z} + \frac{1}{2}$, with the transport closure state acquiring a factor -1 under a 2π rotation. Both families satisfy the same $\mathrm{SU}(2)$ algebra and the same spectral formula $j(j+1)\Phi_0^2$; their difference is topological, not algebraic.

Remark 3.7. *Theorem 3.5 establishes the spectrum for all half-integer s: $s = \frac{1}{2}$ (two-dimensional representation), $s = \frac{3}{2}$ (four-dimensional), $s = \frac{5}{2}$ (six-dimensional), and so on. The present paper develops the $s = \frac{1}{2}$ case in full detail (Sections 4–8), since it is the physically primary case: electrons, protons, neutrons, and neutrinos are all spin-$\frac{1}{2}$ particles. The higher half-integer representations ($s = \frac{3}{2}$ for the Δ baryon, $s = 2$ for the graviton, etc.) use the same algebraic structure with larger matrices and are deferred to later applications.*

Remark 3.8. *For the specific case $s = \frac{1}{2}$, the multiplet has exactly two states: $|\frac{1}{2}, +\frac{1}{2}\rangle = |\uparrow\rangle$ and $|\frac{1}{2}, -\frac{1}{2}\rangle = |\downarrow\rangle$. The matrix elements Eqs.* (13) *and* (14) *reduce to:*

$$\begin{aligned}
\hat{S}_+|\downarrow\rangle &= \sqrt{\tfrac{1}{2}\cdot\tfrac{3}{2} - (-\tfrac{1}{2})\cdot\tfrac{1}{2}}\;\Phi_0|\uparrow\rangle = \sqrt{1}\;\Phi_0|\uparrow\rangle = \Phi_0|\uparrow\rangle,\\
\hat{S}_-|\uparrow\rangle &= \sqrt{\tfrac{1}{2}\cdot\tfrac{3}{2} - \tfrac{1}{2}\cdot(-\tfrac{1}{2})}\;\Phi_0|\downarrow\rangle = \Phi_0|\downarrow\rangle,\\
\hat{S}_+|\uparrow\rangle &= 0, \qquad \hat{S}_-|\downarrow\rangle = 0.
\end{aligned}$$

These are the spin-$\frac{1}{2}$ ladder matrix elements; they determine the Pauli matrix representation of Sec. 4 uniquely. In particular, the eigenvalue of $\hat{S}^2$ on both $|\uparrow\rangle$ and $|\downarrow\rangle$ is $\frac{1}{2}\cdot\frac{3}{2}\Phi_0^2 = \frac{3}{4}\Phi_0^2$, confirming the $j(j+1)\Phi_0^2$ formula.

4 The Pauli Matrices and the Spin-$\frac{1}{2}$ Algebra

The spin-$\frac{1}{2}$ representation established in Theorem 3.5 is two-dimensional: the spin Hilbert space is $\mathbb{C}^2 = \mathbb{C}^2$, with basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ corresponding to $m_s = +\frac{1}{2}$ and $m_s = -\frac{1}{2}$. The spin operators $\hat{\mathbf{S}}_j$ are therefore represented by 2×2 complex matrices on $\mathbb{C}^2$. The present section determines these matrices uniquely from the SU(2) algebra and the ladder matrix elements of Remark 3.8, identifies them as the Pauli matrices, and derives the complete algebraic structure that the Pauli matrices satisfy. The Pauli algebra has two layers: the SU(2) commutation algebra $[\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_k] = 2\mathrm{i}\epsilon_{jkl}\boldsymbol{\sigma}_l$ (shared with the orbital angular momentum algebra of QM5 at the appropriate scale) and the Clifford anticommutation algebra $\{\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_k\} = 2\delta_{jk}\sigma_0$ (new to the spin-$\frac{1}{2}$ representation, not present in the orbital case). Both layers are encoded in a single product formula, derived as a theorem.

4.1 The Spin-$\frac{1}{2}$ Hilbert Space and the Unique Matrix Representation

Definition 4.1 (Spin-$\frac{1}{2}$ Hilbert space and basis)**.** *The* spin-$\frac{1}{2}$ Hilbert space *is $\mathbb{C}^2 = \mathbb{C}^2$ with the standard inner product $\langle\chi, \chi'\rangle_{\mathbb{C}^2} = \sum_{i=1}^{2} \overline{\chi_i}\chi'_i$. The* spin eigenbasis *consists of:*

$$|\uparrow\rangle := \begin{pmatrix}1\\0\end{pmatrix}, \qquad |\downarrow\rangle := \begin{pmatrix}0\\1\end{pmatrix}, \tag{15}$$

satisfying $\hat{S}_3|\uparrow\rangle = +(\Phi_0/2)|\uparrow\rangle$ and $\hat{S}_3|\downarrow\rangle = -(\Phi_0/2)|\downarrow\rangle$. These form a complete orthonormal basis for $\mathbb{C}^2$: $\langle\uparrow,\uparrow\rangle = \langle\downarrow,\downarrow\rangle = 1$, $\langle\uparrow,\downarrow\rangle = 0$, and $|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| = \sigma_0$ (the 2×2 identity).

The spin-$\frac{1}{2}$ operators are 2×2 matrices determined by the conditions: (a) they satisfy $[\hat{\mathbf{S}}_j, \hat{\mathbf{S}}_k] = \mathrm{i}\Phi_0\epsilon_{jkl}\hat{\mathbf{S}}_l$, (b) $\hat{S}_3$ is diagonal in the spin eigenbasis with eigenvalues $\pm\Phi_0/2$, and (c) the ladder matrix elements of Remark 3.8 are satisfied.

Lemma 4.2 (Uniqueness of the spin-$\frac{1}{2}$ matrix representation)**.** *Up to unitary equivalence, the unique 2×2 matrix representation of the* SU(2) *algebra with $\hat{S}_3$ diagonal in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ and spin quantum number $s = \frac{1}{2}$ is:*

$$\hat{\mathbf{S}}_j = \frac{\Phi_0}{2}\boldsymbol{\sigma}_j, \tag{16}$$

where the Pauli matrices *are*

$$\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{17}$$

Proof. The condition $\hat{S}_3|\uparrow\rangle = +(\Phi_0/2)|\uparrow\rangle$ and $\hat{S}_3|\downarrow\rangle = -(\Phi_0/2)|\downarrow\rangle$ in the basis Eq. (15) fixes $\hat{S}_3 = (\Phi_0/2)\sigma_3$ uniquely. The ladder matrix elements of Remark 3.8 give $\hat{S}_+ = \Phi_0|\uparrow\rangle\langle\downarrow|$ and $\hat{S}_- = \Phi_0|\downarrow\rangle\langle\uparrow|$ as the raising and lowering matrices. In the basis Eq. (15): $\hat{S}_+ = \Phi_0\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ and $\hat{S}_- = \Phi_0\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$. Recovering $\hat{S}_1$ and $\hat{S}_2$ from $\hat{S}_1 = (\hat{S}_+ + \hat{S}_-)/2$ and $\hat{S}_2 = (\hat{S}_+ - \hat{S}_-)/(2\mathrm{i})$:

$$\hat{S}_1 = \frac{\Phi_0}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \frac{\Phi_0}{2}\sigma_1, \qquad \hat{S}_2 = \frac{\Phi_0}{2}\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix} = \frac{\Phi_0}{2}\sigma_2,$$

confirming Eq. (16). Uniqueness up to unitary equivalence follows from the fact that the irreducible two-dimensional representation of SU(2) is unique (up to isomorphism), and any two choices of basis are related by a unitary transformation. □

Remark 4.3. *Each Pauli matrix is Hermitian ($\boldsymbol{\sigma}_j^\dagger = \boldsymbol{\sigma}_j$) and traceless ($\mathrm{Tr}(\boldsymbol{\sigma}_j) = 0$). Hermiticity ensures that $\hat{\mathbf{S}}_j = (\Phi_0/2)\boldsymbol{\sigma}_j$ are self-adjoint, as required for physical observables. Tracelessness reflects the equal and opposite eigenvalues $\pm\Phi_0/2$ of each spin component (since the trace equals the sum of eigenvalues). The four matrices $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ form a basis for the space of all 2×2 complex matrices: any 2×2 matrix A can be written uniquely as $A = a_0\sigma_0 + a_j\boldsymbol{\sigma}_j$ with $a_0 = \frac{1}{2}\mathrm{Tr}(A)$ and $a_j = \frac{1}{2}\mathrm{Tr}(A\boldsymbol{\sigma}_j)$.*

4.2 The Pauli Algebra

The Pauli matrices satisfy a rich algebraic structure that encodes simultaneously the SU(2) Lie algebra (via the commutator), the real Clifford algebra $\mathrm{Cl}_{3,0}$ (via the anticommutator), and the group SU(2) itself (via the exponential map).

Theorem 4.4 (The Pauli algebra)**.** *The Pauli matrices satisfy the following identities:*

(i) Product formula:

$$\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k = \delta_{jk}\sigma_0 + \mathrm{i}\epsilon_{jkl}\boldsymbol{\sigma}_l. \tag{18}$$

(ii) Anticommutation (Clifford algebra):

$$\{\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_k\} = 2\delta_{jk}\sigma_0. \tag{19}$$

(iii) Commutation (SU(2) algebra):

$$[\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_k] = 2\mathrm{i}\epsilon_{jkl}\boldsymbol{\sigma}_l. \tag{20}$$

(iv) Square and determinant:

$$\boldsymbol{\sigma}_j^2 = \sigma_0, \qquad \det(\boldsymbol{\sigma}_j) = -1, \qquad j = 1, 2, 3. \tag{21}$$

(v) Trace identities:

$$\mathrm{Tr}(\boldsymbol{\sigma}_j) = 0, \qquad \mathrm{Tr}(\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k) = 2\delta_{jk}, \qquad \mathrm{Tr}(\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k\boldsymbol{\sigma}_l) = 2\mathrm{i}\epsilon_{jkl}. \tag{22}$$

(vi) Completeness (resolution of 2×2 matrices):

$$A = \frac{\mathrm{Tr}(A)}{2}\sigma_0 + \frac{\mathrm{Tr}(A\boldsymbol{\sigma}_j)}{2}\boldsymbol{\sigma}_j \tag{23}$$

for any 2×2 complex matrix A.

Proof. All identities follow by direct matrix computation from Definition 4.1.

(i) Product formula: Compute $\sigma_1\sigma_1$, $\sigma_1\sigma_2$, etc. explicitly:

$$\sigma_1\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma_0,$$

confirming $\delta_{11}\sigma_0 + \mathrm{i}\epsilon_{11l}\boldsymbol{\sigma}_l = \sigma_0 + 0 = \sigma_0$.

$$\sigma_1\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} = \mathrm{i}\sigma_3,$$

confirming $\delta_{12}\sigma_0 + \mathrm{i}\epsilon_{12l}\boldsymbol{\sigma}_l = 0 + \mathrm{i}\sigma_3 = \mathrm{i}\sigma_3$. The remaining six products $\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k$ (three with $j = k$ and three with $j \neq k$) are computed identically; in each case the result matches the right-hand side of Eq. (18), establishing the formula for all index pairs.

(ii) and (iii): Take the symmetric and antisymmetric parts of Eq. (18): $\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k + \boldsymbol{\sigma}_k\boldsymbol{\sigma}_j = 2\delta_{jk}\sigma_0$ (since ϵ_{jkl} is antisymmetric, its contribution cancels in the sum), giving Eq. (19). $\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k - \boldsymbol{\sigma}_k\boldsymbol{\sigma}_j = 2\mathrm{i}\epsilon_{jkl}\boldsymbol{\sigma}_l$ (the δ_{jk} term cancels), giving Eq. (20).

(iv): From Eq. (18) with $j = k$: $\boldsymbol{\sigma}_j^2 = \delta_{jj}\sigma_0 + \mathrm{i}\epsilon_{jjl}\boldsymbol{\sigma}_l = \sigma_0 + 0 = \sigma_0$. Determinant: for σ_3, $\det\sigma_3 = (1)(-1) - (0)(0) = -1$; the others follow by direct computation.

(v): $\mathrm{Tr}(\boldsymbol{\sigma}_j) = 0$ by inspection of each matrix. $\mathrm{Tr}(\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k) = \mathrm{Tr}(\delta_{jk}\sigma_0 + \mathrm{i}\epsilon_{jkl}\boldsymbol{\sigma}_l) = \delta_{jk}\mathrm{Tr}(\sigma_0) + \mathrm{i}\epsilon_{jkl}\mathrm{Tr}(\boldsymbol{\sigma}_l) = 2\delta_{jk} + 0 = 2\delta_{jk}$. $\mathrm{Tr}(\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k\boldsymbol{\sigma}_l)$: use Eq. (18) to write $\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k = \delta_{jk}\sigma_0 + \mathrm{i}\epsilon_{jkm}\boldsymbol{\sigma}_m$, then $\mathrm{Tr}(\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k\boldsymbol{\sigma}_l) = \delta_{jk}\mathrm{Tr}(\boldsymbol{\sigma}_l) + \mathrm{i}\epsilon_{jkm}\mathrm{Tr}(\boldsymbol{\sigma}_m\boldsymbol{\sigma}_l) = 0 + \mathrm{i}\epsilon_{jkm} \cdot 2\delta_{ml} = 2\mathrm{i}\epsilon_{jkl}$.

(vi): Since $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ spans the 4-dimensional space of 2×2 complex matrices, any A expands as $A = a_0\sigma_0 + a_j\boldsymbol{\sigma}_j$. Taking the trace of both sides: $\mathrm{Tr}(A) = 2a_0$ (using $\mathrm{Tr}(\sigma_0) = 2$ and $\mathrm{Tr}(\boldsymbol{\sigma}_j) = 0$), so $a_0 = \mathrm{Tr}(A)/2$. Taking $\mathrm{Tr}(A\boldsymbol{\sigma}_k) = a_0\mathrm{Tr}(\boldsymbol{\sigma}_k) + a_j\mathrm{Tr}(\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k) = 0 + 2a_k\delta_{jk} = 2a_k$, so $a_k = \mathrm{Tr}(A\boldsymbol{\sigma}_k)/2$, confirming Eq. (23). □

Remark 4.5. *The product formula Eq.* (18) *encodes two distinct algebraic structures simultaneously. The antisymmetric part* $[\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_k] = 2\mathrm{i}\epsilon_{jkl}\boldsymbol{\sigma}_l$ *is, up to the factor* $\Phi_0/2$*, the same* SO(3) *commutation algebra satisfied by the orbital angular momentum operators of QM5:* $[\hat{L}_j, \hat{L}_k] = \mathrm{i}\Phi_0\epsilon_{jkl}\hat{L}_l$ *corresponds to* $[\hat{\mathbf{S}}_j, \hat{\mathbf{S}}_k] = \mathrm{i}\Phi_0\epsilon_{jkl}\hat{\mathbf{S}}_l$*, which in Pauli matrix form is* $[(\Phi_0/2)\boldsymbol{\sigma}_j, (\Phi_0/2)\boldsymbol{\sigma}_k] = \mathrm{i}\Phi_0\epsilon_{jkl}(\Phi_0/2)\boldsymbol{\sigma}_l$*, consistent with Eq.* (20) *after dividing by* $\Phi_0/2$*. The symmetric part* $\{\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_k\} = 2\delta_{jk}\sigma_0$ *is the Clifford algebra* Cl_3 *in three Euclidean dimensions: a set of generators that anticommute with each other and square to the identity. This Clifford structure has no orbital analogue: the orbital angular momentum operators* $\hat{L}_j$ *on* $\mathcal{H}$ *satisfy* $\{\hat{L}_j, \hat{L}_k\} \neq 2\delta_{jk}$ *in general (the anticommutator of two differential operators is not a scalar multiple of the identity on an infinite-dimensional space). The Clifford structure is specific to the spin-*$\frac{1}{2}$ *representation and is the algebraic foundation for the Dirac equation of QM11.*

Remark 4.6. *For a unit vector* $\hat{n} = (n_1, n_2, n_3)$ *and angle* $\varphi \in \mathbb{R}$*, the exponential of a Pauli matrix combination takes the closed form:*

$$\exp\Big(-\mathrm{i}\frac{\varphi}{2}\hat{n} \cdot \boldsymbol{\sigma}\Big) = \cos\Big(\frac{\varphi}{2}\Big)\,\sigma_0 - \mathrm{i}\sin\Big(\frac{\varphi}{2}\Big)\,\hat{n} \cdot \boldsymbol{\sigma}, \tag{24}$$

derived using $(\hat{n} \cdot \boldsymbol{\sigma})^2 = \sigma_0$ *(which follows from Eq.* (19) *and* $|\hat{n}|^2 = 1$*) and the Taylor series for the exponential. This is the* SU(2) *element corresponding to a rotation by angle* φ *about the* $\hat{n}$ *axis, confirming that the Pauli matrices are the generators of* SU(2) *in the spin-*$\frac{1}{2}$ *representation. For* $\varphi = 2\pi$*:* $\exp(-\mathrm{i}\pi\hat{n} \cdot \boldsymbol{\sigma}) = \cos(\pi)\sigma_0 - \mathrm{i}\sin(\pi)\hat{n} \cdot \boldsymbol{\sigma} = -\sigma_0$*, recovering the spinor sign change of Theorem 3.3 (ii). For* $\varphi = 4\pi$*:* $\exp(-\mathrm{i}2\pi\hat{n} \cdot \boldsymbol{\sigma}) = +\sigma_0$*, confirming the* 4π *periodicity.*

4.3 Spin-$\frac{1}{2}$ Eigenstates and the Spinor Basis

Proposition 4.7 (Spin-$\frac{1}{2}$ eigenstate structure)**.** *The complete eigenvalue structure of the spin-$\frac{1}{2}$ operators on $\mathbb{C}^2 = \mathbb{C}^2$ is as follows.*

(i) Eigenstates of $\hat{S}_3$:

$$\hat{S}_3|\uparrow\rangle = +\frac{\Phi_0}{2}|\uparrow\rangle, \qquad \hat{S}_3|\downarrow\rangle = -\frac{\Phi_0}{2}|\downarrow\rangle. \tag{25}$$

(ii) Eigenstate of $\hat{S}^2$:

$$\hat{S}^2|\uparrow\rangle = \hat{S}^2|\downarrow\rangle = \frac{3}{4}\Phi_0^2|\uparrow\rangle = \frac{3}{4}\Phi_0^2|\downarrow\rangle. \tag{26}$$

(iii) Action of the ladder operators:

$$\hat{S}_+|\downarrow\rangle = \Phi_0|\uparrow\rangle, \quad \hat{S}_-|\uparrow\rangle = \Phi_0|\downarrow\rangle, \quad \hat{S}_+|\uparrow\rangle = 0, \quad \hat{S}_-|\downarrow\rangle = 0. \tag{27}$$

(iv) Eigenstates of $\hat{S}_1$:

$$\hat{S}_1\begin{pmatrix}1\\+1\end{pmatrix}\frac{1}{\sqrt{2}} = +\frac{\Phi_0}{2}\begin{pmatrix}1\\+1\end{pmatrix}\frac{1}{\sqrt{2}}, \qquad \hat{S}_1\begin{pmatrix}1\\-1\end{pmatrix}\frac{1}{\sqrt{2}} = -\frac{\Phi_0}{2}\begin{pmatrix}1\\-1\end{pmatrix}\frac{1}{\sqrt{2}}. \tag{28}$$

(v) Eigenstates of $\hat{S}_2$:

$$\hat{S}_2\begin{pmatrix}1\\+\mathrm{i}\end{pmatrix}\frac{1}{\sqrt{2}} = +\frac{\Phi_0}{2}\begin{pmatrix}1\\+\mathrm{i}\end{pmatrix}\frac{1}{\sqrt{2}}, \qquad \hat{S}_2\begin{pmatrix}1\\-\mathrm{i}\end{pmatrix}\frac{1}{\sqrt{2}} = -\frac{\Phi_0}{2}\begin{pmatrix}1\\-\mathrm{i}\end{pmatrix}\frac{1}{\sqrt{2}}. \tag{29}$$

Proof. Parts (i) and (ii): $\hat{S}_3 = (\Phi_0/2)\sigma_3 = (\Phi_0/2)\mathrm{diag}(1,-1)$; the diagonal entries are the eigenvalues, with eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$. $\hat{S}^2 = (\Phi_0/2)^2(\sigma_1^2+\sigma_2^2+\sigma_3^2) = (\Phi_0/2)^2\cdot 3\sigma_0 = (3/4)\Phi_0^2\sigma_0$, using $\boldsymbol{\sigma}_j^2 = \sigma_0$ from Eq. (21). *Part (iii):* Direct from $\hat{S}_+ = (\Phi_0/2)(\sigma_1 + \mathrm{i}\sigma_2) = \Phi_0\begin{pmatrix}0&1\\0&0\end{pmatrix}$ and $\hat{S}_- = \Phi_0\begin{pmatrix}0&0\\1&0\end{pmatrix}$. *Parts (iv) and (v):* Solve the eigenvalue equation $\hat{S}_1\chi = \pm(\Phi_0/2)\chi$ using $\hat{S}_1 = (\Phi_0/2)\sigma_1$; the characteristic equation $\det(\sigma_1 - \lambda\sigma_0) = -\lambda^2 + 1 = 0$ gives $\lambda = \pm 1$, with eigenvectors as stated. Similarly for $\hat{S}_2$. □

Remark 4.8. *The general normalized spin-$\frac{1}{2}$ state is $|\chi\rangle = \cos(\theta/2)|\uparrow\rangle + e^{\mathrm{i}\phi}\sin(\theta/2)|\downarrow\rangle$ for $\theta \in [0,\pi]$ and $\phi \in [0, 2\pi)$, a point on the* Bloch sphere S^2. *The expectation values of the spin components are:*

$$\langle\hat{\mathbf{S}}\rangle = \frac{\Phi_0}{2}(\sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta) = \frac{\Phi_0}{2}\hat{n}(\theta,\phi), \tag{30}$$

where $\hat{n}(\theta,\phi)$ is the unit vector in the (θ,ϕ) direction. The spin-up state $|\uparrow\rangle$ corresponds to the north pole ($\theta = 0$) and spin-down $|\downarrow\rangle$ to the south pole ($\theta = \pi$). The eigenstates of $\hat{S}_1$ of Proposition 4.7 (iv) correspond to $\theta = \pi/2$, $\phi = 0$ and $\theta = \pi/2$, $\phi = \pi$ (the equatorial points on the x-axis), and similarly for $\hat{S}_2$. The Bloch sphere parametrizes the projective *space $\mathbb{CP}^1$, the space of rays in $\mathbb{C}^2$, not $\mathbb{C}^2$ itself; the overall phase of $|\chi\rangle$ is physically unobservable and is removed by the projectivization. The time evolution under the Zeeman Hamiltonian $\hat{H}_Z$ of Sec. 6 rotates the Bloch sphere point (θ,ϕ) at constant θ, tracing a circle of latitude on the Bloch sphere at angular velocity ω_L — the Larmor precession of Sec. 6.3.*

Remark 4.9. *For reference, the key properties of the Pauli matrices established in the present section are collected:*

Property	Statement
Hermitian	$\boldsymbol{\sigma}_j^\dagger = \boldsymbol{\sigma}_j$
Traceless	$\mathrm{Tr}(\boldsymbol{\sigma}_j) = 0$
Square	$\boldsymbol{\sigma}_j^2 = \sigma_0$
Determinant	$\det(\boldsymbol{\sigma}_j) = -1$
Anticommutator	$\{\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_k\} = 2\delta_{jk}\sigma_0$
Commutator	$[\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_k] = 2\mathrm{i}\epsilon_{jkl}\boldsymbol{\sigma}_l$
Product	$\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k = \delta_{jk}\sigma_0 + \mathrm{i}\epsilon_{jkl}\boldsymbol{\sigma}_l$
Trace of product	$\mathrm{Tr}(\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k) = 2\delta_{jk}$
Exponential	$\mathrm{e}^{-\mathrm{i}\varphi\hat{n}\cdot\boldsymbol{\sigma}/2} = \cos(\varphi/2)\sigma_0 - \mathrm{i}\sin(\varphi/2)\hat{n}\cdot\boldsymbol{\sigma}$
Eigenvalues	± 1 *for each* $\boldsymbol{\sigma}_j$
Spin eigenvalues	$\pm\Phi_0/2$ *for each* $\hat{\mathbf{S}}_j = (\Phi_0/2)\boldsymbol{\sigma}_j$

These properties are the primary computational tools for all calculations in Secs. 5–8 and throughout QM9-QM11.

5 The Full Spin-$\frac{1}{2}$ Hilbert Space

The spin-$\frac{1}{2}$ Hilbert space $\mathbb{C}^2 = \mathbb{C}^2$ of the preceding section describes the internal spin degree of freedom of a transport closure configuration in isolation from its spatial motion. Physical particles have both spatial and spin degrees of freedom simultaneously, and the full state space that accommodates both is the tensor product $\mathcal{H}_{\text{full}} = \mathcal{H} \otimes \mathbb{C}^2$, the simplest non-trivial application of the QM7 tensor product construction to a mixed finite-dimensional and infinite-dimensional setting. The present section constructs $\mathcal{H}_{\text{full}}$ explicitly, records its inner product and the two-component spinor representation of its elements, establishes the observable algebra on $\mathcal{H}_{\text{full}}$ including the commutation of spatial and spin operators, and derives the Pauli equation — the spin-$\frac{1}{2}$ Schrödinger equation on $\mathcal{H}_{\text{full}}$ — from the QM4 dynamical framework applied to the full space.

5.1 Construction of $\mathcal{H}_{\text{full}} = \mathcal{H} \otimes \mathbb{C}^2$

Definition 5.1 (Full spin-$\frac{1}{2}$ Hilbert space)**.** *The* full single-particle Hilbert space *for a spin-$\frac{1}{2}$ transport closure configuration is*

$$\mathcal{H}_{\text{full}} := \mathcal{H} \otimes \mathbb{C}^2 = \mathcal{H} \otimes \mathbb{C}^2, \tag{31}$$

constructed as the tensor product of the spatial Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C})$ and the spin Hilbert space $\mathbb{C}^2 = \mathbb{C}^2$ via QM7 Definition 3.1. The inner product on $\mathcal{H}_{\text{full}}$ is

$$\langle \chi, \chi' \rangle_{\mathcal{H}_{\text{full}}} = \langle \Psi_\uparrow, \Psi'_\uparrow \rangle_{\mathcal{H}} + \langle \Psi_\downarrow, \Psi'_\downarrow \rangle_{\mathcal{H}} = \int_{\mathbb{R}^3} \left(\overline{\Psi_\uparrow(\mathbf{x})}\Psi'_\uparrow(\mathbf{x}) + \overline{\Psi_\downarrow(\mathbf{x})}\Psi'_\downarrow(\mathbf{x}) \right) \mathrm{d}^3x, \tag{32}$$

where elements of $\mathcal{H}_{\text{full}}$ are identified with pairs $(\Psi_\uparrow, \Psi_\downarrow) \in \mathcal{H} \times \mathcal{H}$ via the spinor representation of Definition 5.3.

Remark 5.2. *The isomorphism $\mathcal{H}_{\text{full}} = \mathcal{H} \otimes \mathbb{C}^2 \cong L^2(\mathbb{R}^3, \mathbb{C}^2)$ follows from the general tensor product isomorphism of QM7 Proposition 3.2 applied to $\mathcal{H}_A = L^2(\mathbb{R}^3)$ and $\mathcal{H}_B = \mathbb{C}^2$: since $\mathbb{C}^2$ is finite-dimensional with standard basis $\{|\uparrow\rangle, |\downarrow\rangle\}$, the product Hilbert space is isomorphic to the space of $\mathbb{C}^2$-valued functions on $\mathbb{R}^3$ that are square-integrable component-wise. Concretely, an element of $\mathcal{H}_{\text{full}}$ is a function $\mathbf{x} \mapsto \chi(\mathbf{x}) \in \mathbb{C}^2$ with $\int_{\mathbb{R}^3} \|\chi(\mathbf{x})\|^2_{\mathbb{C}^2}\, \mathrm{d}^3x < \infty$, where $\|\cdot\|_{\mathbb{C}^2}$ is the standard Euclidean norm on $\mathbb{C}^2$. The inner product Eq. (32) is the $L^2(\mathbb{R}^3, \mathbb{C}^2)$ inner product restricted to two-component complex functions.*

5.2 The Two-Component Spinor Wave Function

Definition 5.3 (Two-component spinor wave function)**.** *A general element of $\mathcal{H}_{\text{full}}$ is a* two-component spinor wave function*:*

$$\chi(\mathbf{x}) = \begin{pmatrix} \Psi_\uparrow(\mathbf{x}) \\ \Psi_\downarrow(\mathbf{x}) \end{pmatrix} = \Psi_\uparrow(\mathbf{x})\,|\uparrow\rangle + \Psi_\downarrow(\mathbf{x})\,|\downarrow\rangle, \tag{33}$$

where $\Psi_\uparrow, \Psi_\downarrow \in \mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C})$ are the spin-up component *and* spin-down component *of the state respectively. The normalization condition $\langle \chi, \chi \rangle_{\mathcal{H}_{\text{full}}} = 1$ reads:*

$$\int_{\mathbb{R}^3} \left(|\Psi_\uparrow(\mathbf{x})|^2 + |\Psi_\downarrow(\mathbf{x})|^2 \right) \mathrm{d}^3x = 1. \tag{34}$$

The joint closure density is $|\chi(\mathbf{x})|^2 = |\Psi_\uparrow(\mathbf{x})|^2 + |\Psi_\downarrow(\mathbf{x})|^2$, the total probability density at position $\mathbf{x}$ summed over spin components.

Proposition 5.4 (Complete orthonormal basis for $\mathcal{H}_{\text{full}}$)**.** *Let $\{\phi_j\}_{j\geq 1}$ be a complete orthonormal basis for $\mathcal{H}$. Then the four families of product states*

$$\{\phi_j \otimes |\uparrow\rangle,\ \phi_j \otimes |\downarrow\rangle\}_{j\geq 1} \tag{35}$$

form a complete orthonormal basis for $\mathcal{H}_{\text{full}}$, with

$$\big\langle \phi_j \otimes |m_s\rangle, \phi_k \otimes |m_s'\rangle \big\rangle_{\mathcal{H}_{\text{full}}} = \delta_{jk}\, \delta_{m_s m_s'}. \tag{36}$$

The resolution of the identity on $\mathcal{H}_{\text{full}}$ is

$$\hat{\mathbf{1}}_{\mathcal{H}_{\text{full}}} = \sum_{j=1}^{\infty} \sum_{m_s=\pm 1/2} |\phi_j \otimes m_s\rangle\langle \phi_j \otimes m_s|. \tag{37}$$

Proof. Orthonormality follows directly from Eq. (32) and the orthonormality of $\{\phi_j\}$ in $\mathcal{H}$ combined with the orthonormality of $\{|\uparrow\rangle, |\downarrow\rangle\}$ in $\mathbb{C}^2$: $\langle \phi_j \otimes m_s, \phi_k \otimes m_s' \rangle_{\mathcal{H}_{\text{full}}} = \langle \phi_j, \phi_k \rangle_{\mathcal{H}} \langle m_s, m_s' \rangle_{\mathbb{C}^2} = \delta_{jk}\delta_{m_s m_s'}$. Completeness: any $\chi \in \mathcal{H}_{\text{full}}$ has components $\Psi_\uparrow, \Psi_\downarrow \in \mathcal{H}$, each of which expands in $\{\phi_j\}$: $\Psi_\uparrow = \sum_j \langle \phi_j, \Psi_\uparrow \rangle\, \phi_j$ and $\Psi_\downarrow = \sum_j \langle \phi_j, \Psi_\downarrow \rangle\, \phi_j$. Therefore $\chi = \sum_j [c_j^+ \phi_j \otimes |\uparrow\rangle + c_j^- \phi_j \otimes |\downarrow\rangle]$ where $c_j^\pm = \left\langle \phi_j, \Psi_\uparrow/\Psi_\downarrow \right\rangle_{\mathcal{H}}$, with $\sum_j (|c_j^+|^2 + |c_j^-|^2) = \|\Psi_\uparrow\|^2 + \|\Psi_\downarrow\|^2 = 1 < \infty$. □

5.3 The Observable Algebra on $\mathcal{H}_{\text{full}}$

Proposition 5.5 (Spatial and spin observables on $\mathcal{H}_{\text{full}}$)**.** *The spatial observable extensions and spin observable extensions on $\mathcal{H}_{\text{full}}$ are:*

$$\hat{A}_{\text{spatial}} \coloneqq A \otimes \sigma_0, \qquad A \textit{ self-adjoint on } \mathcal{H}, \tag{38}$$

$$\hat{B}_{\text{spin}} \coloneqq \hat{\mathbf{1}}_{\mathcal{H}} \otimes B, \qquad B \textit{ Hermitian on } \mathbb{C}^2, \tag{39}$$

acting on the spinor wave function Eq. (33) as:

$$(A \otimes \sigma_0)\chi = \begin{pmatrix} A\Psi_\uparrow \\ A\Psi_\downarrow \end{pmatrix}, \tag{40}$$

$$(\hat{\mathbf{1}} \otimes B)\chi = B \begin{pmatrix} \Psi_\uparrow \\ \Psi_\downarrow \end{pmatrix} = \begin{pmatrix} B_{11}\Psi_\uparrow + B_{12}\Psi_\downarrow \\ B_{21}\Psi_\uparrow + B_{22}\Psi_\downarrow \end{pmatrix}, \tag{41}$$

where B_{ij} are the matrix elements of B in the $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis. All spatial observables commute with all spin observables:

$$[A \otimes \sigma_0,\ \hat{\mathbf{1}} \otimes B] = 0 \quad \textit{on } \mathcal{H}_{\text{full}}. \tag{42}$$

Proof. The actions Eqs. (40) and (41) follow from the tensor product action on simple tensors (QM7 Definition 3.3) extended by linearity to all spinors via the expansion in Proposition 5.4. The commutation Eq. (42) is QM7 Proposition 3.3 in the specific case $\mathcal{H}_A = \mathcal{H}$ and $\mathcal{H}_B = \mathbb{C}^2 = \mathbb{C}^2$: spatial observables act on the first factor and spin observables on the second, and operators on different tensor factors commute. □

Remark 5.6. *The commutation Eq.* (42) *is the algebraic expression of the independence of spatial and spin degrees of freedom. In particular:*

$$[\hat{x}^j \otimes \sigma_0,\ \hat{\mathbf{1}} \otimes \hat{\mathbf{S}}_k] = 0, \tag{43}$$

$$[\hat{p}_j \otimes \sigma_0,\ \hat{\mathbf{1}} \otimes \hat{\mathbf{S}}_k] = 0, \tag{44}$$

$$[\hat{L}_j \otimes \sigma_0,\ \hat{\mathbf{1}} \otimes \hat{\mathbf{S}}_k] = 0. \tag{45}$$

The last relation Eq. (45) *is the key identity for the total angular momentum analysis of Sec. 7: the total angular momentum $\hat{J}_j = \hat{L}_j \otimes \sigma_0 + \hat{\mathbf{1}} \otimes \hat{\mathbf{S}}_j$ satisfies the* SO(3) *algebra because the orbital-spin cross commutators vanish. The relation $[\hat{x}^j, \hat{\mathbf{S}}_k] = 0$ means that the position of the particle carries no spin label: spin and position are independent degrees of freedom, consistent with the tensor product construction.*

5.4 The Total Hamiltonian on $\mathcal{H}_{\mathrm{full}}$

Definition 5.7 (Total spin-$\frac{1}{2}$ Hamiltonian)**.** *For a spin-$\frac{1}{2}$ particle in a potential $V(\mathbf{x})$ and a magnetic field $\mathbf{B}(\mathbf{x})$, the* total Hamiltonian *on $\mathcal{H}_{\mathrm{full}}$ is*

$$\hat{H}_{\mathrm{Pauli}} := \hat{H}_0 \otimes \sigma_0 + \hat{H}_Z, \tag{46}$$

where $\hat{H}_0 = \hat{T} + \hat{V}$ is the spin-independent spatial Hamiltonian (QM4 Theorem 4.2) and $\hat{H}_Z$ is the Zeeman coupling of Sec. 6. The total Hamiltonian is self-adjoint on the domain $(\mathcal{D}(\hat{H}_0) \otimes \mathbb{C}^2) \cap \mathcal{D}(\hat{H}_Z)$ by the self-adjointness of $\hat{H}_0$ (QM4 Theorem 4.2) and the boundedness of the spin operators on the finite-dimensional $\mathbb{C}^2$.

5.5 The Pauli Equation

Theorem 5.8 (The Pauli equation)**.** *The time evolution of a normalized spin-$\frac{1}{2}$ closure state $\chi(t) \in \mathcal{H}_{\mathrm{full}}$ under the total Hamiltonian $\hat{H}_{\mathrm{Pauli}}$ of Definition 5.7 is governed by the* Pauli equation*:*

$$\mathrm{i}\Phi_0\, \partial_t \chi(\mathbf{x}, t) = \hat{H}_{\mathrm{Pauli}} \chi(\mathbf{x}, t), \tag{47}$$

or equivalently in component form, for a uniform magnetic field $\mathbf{B} = B\hat{z}$:

$$\mathrm{i}\Phi_0\, \partial_t \begin{pmatrix} \Psi_\uparrow \\ \Psi_\downarrow \end{pmatrix} = \left[\hat{H}_0 + \frac{g\mu_B B}{2}\sigma_3\right] \begin{pmatrix} \Psi_\uparrow \\ \Psi_\downarrow \end{pmatrix} = \begin{pmatrix} \hat{H}_0\Psi_\uparrow + \frac{g\mu_B B}{2}\Psi_\uparrow \\ \hat{H}_0\Psi_\downarrow - \frac{g\mu_B B}{2}\Psi_\downarrow \end{pmatrix}. \tag{48}$$

The unitary time evolution operator is $U(t) = \mathrm{e}^{-\mathrm{i}\hat{H}_{\mathrm{Pauli}} t/\Phi_0}$, well-defined on $\mathcal{H}_{\mathrm{full}}$ for all $t \in \mathbb{R}$ by Stone's theorem applied to the self-adjoint generator $\hat{H}_{\mathrm{Pauli}}$ (QM4 Theorem 3.1).

Proof. The Pauli equation Eq. (47) is the Schrödinger equation $\mathrm{i}\Phi_0\, \partial_t \chi = \hat{H}_{\mathrm{Pauli}} \chi$ of QM4 Sec. **??** applied to the Hilbert space $\mathcal{H}_{\mathrm{full}}$ with Hamiltonian $\hat{H}_{\mathrm{Pauli}}$. Self-adjointness of $\hat{H}_{\mathrm{Pauli}}$ on $\mathcal{H}_{\mathrm{full}}$ is established in Definition 5.7; Stone's theorem (QM4 Theorem 3.1) then gives the strongly continuous unitary group $U(t) = \mathrm{e}^{-\mathrm{i}\hat{H}_{\mathrm{Pauli}} t/\Phi_0}$ and the unique solution $\chi(t) = U(t)\chi(0)$ for any initial $\chi(0) \in \mathcal{H}_{\mathrm{full}}$. The component form Eq. (48) follows from the action Eq. (41) of σ_3 on the spinor: $(\hat{\mathbf{1}} \otimes \sigma_3)\chi = (\Psi_\uparrow, -\Psi_\downarrow)^{\mathsf{T}}$, giving the diagonal Zeeman splitting. □

Remark 5.9. *The Pauli equation Eq.* (48) *shows that for a uniform magnetic field along $\hat{z}$, the spin-up and spin-down components of the spinor satisfy* decoupled *Schrödinger equations with shifted energies $E \pm g\mu_B B/2$. This decoupling is a consequence of the diagonal form of σ_3 in the $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis: the magnetic field along $\hat{z}$ does not mix the spin components. For a magnetic field not aligned with $\hat{z}$, the off-diagonal Pauli matrices σ_1 and σ_2 enter, coupling the two components and making the Pauli equation a system of two coupled Schrödinger equations. The eigenvalue problem for $\hat{H}_{\mathrm{Pauli}}$ in the coupled case requires diagonalizing the 2×2 Hermitian matrix formed by the Zeeman term, which is accomplished by a rotation in $\mathbb{C}^2$ that aligns the quantization axis with the magnetic field direction.*

Remark 5.10. *The Pauli equation Eq.* (47) *is the non-relativistic equation governing the spin-$\frac{1}{2}$ closure state. It captures the coupling of the spin magnetic moment to an external field (Zeeman effect, Sec. 6) and the spin-orbit coupling (Sec. 7) at the non-relativistic level. In QM11, the Pauli equation will be derived as the non-relativistic limit of the Dirac equation: the Dirac equation is a first-order relativistic equation on a four-component spinor $(\Psi_\uparrow^L, \Psi_\downarrow^L, \Psi_\uparrow^S, \Psi_\downarrow^S)^\mathsf{T}$ (large and small components), and the Pauli equation emerges as the equation for the large components when the non-relativistic limit $c \to \infty$ is taken. The Clifford algebra of the Pauli matrices (Eq.* (19)*) is the starting point for the Dirac algebra: the Dirac matrices γ^μ are constructed from $(\sigma_0, \boldsymbol{\sigma}_j)$, and the anticommutation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ generalizes Eq.* (19) *to the four-dimensional Minkowski setting. The present paper establishes the Pauli equation as a self-contained non-relativistic result; the Dirac derivation is deferred to QM11.*

Remark 5.11. *The Born frequency law of QB6 extended to $\mathcal{H}_{\mathrm{full}}$ gives the joint closure event rate at position $\mathbf{x}$ and spin m_s as:*

$$\rho(\mathbf{x}, m_s, t) = \begin{cases} |\Psi_\uparrow(\mathbf{x}, t)|^2 & \text{if } m_s = +\frac{1}{2}, \\ |\Psi_\downarrow(\mathbf{x}, t)|^2 & \text{if } m_s = -\frac{1}{2}. \end{cases} \tag{49}$$

The marginal position closure density is $|\chi(\mathbf{x}, t)|^2 = |\Psi_\uparrow|^2 + |\Psi_\downarrow|^2$, as in Definition 5.3, and the marginal spin closure rate is $P(m_s = \pm\frac{1}{2}) = \|\Psi_\uparrow/\Psi_\downarrow\|^2_{\mathcal{H}}$, the total probability of measuring spin up or down summed over all positions. These are the primary observable quantities derived from the spinor wave function in the NUVO framework.

6 The Zeeman Effect

The Zeeman effect is the splitting of energy levels when a magnetic-moment-carrying quantum system is placed in an external magnetic field. For a spin-$\frac{1}{2}$ particle, the magnetic moment arises from the spin angular momentum: the spin is a circulating current in the internal transport closure structure, and a circulating current in a magnetic field acquires a potential energy. The coupling of the spin magnetic moment to the field produces the Zeeman Hamiltonian, which splits the degenerate spin-up and spin-down energy levels into two distinct levels. The time evolution of a general spin state under the Zeeman Hamiltonian is Larmor precession: the spin expectation value rotates about the field axis at the Larmor frequency, a result derived directly from the Heisenberg equation of motion applied to the spin operators. The present section establishes both the static (eigenvalue) and dynamic (precession) aspects of the Zeeman effect from the Pauli algebra of Sec. 4 and the QM4 dynamical framework.

6.1 The Magnetic Moment and the Zeeman Hamiltonian

The spin magnetic moment operator for a spin-$\frac{1}{2}$ particle is proportional to the spin angular momentum, with the constant of proportionality determined by the gyromagnetic ratio.

Definition 6.1 (Spin magnetic moment and Zeeman Hamiltonian)**.** *The* spin magnetic moment operator *for a spin-$\frac{1}{2}$ particle on $\mathbb{C}^2$ is*

$$\hat{\boldsymbol{\mu}} \coloneqq -\frac{g\mu_B}{\Phi_0}\hat{\mathbf{S}} = -\frac{g\mu_B}{2}\boldsymbol{\sigma}, \tag{50}$$

where $\mu_B = \mathrm{e}\Phi_0/(2mc)$ is the Bohr magneton*, g is the* Landé g-factor *(with $g = 2$ for the electron at the Dirac level, deferred to QM11), and $\hat{\mathbf{S}} = (\Phi_0/2)\boldsymbol{\sigma}$ from Eq.* (16)*. The* Zeeman Hamiltonian *for a uniform external magnetic field $\mathbf{B} = \mathbf{B}_j\hat{e}_j$ is*

$$\hat{H}_Z \coloneqq -\hat{\boldsymbol{\mu}}\cdot\mathbf{B} = \frac{g\mu_B}{\Phi_0}\hat{\mathbf{S}}\cdot\mathbf{B} = \frac{g\mu_B}{2}\boldsymbol{\sigma}\cdot\mathbf{B}, \tag{51}$$

acting on $\mathbb{C}^2 = \mathbb{C}^2$. For $\mathbf{B} = B\hat{z}$ (field along the z-axis):

$$\hat{H}_Z = \frac{g\mu_B B}{2}\sigma_3 = \frac{g\mu_B B}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{52}$$

Remark 6.2. *The negative sign in Eq.* (50) *reflects the fact that the spin magnetic moment of an electron is* antiparallel *to its spin angular momentum: for an electron with charge $-\mathrm{e}$, the magnetic moment is $\hat{\boldsymbol{\mu}} = -g\mu_B\hat{\mathbf{S}}/\Phi_0$ rather than $+g\mu_B\hat{\mathbf{S}}/\Phi_0$. This antiparallel relationship is a consequence of the negative charge and gives the result that spin-up electrons (with $m_s = +\frac{1}{2}$) have* lower *energy in a field along $+\hat{z}$ (since $E_+ = +g\mu_B B/2 > 0$ for $g > 0$ and $B > 0$, but the convention in Eq.* (52) *follows the standard physical convention where the spin-up state has energy $+g\mu_B B/2$). The specific sign convention is inherited from the choice of orientation in the Bohr magneton definition and does not affect the magnitude of the splitting.*

6.2 Eigenvalues, Eigenstates, and the Energy Splitting

Theorem 6.3 (Zeeman energy splitting)**.** *For a uniform field $\mathbf{B} = B\hat{z}$, the Zeeman Hamiltonian $\hat{H}_Z = (g\mu_B B/2)\sigma_3$ has eigenvalues*

$$E_\pm = \pm\frac{g\mu_B B}{2}, \tag{53}$$

with eigenstates $|\uparrow\rangle$ (for E_+) and $|\downarrow\rangle$ (for E_-). The Zeeman energy splitting *is*

$$\Delta E = E_+ - E_- = g\mu_B B = \Phi_0\omega_L, \tag{54}$$

where

$$\omega_L \coloneqq \frac{g\mu_B B}{\Phi_0} \tag{55}$$

is the Larmor frequency*.*

Proof. From Eq. (52) and the eigenvalues of σ_3 (Proposition 4.7 (i)): $\sigma_3|\uparrow\rangle = +|\uparrow\rangle$ and $\sigma_3|\downarrow\rangle = -|\downarrow\rangle$, so $\hat{H}_Z|\uparrow\rangle = +(g\mu_B B/2)|\uparrow\rangle$ and $\hat{H}_Z|\downarrow\rangle = -(g\mu_B B/2)|\downarrow\rangle$, giving Eq. (53). The splitting Eq. (54) follows immediately. ◻

Remark 6.4. *For an electron ($g = 2$) in a field of $B = 1$ T (one Tesla), the Zeeman splitting is $\Delta E = 2\mu_B \cdot 1 \text{ T} = 2 \times 9.274 \times 10^{-24} \text{ J} \approx 1.16 \times 10^{-4} \text{ eV}$, corresponding to a Larmor frequency $\omega_L = 2\mu_B/\Phi_0 \approx 1.76 \times 10^{11} \text{ rad s}^{-1}$ in the microwave range. This is the basis for electron paramagnetic resonance (EPR) spectroscopy, in which microwave radiation at the Larmor frequency drives transitions between $|\uparrow\rangle$ and $|\downarrow\rangle$. The full Hamiltonian in the presence of both a static field $\mathbf{B}_0$ and an oscillating microwave field $\mathbf{B}_1(t)$ combines the Zeeman Hamiltonian of the present section with a time-dependent perturbation; the resonance condition at ω_L is established by the time-dependent perturbation theory of a later paper.*

Remark 6.5. *When the spin Hamiltonian $\hat{H}_Z$ is combined with the spatial Hamiltonian $\hat{H}_0$ on $\mathcal{H}_{\text{full}}$, the total Hamiltonian $\hat{H}_{\text{Pauli}} = \hat{H}_0 \otimes \sigma_0 + \hat{\mathbf{1}} \otimes \hat{H}_Z$ has eigenvalues $E_n \pm g\mu_B B/2$, where E_n are the eigenvalues of $\hat{H}_0$. Every spatial energy level E_n thus splits into two levels separated by $\Delta E = g\mu_B B$, independently of the spatial quantum numbers. For the hydrogen atom (QM5 Proposition 7.2), each level $E_{n\ell m}$ splits into $E_{n\ell m,+} = E_n + \mu_B Bg/2$ and $E_{n\ell m,-} = E_n - \mu_B Bg/2$, an effect observable in atomic spectra as the anomalous Zeeman splitting (distinguished from the normal Zeeman effect, which involves the orbital magnetic moment and requires the full $\hat{J}_z$ treatment of Sec. 7).*

6.3 Time Evolution: Larmor Precession

The dynamics of a spin-$\frac{1}{2}$ state under the Zeeman Hamiltonian is the quantum analogue of the classical Larmor precession of a magnetic dipole in a field.

Theorem 6.6 (Larmor precession). *Under the Zeeman Hamiltonian $\hat{H}_Z = (g\mu_B B/2)\sigma_3$ for a field $\mathbf{B} = B\hat{z}$, the Heisenberg-picture spin operators evolve as:*

$$\hat{S}_1(t) = \hat{S}_1(0)\cos(\omega_L t) + \hat{S}_2(0)\sin(\omega_L t), \tag{56}$$

$$\hat{S}_2(t) = -\hat{S}_1(0)\sin(\omega_L t) + \hat{S}_2(0)\cos(\omega_L t), \tag{57}$$

$$\hat{S}_3(t) = \hat{S}_3(0), \tag{58}$$

where $\omega_L = g\mu_B B/\Phi_0$ is the Larmor frequency. Consequently, the expectation value $\langle\hat{\mathbf{S}}\rangle(t)$ traces a circle in the plane perpendicular to $\hat{z}$ at angular frequency ω_L, with $\langle\hat{S}_3\rangle$ constant:

$$\langle\hat{S}_1\rangle(t) = \langle\hat{S}_1\rangle(0)\cos(\omega_L t) + \langle\hat{S}_2\rangle(0)\sin(\omega_L t), \tag{59}$$

$$\langle\hat{S}_2\rangle(t) = -\langle\hat{S}_1\rangle(0)\sin(\omega_L t) + \langle\hat{S}_2\rangle(0)\cos(\omega_L t), \tag{60}$$

$$\langle\hat{S}_3\rangle(t) = \langle\hat{S}_3\rangle(0). \tag{61}$$

Proof. Apply the Heisenberg equation of motion (QM4 Remark 7.1), $\mathrm{d}\hat{\mathbf{S}}_j(t)/\mathrm{d}t = (\mathrm{i}/\Phi_0)[\hat{H}_Z, \hat{\mathbf{S}}_j]$, to each spin component.

$\hat{S}_3$ is conserved: $[\hat{H}_Z, \hat{S}_3] = [(g\mu_B B/2)\sigma_3, (\Phi_0/2)\sigma_3] = (g\mu_B B\Phi_0/4)[\sigma_3, \sigma_3] = 0$. Therefore $\mathrm{d}\hat{S}_3/\mathrm{d}t = 0$, giving $\hat{S}_3(t) = \hat{S}_3(0)$, confirming Eq. (58).

Equations of motion for $\hat{S}_1$ and $\hat{S}_2$: Using $\hat{H}_Z = (g\mu_B B/2)\sigma_3$ and $\hat{\mathbf{S}}_j = (\Phi_0/2)\boldsymbol{\sigma}_j$:

$$[\hat{H}_Z, \hat{S}_1] = \left[\frac{g\mu_B B}{2}\sigma_3, \frac{\Phi_0}{2}\sigma_1\right] = \frac{g\mu_B B\Phi_0}{4}[\sigma_3, \sigma_1].$$

From the Pauli algebra Eq. (20): $[\sigma_3, \sigma_1] = 2\mathrm{i}\epsilon_{312}\boldsymbol{\sigma}_2 = 2\mathrm{i}\sigma_2$. Therefore:

$$[\hat{H}_Z, \hat{S}_1] = \frac{g\mu_B B\Phi_0}{4} \cdot 2\mathrm{i}\sigma_2 = \mathrm{i}g\mu_B B \cdot \frac{\Phi_0}{2}\sigma_2 = \mathrm{i}\Phi_0\omega_L\,\hat{S}_2.$$

The Heisenberg equation gives:

$$\frac{\mathrm{d}\hat{S}_1}{\mathrm{d}t} = \frac{\mathrm{i}}{\Phi_0}[\hat{H}_Z, \hat{S}_1] = \frac{\mathrm{i}}{\Phi_0} \cdot \mathrm{i}\Phi_0\omega_L\hat{S}_2 = -\omega_L\hat{S}_2. \tag{62}$$

By the identical argument with σ_1 and σ_2 exchanged, $[\sigma_3, \sigma_2] = 2\mathrm{i}\epsilon_{321}\boldsymbol{\sigma}_1 = -2\mathrm{i}\sigma_1$, giving:

$$\frac{\mathrm{d}\hat{S}_2}{\mathrm{d}t} = \frac{\mathrm{i}}{\Phi_0}[\hat{H}_Z, \hat{S}_2] = +\omega_L\hat{S}_1. \tag{63}$$

The coupled system Eqs. (62) and (63) is the harmonic oscillator equation for the complex combination $\hat{S}_1 + \mathrm{i}\hat{S}_2$: $\mathrm{d}(\hat{S}_1 + \mathrm{i}\hat{S}_2)/\mathrm{d}t = -\omega_L\hat{S}_2 + \mathrm{i}\omega_L\hat{S}_1 = \mathrm{i}\omega_L(\hat{S}_1 + \mathrm{i}\hat{S}_2)$, with solution $(\hat{S}_1 + \mathrm{i}\hat{S}_2)(t) = \mathrm{e}^{\mathrm{i}\omega_L t}(\hat{S}_1 + \mathrm{i}\hat{S}_2)(0)$. Taking real and imaginary parts gives Eqs. (56) and (57). The expectation value equations Eqs. (59)–(61) follow by taking expectation values in any state, since the Heisenberg equations are operator equations. □

Remark 6.7. *The Larmor precession has a natural geometric description on the Bloch sphere of Remark 4.8. A general spin state $|\chi\rangle$ corresponds to a point (θ, ϕ) on the Bloch sphere with $\langle\hat{\mathbf{S}}\rangle = (\Phi_0/2)(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Under the Zeeman evolution, $\langle\hat{S}_3\rangle = (\Phi_0/2)\cos\theta$ is constant (the polar angle θ is preserved) while ϕ advances at rate ω_L (the azimuthal angle precesses uniformly). The Bloch sphere point traces a circle of latitude $\theta = \mathrm{const}$ at angular velocity ω_L about the north-south axis, the axis defined by $\hat{z}$. This is Larmor precession: the classical magnetic dipole in a field $\mathbf{B} = B\hat{z}$ precesses about $\hat{z}$ at the same Larmor frequency $\omega_L = g\mu_B B/\Phi_0$, confirming the quantum-classical correspondence for spin dynamics.*

Remark 6.8. *The Ehrenfest theorem of QM4 Proposition 5.1, extended to the spin Hamiltonian, gives $\mathrm{d}\langle\hat{\mathbf{S}}\rangle/\mathrm{d}t = (\mathrm{i}/\Phi_0)\langle[\hat{H}_Z, \hat{\mathbf{S}}]\rangle$. The result $\mathrm{d}\langle\hat{\mathbf{S}}\rangle/\mathrm{d}t = \omega_L\hat{z} \times \langle\hat{\mathbf{S}}\rangle$ (which follows from taking expectation values of Eqs. (62)–(58)) is precisely the classical equation $\mathrm{d}\mathbf{m}/\mathrm{d}t = \gamma\mathbf{m} \times \mathbf{B}$ for a magnetic dipole $\mathbf{m}$ with gyromagnetic ratio $\gamma = -g\mu_B/\Phi_0$ in a field $\mathbf{B} = B\hat{z}$, with $\gamma B = -\omega_L$. This is an exact correspondence (not just leading order) because the Zeeman Hamiltonian is linear in $\hat{\mathbf{S}}$ and the Heisenberg equations are exact. The result holds for any initial spin state: the quantum Larmor precession is identical to the classical precession for all spin-$\frac{1}{2}$ states, not just coherent states. This is the spin analogue of the statement (established in QM6 Theorem 6.3) that coherent states evolve classically under the harmonic oscillator Hamiltonian: for the Zeeman Hamiltonian,* all *spin-$\frac{1}{2}$ states precess classically, not just a distinguished subfamily. The difference arises because the Zeeman Hamiltonian is linear in the spin operators, just as the harmonic oscillator Hamiltonian is quadratic in position and momentum; for the spin case, the linearity means the Ehrenfest theorem is exact for all states.*

Remark 6.9. *For a general field direction $\mathbf{B} = B\hat{n}$ where $\hat{n} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$ is a unit vector, the Zeeman Hamiltonian $\hat{H}_Z = (g\mu_B B/2)\hat{n}\cdot\boldsymbol{\sigma}$ has eigenvalues $\pm g\mu_B B/2$ and eigenstates:*

$$|\uparrow\rangle_{\hat{n}} = \cos\left(\frac{\vartheta}{2}\right)|\uparrow\rangle + \mathrm{e}^{\mathrm{i}\varphi}\sin\left(\frac{\vartheta}{2}\right)|\downarrow\rangle, \qquad |\downarrow\rangle_{\hat{n}} = -\mathrm{e}^{-\mathrm{i}\varphi}\sin\left(\frac{\vartheta}{2}\right)|\uparrow\rangle + \cos\left(\frac{\vartheta}{2}\right)|\downarrow\rangle, \tag{64}$$

corresponding to spin aligned and antialigned with $\hat{n}$. These are obtained by applying the rotation operator $\mathrm{e}^{-\mathrm{i}\vartheta(\boldsymbol{\sigma}\cdot\hat{n}')/2}$ (for an appropriate perpendicular axis $\hat{n}'$) to $|\uparrow\rangle$ and $|\downarrow\rangle$, using the exponential formula Eq. (24). The Bloch sphere point $\hat{n}(\theta, \phi)$ precesses about $\hat{n}$ (not $\hat{z}$) at the same Larmor frequency $\omega_L = g\mu_B B/\Phi_0$ regardless of the field direction, since the splitting $\Delta E = g\mu_B B$ depends only on the field magnitude.

7 Spin-Orbit Coupling

The Zeeman Hamiltonian of Sec. 6 couples the spin degree of freedom to an external field imposed from outside the system. The spin-orbit coupling is qualitatively different: it is an *internal* coupling between the spin angular momentum and the orbital angular momentum of the same particle, arising from the interaction of the spin magnetic moment with the magnetic field seen by the particle in its own rest frame as it moves through an electric potential. For a particle moving with velocity $\mathbf{v}$ through an electric field $\mathbf{E} = -\nabla V/\mathrm{e}$, the Lorentz transformation to the particle's rest frame generates a magnetic field $\mathbf{B}_{\text{eff}} \propto \mathbf{v} \times \mathbf{E} \propto \mathbf{p} \times \nabla V$, which couples to the spin. For a central potential $V = V(r)$, this effective field is proportional to $\mathbf{L} = \mathbf{x} \times \mathbf{p}$, giving the spin-orbit interaction $H_{\text{SO}} \propto \mathbf{L} \cdot \mathbf{S}$. The present section introduces the spin-orbit Hamiltonian as an operator on $\mathcal{H}_{\text{full}}$, evaluates its eigenvalues in the total angular momentum basis using the fundamental identity $2\mathbf{L} \cdot \mathbf{S} = \hat{J}^2 - \hat{L}^2 - \hat{S}^2$, and applies the result to the hydrogen fine structure as the primary physical application.

7.1 The Total Angular Momentum Operators on $\mathcal{H}_{\text{full}}$

Before introducing the spin-orbit coupling, the total angular momentum structure on $\mathcal{H}_{\text{full}}$ is established.

Definition 7.1 (Total angular momentum on $\mathcal{H}_{\text{full}}$)**.** *The* total angular momentum operators *on* $\mathcal{H}_{\text{full}} = \mathcal{H} \otimes \mathbb{C}^2$ *are*

$$\hat{J}_j := \hat{L}_j \otimes \sigma_0 + \hat{\mathbf{1}}_{\mathcal{H}} \otimes \hat{\mathbf{S}}_j, \qquad j = 1, 2, 3, \tag{65}$$

where $\hat{L}_j$ is the orbital angular momentum of QM5 acting on the spatial factor $\mathcal{H}$, and $\hat{\mathbf{S}}_j = (\Phi_0/2)\boldsymbol{\sigma}_j$ is the spin operator acting on the spin factor $\mathbb{C}^2$. The total angular momentum squared and third component are

$$\hat{J}^2 := \sum_{j=1}^{3} \hat{J}_j^2, \qquad \hat{J}_3 := \hat{J}_3. \tag{66}$$

Proposition 7.2 (Total angular momentum algebra and commutation)**.** *The total angular momentum operators satisfy:*

$$[\hat{J}_j, \hat{J}_k] = \mathrm{i}\Phi_0\, \epsilon_{jkl}\, \hat{J}_l, \tag{67}$$

$$[\hat{J}^2, \hat{J}_j] = 0, \tag{68}$$

$$[\hat{J}^2, \hat{L}^2] = [\hat{J}^2, \hat{S}^2] = 0, \tag{69}$$

$$[\hat{L}^2, \hat{S}^2] = 0. \tag{70}$$

The four operators $\{\hat{J}^2, \hat{J}_3, \hat{L}^2, \hat{S}^2\}$ are mutually commuting and can be simultaneously diagonalized on $\mathcal{H}_{\text{full}}$.

Proof. Eq. (67)*:* By QM7 Theorem 8.1 applied to the present setting with $\hat{L}_j$ in place of $\hat{L}_1^j$ and $\hat{\mathbf{S}}_j$ in place of $\hat{L}_2^j$: since $[\hat{L}_j \otimes \sigma_0, \hat{\mathbf{1}} \otimes \hat{\mathbf{S}}_k] = 0$ (Proposition 5.5) and each factor satisfies the SO(3) algebra (Eqs. (1) and (20)), the total satisfies the same algebra.

Eq. (68)*:* Standard consequence of Eq. (67), by the same argument as QM5 Theorem 3.2.

Eq. (69)*:* $\hat{J}^2 = \hat{L}^2 \otimes \sigma_0 + \hat{\mathbf{1}} \otimes \hat{S}^2 + 2\sum_j \hat{L}_j \otimes \hat{\mathbf{S}}_j$ (from expanding $\hat{J}_j^2$, as in QM7 Proposition 8.2). Since $\hat{L}^2$ commutes with all $\hat{L}_j$ (QM5) and $\hat{S}^2$ commutes with all $\hat{\mathbf{S}}_j$ (by the same algebra), both

$\hat{L}^2 \otimes \sigma_0$ and $\hat{\mathbf{1}} \otimes \hat{S}^2$ commute with all $\hat{L}_j \otimes \hat{\mathbf{S}}_j$ terms via Eq. (42). Therefore $[\hat{J}^2, \hat{L}^2] = 0$ and $[\hat{J}^2, \hat{S}^2] = 0$.

Eq. (70)*:* $\hat{L}^2 \otimes \sigma_0$ and $\hat{\mathbf{1}} \otimes \hat{S}^2$ act on different tensor factors, so they commute by Proposition 5.5. □

Remark 7.3. *Proposition 7.2 establishes that two complete commuting sets exist on* $\mathcal{H}_{\text{full}}$*:*

- Uncoupled basis: *simultaneous eigenstates of* $\{\hat{L}^2, \hat{L}_3, \hat{S}^2, \hat{S}_3\}$*, with quantum numbers* $(\ell, m, \frac{1}{2}, m_s)$*, i.e., product states* $|\,\ell,\, m\,\rangle \otimes |m_s\rangle$*.*
- Coupled basis: *simultaneous eigenstates of* $\{\hat{J}^2, \hat{J}_3, \hat{L}^2, \hat{S}^2\}$*, with quantum numbers* $(j, m_j, \ell, \frac{1}{2})$*, written* $|j, m_j\rangle$ *(with* ℓ *and* $s = \frac{1}{2}$ *suppressed when clear from context).*

The transformation between these two bases is exactly the Clebsch-Gordan decomposition of Sec. 8, with explicit coefficients given in Theorem 8.4. The spin-orbit Hamiltonian is diagonal in the coupled basis (as shown in Proposition 7.7) but not in the uncoupled basis, making the coupled basis the natural one for the fine structure analysis of Sec. 7.4.

7.2 The Spin-Orbit Hamiltonian

Definition 7.4 (Spin-orbit Hamiltonian)**.** *The* spin-orbit Hamiltonian *for a particle in a central potential* $V = V(r)$ *is*

$$\hat{H}_{\text{SO}} \coloneqq \xi(r)\, \hat{\mathbf{L}} \cdot \hat{\mathbf{S}}, \tag{71}$$

where

$$\xi(r) \coloneqq \frac{1}{2m^2c^2}\,\frac{1}{r}\,\frac{\mathrm{d}V}{\mathrm{d}r} \tag{72}$$

is the radial spin-orbit coupling function *and* $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} = \sum_j \hat{L}_j \otimes \hat{\mathbf{S}}_j$ *is the orbital-spin dot product operator on* $\mathcal{H}_{\text{full}}$*. For the hydrogen potential* $V(r) = -\mathrm{e}^2/r$*:*

$$\xi(r) = \frac{1}{2m^2c^2}\,\frac{\mathrm{e}^2}{r^3}. \tag{73}$$

Remark 7.5. *The spin-orbit coupling function Eq.* (72) *arises from the non-relativistic reduction of the Dirac equation: a spin-*$\frac{1}{2}$ *particle moving through an electric field* $\mathbf{E} = -(1/\mathrm{e})\nabla V$ *experiences, in its instantaneous rest frame, an effective magnetic field* $\mathbf{B}_{\text{eff}} = -(\mathbf{v} \times \mathbf{E})/c^2 = (\mathbf{p} \times \nabla V)/(m\mathrm{e}c^2)$*. For a central potential:* $\nabla V = (1/r)(\mathrm{d}V/\mathrm{d}r)\mathbf{x}$*, so* $\mathbf{p} \times \nabla V = -(1/r)(\mathrm{d}V/\mathrm{d}r)(\mathbf{x} \times \mathbf{p}) = -(1/r)(\mathrm{d}V/\mathrm{d}r)\Phi_0\hat{\mathbf{L}}$ *(where* $\mathbf{L} = \mathbf{x} \times \mathbf{p}/\Phi_0$ *is the dimensionless orbital angular momentum). The coupling of this effective field to the spin magnetic moment* $\hat{\boldsymbol{\mu}} = -g\mu_B\hat{\mathbf{S}}/\Phi_0$ *gives* $-\hat{\boldsymbol{\mu}} \cdot \mathbf{B}_{\text{eff}} \propto \xi(r)\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$*, with an additional factor of* $\frac{1}{2}$ *from the Thomas precession correction (a relativistic kinematic effect). The full derivation, including the Thomas factor, is deferred to QM11 where it emerges naturally from the Dirac equation. The present paper takes Eq.* (72) *as the correct non-relativistic form and derives its consequences.*

7.3 Eigenvalues of the Spin-Orbit Operator

The key to evaluating the spin-orbit coupling is the algebraic identity that expresses $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$ in terms of the total, orbital, and spin angular momentum squares.

Lemma 7.6 (The $\mathbf{L} \cdot \mathbf{S}$ identity)**.** *On* $\mathcal{H}_{\text{full}}$*:*

$$\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} = \frac{1}{2}\big(\hat{J}^2 - \hat{L}^2 \otimes \sigma_0 - \hat{\mathbf{1}} \otimes \hat{S}^2\big). \tag{74}$$

Proof. Expand $\hat{J}^2 = \sum_j \hat{J}_j^2$:

$$\begin{aligned}\hat{J}_j^2 &= \left(\hat{L}_j \otimes \sigma_0 + \hat{\mathbf{1}} \otimes \hat{\mathbf{S}}_j\right)^2 \\ &= (\hat{L}_j)^2 \otimes \sigma_0 + \hat{L}_j \otimes \hat{\mathbf{S}}_j + \hat{L}_j \otimes \hat{\mathbf{S}}_j + \hat{\mathbf{1}} \otimes (\hat{\mathbf{S}}_j)^2,\end{aligned}$$

using $[\hat{L}_j \otimes \sigma_0, \hat{\mathbf{1}} \otimes \hat{\mathbf{S}}_j] = 0$ (Proposition 5.5) to expand the cross terms. Summing over j:

$$\hat{J}^2 = \hat{L}^2 \otimes \sigma_0 + 2\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} + \hat{\mathbf{1}} \otimes \hat{S}^2.$$

Solving for $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$ gives Eq. (74). □

Proposition 7.7 (Eigenvalues of $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$ in the coupled basis)**.** *In the coupled basis $|j, m_j\rangle$ of Definition* **??**, *the spin-orbit operator $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$ is diagonal with eigenvalues:*

$$\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}\,|j, m_j\rangle = \frac{\Phi_0^2}{2}\Big[j(j+1) - \ell(\ell+1) - \frac{3}{4}\Big]|j, m_j\rangle. \tag{75}$$

For the two values $j = \ell \pm \frac{1}{2}$:

$$\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}\,|j = \ell + \tfrac{1}{2}, m_j\rangle = +\frac{\Phi_0^2 \ell}{2}\,|j = \ell + \tfrac{1}{2}, m_j\rangle, \tag{76}$$

$$\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}\,|j = \ell - \tfrac{1}{2}, m_j\rangle = -\frac{\Phi_0^2(\ell+1)}{2}\,|j = \ell - \tfrac{1}{2}, m_j\rangle. \tag{77}$$

Proof. Apply the identity Eq. (74) to $|j, m_j\rangle$. By Eqs. (**??**)–(**??**): $\hat{J}^2|j, m_j\rangle = j(j+1)\Phi_0^2|j, m_j\rangle$, $\hat{L}^2|j, m_j\rangle = \ell(\ell+1)\Phi_0^2|j, m_j\rangle$, and $\hat{S}^2|j, m_j\rangle = \frac{1}{2} \cdot \frac{3}{2}\Phi_0^2|j, m_j\rangle = \frac{3}{4}\Phi_0^2|j, m_j\rangle$. Substituting into Eq. (74):

$$\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}|j, m_j\rangle = \frac{\Phi_0^2}{2}\big[j(j+1) - \ell(\ell+1) - \tfrac{3}{4}\big]|j, m_j\rangle,$$

giving Eq. (75). For $j = \ell + \frac{1}{2}$: $j(j+1) = (\ell + \frac{1}{2})(\ell + \frac{3}{2}) = \ell^2 + 2\ell + \frac{3}{4}$, so $j(j+1) - \ell(\ell+1) - \frac{3}{4} = \ell^2 + 2\ell + \frac{3}{4} - \ell^2 - \ell - \frac{3}{4} = \ell$, giving Eq. (76). For $j = \ell - \frac{1}{2}$: $j(j+1) = (\ell - \frac{1}{2})(\ell + \frac{1}{2}) = \ell^2 - \frac{1}{4}$, so $j(j+1) - \ell(\ell+1) - \frac{3}{4} = \ell^2 - \frac{1}{4} - \ell^2 - \ell - \frac{3}{4} = -\ell - 1 = -(\ell+1)$, giving Eq. (77). □

Remark 7.8. *Proposition 7.7 confirms that the coupled basis $|j, m_j\rangle$ is the correct basis for the spin-orbit problem: $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$ is diagonal in this basis with eigenvalue depending only on j (and the fixed ℓ and $s = \frac{1}{2}$), not on m_j. The $2j+1$ states within each j-multiplet are therefore degenerate in the spin-orbit Hamiltonian, as required by the rotational symmetry $[\hat{H}_{\mathrm{SO}}, \hat{J}_j] = 0$ (which follows from $[\xi(r)\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}, \hat{J}_j] = \xi(r)[\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}, \hat{J}_j] = 0$, since $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$ is a scalar under rotations). In the uncoupled basis $|\,\ell, m\,\rangle \otimes |m_s\rangle$, by contrast, $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$ is not diagonal: its off-diagonal matrix elements mix states with the same $M = m + m_s$ but different (m, m_s) splits, requiring the Clebsch-Gordan rotation of Sec. 8 to diagonalize.*

7.4 Application to the Hydrogen Fine Structure

The primary physical application of the spin-orbit coupling is the hydrogen fine structure: the splitting of each $n\ell$ energy level of QM5 into two levels by the $\hat{H}_{\mathrm{SO}}$ perturbation.

Proposition 7.9 (Hydrogen spin-orbit energy correction)**.** *For the hydrogen atom with spin-orbit Hamiltonian* $\hat{H}_{\rm SO} = \xi_{\rm H}(r)\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}$ *where* $\xi_{\rm H}(r) = {\rm e}^2/(2m^2c^2r^3)$*, the first-order energy correction in the state* $|n,\ell,j,m_j\rangle$ *is*

$$E^{(1)}_{n\ell j} = \langle n,\ell,j,m_j|\hat{H}_{\rm SO}|n,\ell,j,m_j\rangle = \xi_{n\ell}\cdot\frac{\Phi_0^2}{2}\Big[j(j+1)-\ell(\ell+1)-\frac{3}{4}\Big], \tag{78}$$

where $\xi_{n\ell} = \langle n\ell|\xi_{\rm H}(r)|n\ell\rangle$ *is the radial expectation value:*

$$\xi_{n\ell} = \frac{{\rm e}^2}{2m^2c^2}\left\langle\frac{1}{r^3}\right\rangle_{n\ell} = \frac{{\rm e}^2}{2m^2c^2}\cdot\frac{1}{a_0^3\,n^3\,\ell(\ell+\frac{1}{2})(\ell+1)}, \tag{79}$$

with $a_0 = \Phi_0^2/(m{\rm e}^2)$ *the Bohr radius. The correction Eq.* (78) *is independent of* m_j*, confirming the* $(2j+1)$*-fold degeneracy within each* j*-multiplet.*

Proof. The first-order energy correction is $E^{(1)} = \langle\hat{H}_{\rm SO}\rangle = \langle\xi(r)\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}\rangle$. Since $\xi(r)$ is a purely radial operator acting on the spatial factor and $\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}$ acts on both spatial (through $\hat{L}_j$) and spin factors, the expectation value separates in the coupled basis: $\langle\xi(r)\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}\rangle = \langle\xi(r)\rangle_{n\ell}\cdot\langle\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}\rangle_j$, where the radial expectation is over the hydrogenic radial wave function $R_{n\ell}(r)$ of QM5 Proposition 7.4 and the angular-spin expectation is the eigenvalue from Proposition 7.7. The radial expectation value $\langle 1/r^3\rangle_{n\ell}$ for hydrogenic states is the standard result [1]:

$$\left\langle\frac{1}{r^3}\right\rangle_{n\ell} = \frac{1}{a_0^3\,n^3\,\ell(\ell+\frac{1}{2})(\ell+1)},$$

valid for $\ell \geq 1$ (the $\ell = 0$ case is handled separately below). Combining with Eq. (75) gives Eq. (78). Independence of m_j: $\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}$ is diagonal in j (Proposition 7.7) and its eigenvalue Eq. (75) depends only on j, ℓ, and s, not on m_j. □

Remark 7.10. *For* $\ell = 0$ *(the s-wave states), the spin-orbit correction Eq.* (78) *formally gives* $E^{(1)}_{n,\ell=0,j=1/2} = 0$*: the spin-orbit coupling vanishes for s-wave states. This is consistent with the physical picture: an* $\ell = 0$ *state has no orbital angular momentum, so there is no internal magnetic field to couple to the spin. Formally, the denominator of Eq.* (79) *contains* $\ell(\ell+\frac{1}{2})(\ell+1)$*, which vanishes at* $\ell = 0$*, while the numerator from the* $\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}$ *eigenvalue Eq.* (75) *also vanishes:* $(\Phi_0^2/2)[j(j+1)-0-\frac{3}{4}] = 0$ *for* $j = \frac{1}{2}$*. The product is of the form* $0\cdot\infty$ *and requires a careful limiting procedure; the correct result, confirmed by the full Dirac calculation, is* $E^{(1)}_{\ell=0} = 0$*.*

Theorem 7.11 (Hydrogen fine structure from spin-orbit coupling)**.** *The spin-orbit coupling splits each energy level* E_n *of the hydrogen atom (QM5 Theorem 7.2) with* $\ell \geq 1$ *into two levels corresponding to* $j = \ell+\frac{1}{2}$ *and* $j = \ell-\frac{1}{2}$*:*

$$E_{n\ell,j=\ell+1/2} = E_n + \frac{E^{\rm FS}_{n\ell\ell}}{2\ell+1}, \tag{80}$$

$$E_{n\ell,j=\ell-1/2} = E_n - \frac{E^{\rm FS}_{n\ell(}\ell+1)}{2\ell+1}, \tag{81}$$

where the fine structure energy scale *is*

$$E^{\rm FS}_{n\ell:=}\xi_{n\ell}\cdot\Phi_0^2\cdot\ell(\ell+1) = \frac{\alpha^2|E_n|}{n}\cdot\frac{1}{\ell(\ell+\frac{1}{2})(\ell+1)}, \tag{82}$$

with $\alpha = \mathrm{e}^2/(\Phi_0 c)$ the fine structure constant and $|E_n| = m\mathrm{e}^4/(2\Phi_0^2 n^2)$ the magnitude of the n-th hydrogen energy level. The splitting between the two j-levels is

$$\Delta E_{n\ell} = E_{n\ell,j=\ell+1/2} - E_{n\ell,j=\ell-1/2} = E^{\mathrm{FS}}_{n\ell=}\frac{\alpha^2|E_n|}{n\,\ell(\ell+\frac{1}{2})(\ell+1)}. \tag{83}$$

Proof. Substitute the spin-orbit eigenvalues from Proposition 7.7 into the first-order correction formula Eq. (78) and add to the unperturbed energy E_n. For $j = \ell + \frac{1}{2}$: $E^{(1)} = \xi_{n\ell} \cdot (\Phi_0^2/2) \cdot \ell = \xi_{n\ell}\Phi_0^2\ell/2$. For $j = \ell - \frac{1}{2}$: $E^{(1)} = \xi_{n\ell} \cdot (\Phi_0^2/2) \cdot (-(\ell+1)) = -\xi_{n\ell}\Phi_0^2(\ell+1)/2$. The total energies are $E_n + E^{(1)}$ as in Eqs. (80) and (81) (with $E^{\mathrm{FS}}_{n\ell/}(2\ell+1) = \xi_{n\ell}\Phi_0^2/2$). The splitting Eq. (83) follows by subtracting, and the expression in terms of α is obtained by substituting Eq. (79): $\xi_{n\ell}\Phi_0^2/2 = \mathrm{e}^2\Phi_0^2/(4m^2c^2a_0^3n^3\ell(\ell+\frac{1}{2})(\ell+1))$. Using $a_0 = \Phi_0^2/(m\mathrm{e}^2)$ and $\alpha = \mathrm{e}^2/(\Phi_0 c)$: $\mathrm{e}^2\Phi_0^2/(4m^2c^2a_0^3) = \alpha^2 m\mathrm{e}^4/(4\Phi_0^2) \cdot 1/n^3 = \alpha^2|E_n|/(2n)$, giving Eq. (82). $\square$

Remark 7.12. *The fine structure splitting Eq. (83) is of order $\alpha^2|E_n|/n$, where $\alpha \approx 1/137$ is the fine structure constant. The factor $\alpha^2 \approx 5 \times 10^{-5}$ means the spin-orbit splitting is roughly α^2 times smaller than the principal energy spacing $|E_n| - |E_{n+1}|$. For the hydrogen $n = 2$ level: $E_2 = -3.4$ eV, $\ell = 1$ (p-states), $j = 3/2$ and $j = 1/2$ (the $2p_{3/2}$ and $2p_{1/2}$ levels), splitting* $\Delta E_{21} \approx 4.5 \times 10^{-5}$ eV ≈ 10.9 GHz. *This fine structure is directly observable in atomic spectra and was one of the earliest quantitative confirmations that spin is a real physical degree of freedom, with the spectroscopic splitting matching the prediction $\Delta E \propto \alpha^2$.*

Remark 7.13. *The fine structure removes the ℓ-degeneracy of the hydrogen levels: the $2p$ level ($n = 2$, $\ell = 1$) splits into $2p_{3/2}$ ($j = 3/2$, four states) and $2p_{1/2}$ ($j = 1/2$, two states). The $2s$ level ($n = 2$, $\ell = 0$) has $j = 1/2$ only and does not split. The degeneracy structure is now labeled by (j, m_j) rather than (ℓ, m, m_s): the spin-orbit coupling promotes the total angular momentum j to the good quantum number, replacing the individual orbital and spin components. The remaining degeneracy — the $(2j+1)$-fold degeneracy in m_j — is the rotational degeneracy that is preserved as long as no external field is applied. The Zeeman effect of Sec. 6 lifts this remaining degeneracy when a field is applied, giving the full Zeeman-plus-fine-structure pattern.*

8 Clebsch-Gordan Decomposition for $\ell \otimes \frac{1}{2}$

The Clebsch-Gordan decomposition of $\mathcal{H}_\ell \otimes \mathbb{C}^2$ into total angular momentum sectors is the structural result that makes the spin-orbit coupling analysis of Sec. 7 precise: the coupled basis $|j, m_j\rangle$ in which $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$ is diagonal is defined by the decomposition, and its relation to the uncoupled product basis $|\,\ell,\, m\,\rangle \otimes |m_s\rangle$ is given by the explicit Clebsch-Gordan coefficients derived here. QM7 Proposition 8.1 established the decomposition structure and proved the dimension count; the explicit CG coefficients for the $\ell \otimes \frac{1}{2}$ case were deferred. The present section completes that derivation by the ladder operator method: the highest-weight state of the $j = \ell + \frac{1}{2}$ multiplet is identified uniquely from the product basis, the lowering operator $\hat{J}_- = \hat{L}_- \otimes \sigma_0 + \hat{\mathbf{1}} \otimes \hat{S}_-$ is applied repeatedly to generate all states of that multiplet, and the orthogonal complement in each magnetic subspace gives the $j = \ell - \frac{1}{2}$ states.

8.1 The Magnetic Quantum Number Constraint and the Subspace Structure

The structure of $\mathcal{H}_\ell \otimes \mathbb{C}^2$ is organized by the eigenvalue of $\hat{J}_3 = \hat{L}_3 \otimes \sigma_0 + \hat{\mathbf{1}} \otimes \hat{S}_3$.

Lemma 8.1 (m_j conservation and subspace decomposition)**.** *In the product basis $\{|\,\ell,\, m\,\rangle \otimes |m_s\rangle\}$, the eigenvalue of $\hat{J}_3$ is $m_j = m + m_s$. The product space $\mathcal{H}_\ell \otimes \mathbb{C}^2$ decomposes into mutually orthogonal m_j-eigenspaces:*

$$\mathcal{H}_\ell \otimes \mathbb{C}^2 = \bigoplus_{m_j} V_{m_j}, \tag{84}$$

where m_j ranges over $\{-\ell - \frac{1}{2}, -\ell + \frac{1}{2}, \ldots, \ell + \frac{1}{2}\}$. The dimension of each V_{m_j} is:

$$\dim V_{m_j} = \begin{cases} 1 & \text{if } |m_j| = \ell + \frac{1}{2} \text{ (extremal values)}, \\ 2 & \text{if } |m_j| \leq \ell - \frac{1}{2} \text{ (interior values)}. \end{cases} \tag{85}$$

Proof. The eigenvalue of $\hat{J}_3$ on $|\,\ell,\, m\,\rangle \otimes |m_s\rangle$ is $m\Phi_0 + m_s\Phi_0 = (m + m_s)\Phi_0$, so $m_j = m + m_s$. The maximum value is $m_{\max} + m_{s\max} = \ell + \frac{1}{2}$, achieved uniquely by $|\,\ell,\, \ell\,\rangle \otimes |\uparrow\rangle$. The minimum is $-\ell - \frac{1}{2}$, achieved uniquely by $|\,\ell,\, -\ell\,\rangle \otimes |\downarrow\rangle$. For an interior value $|m_j| \leq \ell - \frac{1}{2}$, there are exactly two product states with $m + m_s = m_j$: either $(m, m_s) = (m_j - \frac{1}{2}, +\frac{1}{2})$ or $(m, m_s) = (m_j + \frac{1}{2}, -\frac{1}{2})$, both valid since $|m_j \mp \frac{1}{2}| \leq \ell$. □

Remark 8.2. *Lemma 8.1 already determines the decomposition structure without further computation. The extremal subspaces $V_{\ell+1/2}$ and $V_{-\ell-1/2}$ are one-dimensional; by the* SO(3) *algebra of $\hat{J}$, they must belong to the $j = \ell + \frac{1}{2}$ multiplet (the largest possible $|m_j| = j_{\max}$). The interior subspaces are two-dimensional, each containing one state from the $j = \ell + \frac{1}{2}$ multiplet and one from $j = \ell - \frac{1}{2}$. This counting gives the dimension decomposition: $(2\ell + 2)$ states in $\mathcal{H}_{\ell+1/2}$ and (2ℓ) states in $\mathcal{H}_{\ell-1/2}$, with $(2\ell + 2) + (2\ell) = (2\ell + 1) \cdot 2$, confirming QM7 Proposition 8.1.*

8.2 Proof of the Decomposition by the Ladder Method

Theorem 8.3 (Clebsch-Gordan decomposition for $\mathcal{H}_\ell \otimes \mathbb{C}^2$)**.** *For $\ell \geq 1$:*

$$\mathcal{H}_\ell \otimes \mathbb{C}^2 \cong \mathcal{H}_{\ell+1/2} \oplus \mathcal{H}_{\ell-1/2}, \tag{86}$$

where $\mathcal{H}_j$ denotes the $(2j+1)$-dimensional irreducible SU(2) *representation with total angular momentum j. For $\ell = 0$: $\mathcal{H}_0 \otimes \mathbb{C}^2 \cong \mathcal{H}_{1/2}$.*

Proof. We prove the case $\ell \geq 1$; the case $\ell = 0$ is immediate since $\mathcal{H}_0$ is one-dimensional and $\mathcal{H}_0 \otimes \mathbb{C}^2 \cong \mathbb{C}^2 = \mathcal{H}_{1/2}$.

Step 1: The highest-weight state of $\mathcal{H}_{\ell+1/2}$. The subspace $V_{\ell+1/2}$ is one-dimensional (Lemma 8.1), spanned by:

$$|j = \ell + \tfrac{1}{2},\; m_j = \ell + \tfrac{1}{2}\rangle := |\,\ell,\, \ell\,\rangle \otimes |\uparrow\rangle. \tag{87}$$

This is the unique normalized state with $m_j = \ell + \frac{1}{2}$, and it satisfies $\hat{J}_+|j, \ell + \frac{1}{2}\rangle = 0$ (confirmed by $\hat{J}_+ = \hat{L}_+ \otimes \sigma_0 + \mathbf{1} \otimes \hat{S}_+$: $\hat{L}_+|\,\ell,\, \ell\,\rangle = 0$ and $\hat{S}_+|\uparrow\rangle = 0$). It is therefore the highest-weight state of an irreducible multiplet with $j = \ell + \frac{1}{2}$.

Step 2: The lowering operator on the product basis. The total lowering operator is $\hat{J}_- = \hat{L}_- \otimes \sigma_0 + \mathbf{1} \otimes \hat{S}_-$, with actions:

$$\hat{L}_-|\,\ell,\, m\,\rangle = \sqrt{\ell(\ell+1) - m(m-1)}\;\Phi_0\,|\,\ell,\, m-1\,\rangle,$$
$$\hat{S}_-|\uparrow\rangle = \Phi_0|\downarrow\rangle, \qquad \hat{S}_-|\downarrow\rangle = 0.$$

Applied to the highest-weight state Eq. (87):

$$\begin{aligned}\hat{J}_-\big(|\,\ell,\,\ell\,\rangle\otimes|\!\uparrow\rangle\big) &= (\hat{L}_-|\,\ell,\,\ell\,\rangle)\otimes|\!\uparrow\rangle + |\,\ell,\,\ell\,\rangle\otimes(\hat{S}_-|\!\uparrow\rangle)\\ &= \sqrt{2\ell}\,\Phi_0\,|\,\ell,\,\ell-1\,\rangle\otimes|\!\uparrow\rangle + \Phi_0\,|\,\ell,\,\ell\,\rangle\otimes|\!\downarrow\rangle.\end{aligned}$$

The total lowering formula gives $\hat{J}_-|j,\ell+\frac{1}{2}\rangle = \sqrt{(2\ell+1)}\,\Phi_0\,|j,\ell-\frac{1}{2}\rangle$. Therefore:

$$|j=\ell+\tfrac{1}{2},\ m_j=\ell-\tfrac{1}{2}\rangle = \sqrt{\frac{2\ell}{2\ell+1}}\,|\,\ell,\,\ell-1\,\rangle\otimes|\!\uparrow\rangle + \sqrt{\frac{1}{2\ell+1}}\,|\,\ell,\,\ell\,\rangle\otimes|\!\downarrow\rangle. \tag{88}$$

Step 3: The highest-weight state of $\mathcal{H}_{\ell-1/2}$. The subspace $V_{\ell-1/2}$ is two-dimensional, spanned by $|\,\ell,\,\ell-1\,\rangle\otimes|\!\uparrow\rangle$ and $|\,\ell,\,\ell\,\rangle\otimes|\!\downarrow\rangle$. The state Eq. (88) accounts for one dimension; the orthogonal complement in $V_{\ell-1/2}$, normalized and with the Condon-Shortley phase convention $\langle\ell,\ell-1;\frac{1}{2},+\frac{1}{2}|\ell-\frac{1}{2},\ell-\frac{1}{2}\rangle<0$, is:

$$|j=\ell-\tfrac{1}{2},\ m_j=\ell-\tfrac{1}{2}\rangle = -\sqrt{\frac{1}{2\ell+1}}\,|\,\ell,\,\ell-1\,\rangle\otimes|\!\uparrow\rangle + \sqrt{\frac{2\ell}{2\ell+1}}\,|\,\ell,\,\ell\,\rangle\otimes|\!\downarrow\rangle. \tag{89}$$

Verification: $\hat{J}_+|j=\ell-\frac{1}{2},m_j=\ell-\frac{1}{2}\rangle=0$ (confirming this is the highest-weight state of the $j=\ell-\frac{1}{2}$ multiplet), by direct computation using $\hat{J}_+=\hat{L}_+\otimes\sigma_0+\mathbf{1}\otimes\hat{S}_+$:

$$\begin{aligned}&\hat{J}_+\big(-\tfrac{1}{\sqrt{2\ell+1}}|\,\ell,\,\ell-1\,\rangle\otimes|\!\uparrow\rangle+\tfrac{\sqrt{2\ell}}{\sqrt{2\ell+1}}|\,\ell,\,\ell\,\rangle\otimes|\!\downarrow\rangle\big)\\ &\qquad= -\frac{\sqrt{2\ell}}{\sqrt{2\ell+1}}\Phi_0\,|\,\ell,\,\ell\,\rangle\otimes|\!\uparrow\rangle + \frac{\sqrt{2\ell}}{\sqrt{2\ell+1}}\Phi_0\,|\,\ell,\,\ell\,\rangle\otimes|\!\uparrow\rangle = 0.\checkmark\end{aligned}$$

The two multiplets $\mathcal{H}_{\ell+1/2}$ and $\mathcal{H}_{\ell-1/2}$ are orthogonal (their highest-weight states are orthogonal by construction) and together account for all $(2\ell+2)+(2\ell)=(2\ell+1)\times 2$ dimensions of $\mathcal{H}_\ell\otimes\mathbb{C}^2$. $\square$

8.3 The Explicit Clebsch-Gordan Coefficients

The general coupled basis states, obtained by applying $\hat{J}_-$ repeatedly to the highest-weight states Eqs. (87) and (89), are given in the following theorem.

Theorem 8.4 (Explicit CG coefficients for $\ell\otimes\frac{1}{2}$)**.** *For $\ell\geq 1$ and $m_j\in\{-j,\ldots,+j\}$, the coupled basis states are:*

Multiplet $j=\ell+\frac{1}{2}$ (m_j ranges over $\{-\ell-\frac{1}{2},\ldots,+\ell+\frac{1}{2}\}$):

$$\big|\ell+\tfrac{1}{2},m_j\big\rangle = \sqrt{\frac{\ell+m_j+\frac{1}{2}}{2\ell+1}}\,|\,\ell,\,m_j-\frac{1}{2}\,\rangle\otimes|\!\uparrow\rangle + \sqrt{\frac{\ell-m_j+\frac{1}{2}}{2\ell+1}}\,|\,\ell,\,m_j+\frac{1}{2}\,\rangle\otimes|\!\downarrow\rangle. \tag{90}$$

Multiplet $j=\ell-\frac{1}{2}$ (m_j ranges over $\{-\ell+\frac{1}{2},\ldots,+\ell-\frac{1}{2}\}$):

$$\big|\ell-\tfrac{1}{2},m_j\big\rangle = -\sqrt{\frac{\ell-m_j+\frac{1}{2}}{2\ell+1}}\,|\,\ell,\,m_j-\frac{1}{2}\,\rangle\otimes|\!\uparrow\rangle + \sqrt{\frac{\ell+m_j+\frac{1}{2}}{2\ell+1}}\,|\,\ell,\,m_j+\frac{1}{2}\,\rangle\otimes|\!\downarrow\rangle. \tag{91}$$

In both formulae, any term involving $|\,\ell,\,m\,\rangle$ with $|m|>\ell$ is zero.

Proof. We establish Eq. (90) by induction via the lowering operator; Eq. (91) follows by the orthogonality argument.

Base case for Eq. (90)*:* At $m_j = \ell + \frac{1}{2}$: $(\ell + m_j + \frac{1}{2})/(2\ell + 1) = (2\ell + 1)/(2\ell + 1) = 1$ and $(\ell - m_j + \frac{1}{2})/(2\ell + 1) = 0$, giving $|\ell + \frac{1}{2}, \ell + \frac{1}{2}\rangle = |\,\ell,\, \ell\,\rangle \otimes |\uparrow\rangle$, which is Eq. (87).

Inductive step: Assume Eq. (90) holds for m_j. Apply $\hat{J}_-$:

$$\hat{J}_-|\ell + \tfrac{1}{2}, m_j\rangle = \sqrt{(\ell + \tfrac{1}{2})(\ell + \tfrac{3}{2}) - m_j(m_j - 1)}\; \Phi_0\, |\ell + \tfrac{1}{2}, m_j - 1\rangle.$$

On the left-hand side, using $\hat{J}_- = \hat{L}_- \otimes \sigma_0 + \hat{\mathbf{1}} \otimes \hat{S}_-$ and the expansion Eq. (90):

$$\begin{aligned}\hat{J}_-|\ell + \tfrac{1}{2}, m_j\rangle &= \sqrt{\frac{\ell + m_j + \frac{1}{2}}{2\ell + 1}} \cdot \sqrt{\ell(\ell + 1) - (m_j - \tfrac{1}{2})(m_j - \tfrac{3}{2})}\; \Phi_0\,|\,\ell,\, m_j - \frac{3}{2}\,\rangle \otimes |\uparrow\rangle \\ &+ \sqrt{\frac{\ell - m_j + \frac{1}{2}}{2\ell + 1}} \cdot \sqrt{\ell(\ell + 1) - (m_j + \tfrac{1}{2})(m_j - \tfrac{1}{2})}\; \Phi_0\,|\,\ell,\, m_j - \frac{1}{2}\,\rangle \otimes |\uparrow\rangle \\ &+ \sqrt{\frac{\ell + m_j + \frac{1}{2}}{2\ell + 1}}\; \Phi_0\,|\,\ell,\, m_j - \frac{1}{2}\,\rangle \otimes |\downarrow\rangle.\end{aligned}$$

Using the algebraic identities $\ell(\ell{+}1){-}(m_j{-}\frac{1}{2})(m_j{-}\frac{3}{2}) = (\ell{+}m_j{-}\frac{1}{2})(\ell{-}m_j{+}\frac{3}{2})$ and $\ell(\ell{+}1){-}(m_j{+}\frac{1}{2})(m_j{-}\frac{1}{2}) = (\ell{+}m_j{+}\frac{1}{2})(\ell{-}m_j{+}\frac{1}{2})$, and collecting the coefficients of $|\,\ell,\, m_j{-}3/2\,\rangle{\otimes}|\uparrow\rangle$ and $|\,\ell,\, m_j{-}1/2\,\rangle{\otimes}|\downarrow\rangle$: the coefficient of $|\,\ell,\, m_j{-}3/2\,\rangle{\otimes}|\uparrow\rangle$ is $\sqrt{(\ell + m_j - \frac{1}{2})(\ell - m_j + \frac{3}{2})(\ell + m_j + \frac{1}{2})/(2\ell + 1)}\; \Phi_0$, and the coefficient of $|\,\ell,\, m_j - 1/2\,\rangle \otimes |\downarrow\rangle$ is $\sqrt{(\ell + m_j + \frac{1}{2})^2/(2\ell + 1)/(1)}\; \Phi_0$. Dividing by the total $\hat{J}_-$ factor $\sqrt{(\ell + \frac{1}{2})(\ell + \frac{3}{2}) - m_j(m_j - 1)} = \sqrt{(\ell + m_j + \frac{1}{2})(\ell - m_j + \frac{3}{2})}$ and simplifying confirms Eq. (90) with m_j replaced by $m_j - 1$.

Equation (91) *by orthogonality:* For each interior value of m_j (with $|m_j| \le \ell - \frac{1}{2}$), the two-dimensional subspace V_{m_j} is spanned by $|\,\ell,\, m_j - 1/2\,\rangle \otimes |\uparrow\rangle$ and $|\,\ell,\, m_j + 1/2\,\rangle \otimes |\downarrow\rangle$. The state $|\ell + \frac{1}{2}, m_j\rangle$ from Eq. (90) occupies one dimension; the orthogonal normalized complement with the Condon-Shortley phase convention (requiring the coefficient of $|\,\ell,\, m_j - 1/2\,\rangle \otimes |\uparrow\rangle$ to be real and negative when non-zero) is precisely Eq. (91). The phase convention and normalization are verified: $\langle \ell + \frac{1}{2}, m_j, \ell - \frac{1}{2}, m_j\rangle = -\sqrt{(\ell + m_j + \frac{1}{2})(\ell - m_j + \frac{1}{2})}/(2\ell{+}1) + \sqrt{(\ell - m_j + \frac{1}{2})(\ell + m_j + \frac{1}{2})}/(2\ell{+}1) = 0$ (orthogonality) and $\||\ell - \frac{1}{2}, m_j\rangle\|^2 = 1$ (normalization). □

8.4 The CG Coefficient Table and Inverse Transform

The CG coefficients of Theorem 8.4 are assembled in the following proposition, which records both the forward transform (uncoupled to coupled) and the inverse transform (coupled to uncoupled).

Proposition 8.5 (CG coefficients for $\ell \otimes \frac{1}{2}$: complete table)**.** *The non-zero CG coefficients for the* *coupling $\ell \otimes \frac{1}{2}$ are:*

$$\left\langle \ell, m_j - \tfrac{1}{2};\, \tfrac{1}{2}, +\tfrac{1}{2} \,\middle|\, \ell + \tfrac{1}{2}, m_j \right\rangle = \sqrt{\frac{\ell + m_j + \frac{1}{2}}{2\ell + 1}}, \tag{92}$$

$$\left\langle \ell, m_j + \tfrac{1}{2};\, \tfrac{1}{2}, -\tfrac{1}{2} \,\middle|\, \ell + \tfrac{1}{2}, m_j \right\rangle = \sqrt{\frac{\ell - m_j + \frac{1}{2}}{2\ell + 1}}, \tag{93}$$

$$\left\langle \ell, m_j - \tfrac{1}{2};\, \tfrac{1}{2}, +\tfrac{1}{2} \,\middle|\, \ell - \tfrac{1}{2}, m_j \right\rangle = -\sqrt{\frac{\ell - m_j + \frac{1}{2}}{2\ell + 1}}, \tag{94}$$

$$\left\langle \ell, m_j + \tfrac{1}{2};\, \tfrac{1}{2}, -\tfrac{1}{2} \,\middle|\, \ell - \tfrac{1}{2}, m_j \right\rangle = \sqrt{\frac{\ell + m_j + \frac{1}{2}}{2\ell + 1}}. \tag{95}$$

The inverse transform (coupled to uncoupled) is:

$$|\, \ell,\, m_j - \frac{1}{2}\,\rangle \otimes |\uparrow\rangle = \sqrt{\frac{\ell + m_j + \frac{1}{2}}{2\ell + 1}}\; |\ell + \tfrac{1}{2}, m_j\rangle - \sqrt{\frac{\ell - m_j + \frac{1}{2}}{2\ell + 1}}\; |\ell - \tfrac{1}{2}, m_j\rangle, \tag{96}$$

$$|\, \ell,\, m_j + \frac{1}{2}\,\rangle \otimes |\downarrow\rangle = \sqrt{\frac{\ell - m_j + \frac{1}{2}}{2\ell + 1}}\; |\ell + \tfrac{1}{2}, m_j\rangle + \sqrt{\frac{\ell + m_j + \frac{1}{2}}{2\ell + 1}}\; |\ell - \tfrac{1}{2}, m_j\rangle. \tag{97}$$

Proof. The forward transform Eqs. (92) through (95) are the inner products $\langle \ell, m; \frac{1}{2}, m_s | j, m_j\rangle$ read from Theorem 8.4. The inverse transform follows by the unitarity of the CG transformation (the 4 coefficients for a given m_j form a 2×2 unitary matrix rotating the uncoupled basis $\{|\, \ell,\, m_j - 1/2\,\rangle \otimes |\uparrow\rangle, |\, \ell,\, m_j + 1/2\,\rangle \otimes |\downarrow\rangle\}$ to the coupled basis $\{|\ell + \frac{1}{2}, m_j\rangle, |\ell - \frac{1}{2}, m_j\rangle\}$): explicitly:

$$\begin{pmatrix} |\ell + \frac{1}{2}, m_j\rangle \\ |\ell - \frac{1}{2}, m_j\rangle \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\ell + m_j + \frac{1}{2}}{2\ell+1}} & \sqrt{\frac{\ell - m_j + \frac{1}{2}}{2\ell+1}} \\ -\sqrt{\frac{\ell - m_j + \frac{1}{2}}{2\ell+1}} & \sqrt{\frac{\ell + m_j + \frac{1}{2}}{2\ell+1}} \end{pmatrix} \begin{pmatrix} |\, \ell,\, m_j - \frac{1}{2}\,\rangle \otimes |\uparrow\rangle \\ |\, \ell,\, m_j + \frac{1}{2}\,\rangle \otimes |\downarrow\rangle \end{pmatrix}.$$

This matrix is real orthogonal (its columns and rows are orthonormal, using $(\ell + m_j + \frac{1}{2}) + (\ell - m_j + \frac{1}{2}) = 2\ell + 1$). The inverse is its transpose, giving Eqs. (96) and (97). □

Remark 8.6. *The CG coefficients of Proposition 8.5 take especially simple forms at the extremal and near-extremal values of m_j. At $m_j = \ell + \frac{1}{2}$ (maximum): Eq. (90) gives $|\ell + \frac{1}{2}, \ell + \frac{1}{2}\rangle = |\, \ell,\, \ell\,\rangle \otimes |\uparrow\rangle$ (coefficient 1, as established in Step 1 of the proof of Theorem 8.3). At $m_j = -(\ell + \frac{1}{2})$ (minimum): $|\ell + \frac{1}{2}, -\ell - \frac{1}{2}\rangle = |\, \ell,\, -\ell\,\rangle \otimes |\downarrow\rangle$ (coefficient 1; both orbital and spin are at their minimum). At $m_j = \frac{1}{2}$ (for $\ell = 1$, relevant for $p_{1/2}$ and $p_{3/2}$):*

$$|p_{3/2}, +\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|\, 1,\, 0\,\rangle \otimes |\uparrow\rangle + \sqrt{\tfrac{1}{3}}\,|\, 1,\, 1\,\rangle \otimes |\downarrow\rangle,$$
$$|p_{1/2}, +\tfrac{1}{2}\rangle = -\sqrt{\tfrac{1}{3}}\,|\, 1,\, 0\,\rangle \otimes |\uparrow\rangle + \sqrt{\tfrac{2}{3}}\,|\, 1,\, 1\,\rangle \otimes |\downarrow\rangle,$$

the standard p-wave CG coefficients used in hydrogen fine structure calculations.

Remark 8.7. *The CG coefficients satisfy two orthogonality relations that express the unitarity of the basis transformation.* Forward completeness *(sum over coupled states, fixed uncoupled):*

$$\sum_{j=\ell\pm 1/2} \left| \left\langle \ell, m; \tfrac{1}{2}, m_s \,\middle|\, j, m_j \right\rangle \right|^2 = 1, \tag{98}$$

which for fixed (m, m_s) with $m_j = m + m_s$ gives $(\ell \pm m_j + \frac{1}{2})/(2\ell+1) + (\ell \mp m_j + \frac{1}{2})/(2\ell+1) = 1$, verified immediately. Inverse completeness *(sum over uncoupled states, fixed coupled):*

$$\sum_{m,m_s\colon m+m_s=m_j} \left| \left\langle \ell, m; \tfrac{1}{2}, m_s \,\middle|\, j, m_j \right\rangle \right|^2 = 1, \tag{99}$$

which gives $(\ell + m_j + \frac{1}{2})/(2\ell+1) + (\ell - m_j + \frac{1}{2})/(2\ell+1) = 1$, again verified immediately. Both completeness relations confirm that the CG matrix is unitary, as asserted in the proof of Proposition 8.5.

Remark 8.8. *Theorem 8.4 completes the derivation deferred in QM7 Proposition 8.1 for the primary physical case. The explicit coefficients Eqs.* (90) *and* (91) *are the specific tools used in: the spin-orbit energy calculation of Sec. 7 (converting the $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$ eigenvalue from the coupled basis back to the uncoupled for matrix element calculations); the hydrogen fine structure selection rules for electric dipole transitions (which require evaluating matrix elements of $\hat{x}_j$ between coupled states, expressible as products of a spatial matrix element and a CG coefficient); and the Bell state analysis of QM9 (where the singlet $|j = 0\rangle$ for $s_1 = s_2 = \frac{1}{2}$ is exactly the CG decomposition of $\frac{1}{2} \otimes \frac{1}{2}$ at $j = 0$, a special case of the general $\ell_1 \otimes \ell_2$ theory whose $\ell \otimes \frac{1}{2}$ instance is completed here).*

9 Interpretive Clarifications and Scope

The present section collects the interpretive constraints governing the spin analysis of the preceding sections and records the precise boundary between what the present paper establishes and what is deferred. Three items are addressed: the place of QM8 in the holonomy quantization sequence that runs through the entire NUVO program, the scope of the exchange symmetry analysis relative to the spin-statistics theorem, and the complete inventory of what the present paper establishes and does not establish.

9.1 The Fourth Holonomy Quantization

The derivation of the spin spectrum in Sec. 3 is the fourth instance of the holonomy quantization principle in the NUVO program. Collecting all four in chronological order of derivation makes the structural unity of the program explicit.

First instance: radial holonomy (Q-series). The closed radial transport path in the hydrogenic exchange sector has holonomy quantized by the condition $\Delta\phi_{\text{radial}} \in 2\pi\Phi_0\mathbb{Z}$, selecting the principal quantum number $n \in \mathbb{Z}_{>0}$. The configuration space for this holonomy is the positive real line, and the quantization is the condition that the accumulated radial transport phase is a multiple of $2\pi\Phi_0$.

Second instance: azimuthal holonomy (QM5). The closed azimuthal transport path in SO(3) has holonomy $\mathrm{e}^{\mathrm{i}2\pi m} = +1$ (the single-valuedness of the orbital closure state under a 2π rotation), selecting $m \in \mathbb{Z}$ and thereby $\ell \in \{0, 1, 2, \ldots\}$. The fundamental group of the azimuthal configuration space (the circle S^1) is $\pi_1(S^1) = \mathbb{Z}$; the integer winding numbers are the orbital magnetic quantum numbers.

Third instance: exchange holonomy (QM7). The closed exchange path in the symmetrized configuration space $(\mathbb{R}^3 \times \mathbb{R}^3)/\mathrm{Sym}_2$ has holonomy $\pi \in \{+1, -1\}$ (from $\pi^2 = 1$), selecting the exchange parity of the particle type. The fundamental group $\pi_1((\mathbb{R}^3 \times \mathbb{R}^3)/\mathrm{Sym}_2) = \mathbb{Z}_2$ contains exactly two elements, giving exactly two possible exchange parities.

Fourth instance: double-cover holonomy (QM8). The closed rotation path in SU(2) has holonomy quantized by $\mathrm{e}^{\mathrm{i}4\pi m_s} = +1$ (the single-valuedness of the spin closure state under a 4π rotation), selecting $m_s \in \frac{1}{2}\mathbb{Z}$ and thereby $s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$. The fundamental group $\pi_1(\mathrm{SU}(2)) = 0$ (simply connected) means every loop in SU(2) is contractible; the quantization arises from the 4π periodicity of the covering map $\pi : \mathrm{SU}(2) \to \mathrm{SO}(3)$.

The pattern across the four instances is:

Instance	Series	Config. space	π_1	Quantum number
Radial	Q	$\mathbb{R}_{>0}$	0	$n \in \mathbb{Z}_{>0}$
Azimuthal	QM5	S^1	$\mathbb{Z}$	$m \in \mathbb{Z}$
Exchange	QM7	$(\mathbb{R}^3 \times \mathbb{R}^3)/\mathrm{Sym}_2$	$\mathbb{Z}_2$	$\pi \in \{+1, -1\}$
Double-cover	QM8	$\mathrm{SO}(3) \cong \mathbb{RP}^3$	$\mathbb{Z}_2$	$s \in \frac{1}{2}\mathbb{Z}_{\geq 0}$

The radial and azimuthal cases produce countably infinite spectra ($n \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}$), reflecting the infinite fundamental groups 0 and $\mathbb{Z}$ respectively. The exchange and double-cover cases both have $\pi_1 = \mathbb{Z}_2$ and produce two-element discrete choices ($\pi \in \{+1, -1\}$ and the integer/half-integer dichotomy for s), though the physical content of the two $\mathbb{Z}_2$ holonomies is entirely different: the exchange holonomy classifies particle species, while the double-cover holonomy classifies representation types of the rotation group.

Remark 9.1. *QM11 and the RQM-series will add a fifth instance: the holonomy of the relativistic rotation group* SO(3, 1) *and its double cover* $\mathrm{SL}(2, \mathbb{C})$ *quantize the relativistic spin representations and, through the CPT theorem and the analyticity of the n-point functions, connect the spin quantum number to the exchange parity. The spin-statistics theorem — that integer-s representations correspond to $\pi = +1$ (bosons) and half-integer-s representations correspond to $\pi = -1$ (fermions) — is the statement that the third and fourth holonomy quantizations are not independent but are linked by the relativistic structure. In the non-relativistic NUVO program, the two $\mathbb{Z}_2$ holonomies appear as independent choices; their identification is the content of the relativistic extension.*

9.2 Spin Without the Spin-Statistics Theorem

QM8 derives the complete spin-$\frac{1}{2}$ structure: the double-cover holonomy, the Pauli algebra, the full Hilbert space $\mathcal{H}_{\text{full}}$, the Zeeman effect, the spin-orbit coupling, and the Clebsch-Gordan decomposition. None of these results depends on knowing whether a spin-$\frac{1}{2}$ particle is a boson or a fermion. The exchange symmetry of a many-particle system of spin-$\frac{1}{2}$ configurations is determined by the exchange holonomy $\pi \in \{+1, -1\}$ of QM7 Theorem 5.1, which is a separate structural property not derived here. The spin-statistics theorem asserting $\pi = -1$ for spin-$\frac{1}{2}$ particles (fermions) requires the relativistic framework; within the non-relativistic QM-series, the exchange parity and the spin quantum number remain logically independent.

This interpretive boundary is not merely formal. The non-relativistic quantum mechanics of a single spin-$\frac{1}{2}$ particle is fully captured by $\mathcal{H}_{\text{full}} = \mathcal{H} \otimes \mathbb{C}^2$ and the Pauli equation, without any reference to exchange symmetry. The exchange symmetry becomes relevant only for *multi-particle* systems of identical spin-$\frac{1}{2}$ configurations, where the many-particle state must lie in the symmetric or antisymmetric sector of the tensor product of copies of $\mathcal{H}_{\text{full}}$. QM9 will treat such multi-particle systems — including the spin-entangled Bell states — using the QM7 tensor product framework and the QM8 spin structure together, but without invoking the spin-statistics theorem. The antisymmetry of the electron wave function (which appears in the Pauli exclusion principle for electrons) is, in the NUVO program, a consequence of $\pi = -1$ established by the relativistic extension, not of the spin-$\frac{1}{2}$ structure alone.

9.3 Scope of the Present Construction

The present paper establishes the following results, available as inputs to subsequent QM-series papers.

Double-cover holonomy and spin spectrum: Theorem 3.3 (double-cover holonomy selects $m_s \in \frac{1}{2}\mathbb{Z}$; the 2π rotation factor is ± 1 for integer/half-integer m_s; spinor behavior for half-integer m_s) and Theorem 3.5 (complete spin spectrum $j(j+1)\Phi_0^2$ and $m_s\Phi_0$ for $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, derived by the QM5 ladder argument applied to the SU(2) algebra).

Pauli matrices and the spin-$\frac{1}{2}$ algebra: Lemma 4.2 (uniqueness of the Pauli representation), Definition (Pauli matrices σ_1, σ_2, σ_3 and relation $\hat{\mathbf{S}}_j = (\Phi_0/2)\boldsymbol{\sigma}_j$), Theorem 4.4 (Pauli algebra: product formula, anticommutator, commutator, trace, square, exponential map), Proposition 4.7 (eigenstates of $\hat{S}_3$, $\hat{S}_1$, $\hat{S}_2$, $\hat{S}^2$, ladder actions), and Remark 4.8 (Bloch sphere parametrization of general spin-$\frac{1}{2}$ states).

Full spin-$\frac{1}{2}$ Hilbert space: Definition 5.1 ($\mathcal{H}_{\text{full}} = \mathcal{H} \otimes \mathbb{C}^2 \cong L^2(\mathbb{R}^3, \mathbb{C}^2)$), Definition 5.3 (two-component spinor wave function, normalization, joint closure density), Proposition 5.4 (product ONB $\{\phi_j \otimes |m_s\rangle\}$, resolution of identity), Proposition 5.5 (spatial-spin observable algebra including $[A \otimes \sigma_0, \hat{\mathbf{1}} \otimes B] = 0$), and Theorem 5.8 (the Pauli equation and its component form for a uniform field).

Zeeman effect: Definition 6.1 (spin magnetic moment and Zeeman Hamiltonian), Theorem 6.3 (eigenvalues $\pm g\mu_B B/2$, splitting $\Delta E = \Phi_0\omega_L$, Larmor frequency $\omega_L = g\mu_B B/\Phi_0$), and Theorem 6.6 (Larmor precession: $\hat{S}_1$ and $\hat{S}_2$ rotate at ω_L, $\hat{S}_3$ conserved; exact correspondence with classical magnetic dipole precession for all spin states).

Spin-orbit coupling: Definition 7.1 and Proposition 7.2 (total angular momentum $\hat{J}_j$ algebra, commutation $[\hat{J}^2, \hat{L}^2] = [\hat{J}^2, \hat{S}^2] = 0$, simultaneous diagonalizability of $\{\hat{J}^2, \hat{J}_3, \hat{L}^2, \hat{S}^2\}$), Lemma 7.6 ($\hat{\mathbf{L}}\cdot\hat{\mathbf{S}} = (\hat{J}^2 - \hat{L}^2 - \hat{S}^2)/2$), Proposition 7.7 ($\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}$ eigenvalues in coupled basis: $(\Phi_0^2/2)[j(j+1) - \ell(\ell+1) - \frac{3}{4}]$, giving $+\Phi_0^2\ell/2$ for $j = \ell + \frac{1}{2}$ and $-\Phi_0^2(\ell+1)/2$ for $j = \ell - \frac{1}{2}$), Proposition 7.9 (first-order spin-orbit energy correction with the radial expectation $\langle 1/r^3\rangle_{n\ell}$), and Theorem 7.11 (hydrogen fine structure: $n\ell$ level splits into $j = \ell \pm \frac{1}{2}$ with splitting $\propto \alpha^2|E_n|/n$).

Clebsch-Gordan decomposition for $\ell \otimes \frac{1}{2}$: Lemma 8.1 (m_j conservation and two-dimensional interior subspaces), Theorem 8.3 (decomposition $\mathcal{H}_\ell \otimes \mathbb{C}^2 \cong \mathcal{H}_{\ell+1/2} \oplus \mathcal{H}_{\ell-1/2}$ for $\ell \geq 1$, proved by the ladder method), Theorem 8.4 (explicit coupled basis states Eqs. (90) and (91)), and Proposition 8.5 (complete CG coefficient table and inverse transform).

The following topics are outside the scope of the present paper.

The spin-statistics theorem. As established in QM7 Sec. **??** and reiterated above: the identification of half-integer-s particles as fermions ($\pi = -1$) requires the relativistic framework of the RQM-series.

Higher spin representations ($s \geq 1$). Theorem 3.5 establishes the spectrum for all s, but the explicit matrix representations of $\hat{\mathbf{S}}_j$ for $s = 1$ (three-dimensional, spin-1 photon), $s = \frac{3}{2}$ (four-dimensional, spin-$\frac{3}{2}$ baryon), and higher are not developed. The general CG decomposition $\ell_1 \otimes \ell_2$ for $\ell_2 > \frac{1}{2}$ is similarly deferred.

The g-factor $g = 2$ and its radiative corrections. The value $g = 2$ for the electron is the Dirac prediction, derived in QM11 from the relativistic transport closure structure. The anomalous magnetic moment $g - 2 = \alpha/\pi + \ldots$ from quantum electrodynamics is beyond the scope of the QM-series.

The relativistic spin-orbit coupling. The non-relativistic spin-orbit Hamiltonian $\hat{H}_{\text{SO}} = \xi(r)\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}$ introduced in Sec. 7 is the leading term in the non-relativistic reduction of the Dirac equation; the full derivation including the Thomas precession factor $\frac{1}{2}$ and the higher-order relativistic corrections is deferred to QM11.

Time-dependent fields. The Zeeman effect of Sec. 6 is derived for a static uniform field. The response to a time-dependent oscillating field — the basis of magnetic resonance spectroscopy — requires time-dependent perturbation theory and is deferred.

10 Conclusion

10.1 Summary of Results

The present paper has derived the spin degree of freedom from the double-cover holonomy of SU(2) and developed its complete non-relativistic structure within the scalar–conformal NUVO framework. The twelve principal results are as follows.

Double-cover holonomy (Theorem 3.3). Rotation paths in SU(2) satisfy a 4π periodicity condition; the holonomy quantization selects $m_s \in \frac{1}{2}\mathbb{Z}$. Integer m_s gives the orbital (SO(3)) representations of QM5; half-integer m_s gives the spin (SU(2)) representations, with a -1 factor under 2π rotation (spinor behavior).

Complete spin spectrum (Theorem 3.5). Applying the QM5 ladder argument to the SU(2) algebra: $\hat{S}^2|s, m_s\rangle = s(s+1)\Phi_0^2|s, m_s\rangle$ and $\hat{S}_3|s, m_s\rangle = m_s\Phi_0|s, m_s\rangle$ for $s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ and $m_s \in \{-s, \ldots, +s\}$. The ladder matrix elements $\hat{S}_+|s, m_s\rangle = \sqrt{s(s+1) - m_s(m_s+1)}\ \Phi_0|s, m_s{+}1\rangle$ are derived.

Pauli matrices and uniqueness (Lemma 4.2). The unique (up to unitary equivalence) 2×2 matrix representation of the SU(2) spin-$\frac{1}{2}$ algebra with $\hat{S}_3$ diagonal is $\mathbf{S}_j = (\Phi_0/2)\boldsymbol{\sigma}_j$, where the Pauli matrices are given by Eq. (17).

Pauli algebra (Theorem 4.4). The product formula $\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k = \delta_{jk}\sigma_0 + \mathrm{i}\epsilon_{jkl}\boldsymbol{\sigma}_l$ encodes both the SU(2) commutator $[\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_k] = 2\mathrm{i}\epsilon_{jkl}\boldsymbol{\sigma}_l$ and the Clifford anticommutator $\{\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_k\} = 2\delta_{jk}\sigma_0$; each Pauli matrix squares to the identity and is traceless; the exponential formula $\mathrm{e}^{-\mathrm{i}\varphi\hat{n}\cdot\boldsymbol{\sigma}/2} = \cos(\varphi/2)\sigma_0 - \mathrm{i}\sin(\varphi/2)\hat{n}\cdot\boldsymbol{\sigma}$ gives the SU(2) element for a rotation by φ about $\hat{n}$.

Spin-$\frac{1}{2}$ eigenstates and Bloch sphere (Proposition 4.7 and Remark 4.8). The eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ of $\hat{S}_3$ with eigenvalues $\pm\Phi_0/2$; both are eigenstates of $\hat{S}^2$ with eigenvalue $\frac{3}{4}\Phi_0^2$. The general normalized state is parametrized by the Bloch sphere (θ, ϕ) with $\langle\hat{\mathbf{S}}\rangle = (\Phi_0/2)\hat{n}(\theta, \phi)$.

Full spin-$\frac{1}{2}$ Hilbert space and Pauli equation (Definition 5.1 and Theorem 5.8). $\mathcal{H}_{\mathrm{full}} = \mathcal{H} \otimes \mathbb{C}^2 \cong L^2(\mathbb{R}^3, \mathbb{C}^2)$; elements are two-component spinors $(\Psi_\uparrow, \Psi_\downarrow)^\mathsf{T}$; spatial and spin observables commute. The Pauli equation $\mathrm{i}\Phi_0\partial_t\chi = \hat{H}_{\mathrm{Pauli}}\chi$ follows from Stone's theorem on $\mathcal{H}_{\mathrm{full}}$.

Zeeman energy splitting (Theorem 6.3). The Zeeman Hamiltonian $\hat{H}_Z = (g\mu_B/\Phi_0)B\hat{S}_3$ has eigenvalues $E_\pm = \pm g\mu_B B/2$ with splitting $\Delta E = g\mu_B B = \Phi_0\omega_L$.

Larmor precession (Theorem 6.6). The Heisenberg equations for $\hat{S}_1$, $\hat{S}_2$, $\hat{S}_3$ give exact Larmor precession at frequency ω_L for all spin-$\frac{1}{2}$ states; $\hat{S}_3$ is conserved; the quantum precession is identical to the classical magnetic dipole precession.

Spin-orbit coupling eigenvalues (Lemma 7.6 and Proposition 7.7). The identity $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} = (\hat{J}^2 - \hat{L}^2 - \hat{S}^2)/2$ gives the eigenvalues $+\Phi_0^2\ell/2$ for $j = \ell + \frac{1}{2}$ and $-\Phi_0^2(\ell+1)/2$ for $j = \ell - \frac{1}{2}$, diagonal in the coupled basis $|j, m_j\rangle$.

Hydrogen fine structure (Theorem 7.11). The spin-orbit coupling splits the $n\ell$-level into $j = \ell \pm \frac{1}{2}$ with splitting $\Delta E \propto \alpha^2|E_n|/(n\,\ell(\ell+\frac{1}{2})(\ell+1))$; the fine structure constant $\alpha \approx 1/137$ sets the scale.

Clebsch-Gordan decomposition for $\ell\otimes\frac{1}{2}$ (Theorems 8.3 and 8.4). $\mathcal{H}_\ell \otimes \mathbb{C}^2 \cong \mathcal{H}_{\ell+1/2} \oplus \mathcal{H}_{\ell-1/2}$ for $\ell \geq 1$, proved by the ladder method from the unique highest-weight state. The explicit coupled basis states are given by Eqs. (90) and (91), completing QM7 Proposition 8.1 for this case.

CG coefficient table and inverse transform (Proposition 8.5). The complete 2×2 unitary CG matrix for each m_j-subspace is recorded explicitly, together with the inverse transform (coupled to uncoupled) in Eqs. (96) and (97); the forward and inverse completeness relations confirm unitarity.

10.2 Programmatic Significance

The results of the present paper are of broad programmatic significance on three grounds.

The first is the completion of the holonomy quantization program for the non-relativistic QM-series. QM8 is the fourth and final instance of the holonomy quantization principle in the current series, following the radial (Q-series), azimuthal (QM5), and exchange (QM7) instances. The spin quantum number $s \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ is not postulated as an additional degree of freedom but derived from the topology of the rotation group double cover SU(2). All discrete quantum numbers appearing in the non-relativistic NUVO program — n, ℓ, m, π, s, m_s — arise from holonomy quantization of closed paths in appropriate configuration spaces, with the fundamental group of the configuration space determining the structure of the discrete set. This topological unity is the distinguishing feature of the NUVO derivation relative to standard formulations, in which some quantum numbers (orbital) are derived from differential equations and others (spin) are introduced empirically.

The second ground of significance is the Clifford algebra structure of the Pauli matrices and its forward propagation. The anticommutation relation $\{\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_k\} = 2\delta_{jk}\sigma_0$ established in Theorem 4.4 is absent from the orbital angular momentum algebra of QM5: the orbital operators $\hat{L}_j$ on $\mathcal{H}$ satisfy no such Clifford relation. The Clifford structure is specific to the spin-$\frac{1}{2}$ representation and is the algebraic foundation for the Dirac equation of QM11: the Dirac matrices γ^μ satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, which extends the Clifford algebra of the Pauli matrices to the four-dimensional Minkowski setting. Establishing the Clifford structure here as a consequence of the SU(2) double cover — not as an additional axiom — makes the transition to the Dirac equation in QM11 a natural algebraic extension rather than a discontinuous postulate.

The third ground is the completion of the Clebsch-Gordan program for the primary physical case. QM7 Proposition 8.1 established the decomposition structure for the general tensor product $\mathcal{H}_{\ell_1} \otimes \mathcal{H}_{\ell_2}$ and deferred the explicit coefficients. Theorem 8.4 of the present paper derives these coefficients in full for $\ell_2 = \frac{1}{2}$, the case that arises in all atomic fine structure calculations, magnetic resonance spectroscopy, and the spin-entangled Bell states of QM9. The CG coefficients for $\ell \otimes \frac{1}{2}$, expressed in the compact closed form of Eqs. (90) and (91), are a primary computational tool for all subsequent papers in the series.

10.3 Transition to QM9

QM9 develops the entanglement theory promised in the transition paragraph of QM7 and first encountered concretely in the coupled oscillator ground state of QM7 Sec. **??**. The primary objects of QM9 are states in the two-particle Hilbert space $\mathcal{H}_{\text{full}} \otimes \mathcal{H}_{\text{full}}$ (for two spin-$\frac{1}{2}$ particles, using the QM8 full Hilbert space $\mathcal{H}_{\text{full}} = \mathcal{H} \otimes \mathbb{C}^2$) that do not factorize as product states.

The first new tool is the *Schmidt decomposition*: any pure state $\Psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ (for Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$) can be written $\Psi = \sum_k \lambda_k \phi_k \otimes \psi_k$ with $\lambda_k > 0$ and $\{\phi_k\}$, $\{\psi_k\}$ orthonormal in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. The Schmidt coefficients λ_k measure the degree of entanglement: Ψ is a product state if and only if there is a single non-zero Schmidt coefficient ($\lambda_1 = 1$, all others zero), and the entanglement is quantified by the *von Neumann entanglement entropy* $S(\rho_A) = -\text{Tr}(\rho_A \log \rho_A)$ where $\rho_A = \text{Tr}_B(|\Psi\rangle\langle\Psi|)$ is the reduced density matrix of subsystem A.

The second new structure is the *Bell states*: the four maximally entangled states of $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^2 \otimes \mathbb{C}^2$, which are exactly the singlet ($j = 0$) and triplet ($j = 1$) states of the $\frac{1}{2} \otimes \frac{1}{2}$ Clebsch-Gordan decomposition previewed in QM7 Remark 8.3. The singlet $|j = 0, m_j = 0\rangle = (|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle)/\sqrt{2}$ is the primary example of a maximally entangled state: its Schmidt decomposition has two equal coefficients $\lambda_1 = \lambda_2 = 1/\sqrt{2}$, and its entanglement entropy is $S = \log 2$ (one ebit). The CG coefficients of Theorem 8.4 for $\ell = \frac{1}{2}$ (the $\frac{1}{2} \otimes \frac{1}{2}$ case) give the explicit form of all four Bell states in the product basis, making QM8 the direct algebraic input to the Bell state analysis of QM9.

References

[1] Albert Messiah. *Quantum Mechanics*, volume II. North-Holland, Amsterdam, 1962. Reference for: the radial expectation value $\langle 1/r^3\rangle_{n\ell}$ for hydrogenic states (Chapter XVI, §16.16, used in QM8 Proposition 7.2); the hydrogen fine structure calculation including the spin-orbit correction and the Thomas precession factor (Chapter XVI, §§16.13–16.16); the Clebsch-Gordan coefficients and their symmetry properties (Chapter XIII).