QM9 — Entanglement: Schmidt Decomposition, Density Matrices, and Bell States

in Scalar–Conformal NUVO Systems *Preprint, Version 1.0*[*]

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Notation and Conventions

- $\mathcal{M}$ denotes the spacetime manifold.
- η denotes the reference Lorentzian metric (typically Minkowski in a global chart).
- g denotes the physical metric.
- The scalar field $\Lambda : \mathcal{M} \to \mathbb{R}_{>0}$ is the NUVO modulation field.
- The physical metric is scalar–conformal:

$$g_{\mu\nu} = \Lambda^2 \, \eta_{\mu\nu}.$$

- $\Lambda_0 > 0$ denotes the baseline scalar availability level supported by the intrinsic delivery structure of the underlying field. In the absence of localized structural occupation the scalar field satisfies $\Lambda(x) = \Lambda_0$.
- The dimensionless scalar diagnostic is

$$\lambda(x) \coloneqq \frac{\Lambda(x)}{\Lambda_0}.$$

- The scalar field represents the *locally available structural capacity* of the underlying delivery field. Localized structures may reduce this availability through occupation or transport, but the intrinsic delivery baseline Λ_0 remains fixed.
- Greek indices $\mu, \nu, \ldots$ range over spacetime coordinates $0, 1, 2, 3$.
- We use the Einstein summation convention unless explicitly stated otherwise.

Remark 0.1. *Unless otherwise stated, the background signature is* $(-, +, +, +)$.

[*]Bibliography is provisional. Cross-references to companion NUVO-series papers (M-, SR-, Q-, QB-, QM-series) will be updated with Zenodo DOIs in subsequent versions.

Program scope.

Abstract

The tensor product construction of QM7 produces a Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ containing states that do not factorize as $\Psi_A \otimes \Psi_B$. These *entangled* states, introduced concretely in the coupled oscillator ground state of QM7 and the Bell states previewed in QM8, are the subject of the present paper. Their mathematical structure is analyzed by the Schmidt decomposition, their information content by the reduced density matrix and von Neumann entropy, and their operational content by the Bell inequalities.

The *Schmidt decomposition* establishes that every pure state $|\Psi\rangle \in \mathcal{H}_{AB}$ can be written as $|\Psi\rangle = \sum_{k=1}^{r} \lambda_k |\phi_k\rangle \otimes |\psi_k\rangle$ with $\lambda_k > 0$ and $\{|\phi_k\rangle\}$, $\{|\psi_k\rangle\}$ orthonormal families in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. The Schmidt rank r is the number of non-zero Schmidt coefficients; Ψ is a product state if and only if $r = 1$.

The *reduced density matrix* $\hat{\rho}_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$ describes the subsystem A after tracing out subsystem B; it is a positive trace-class operator on $\mathcal{H}_A$ with $\mathrm{Tr}(\hat{\rho}_A) = 1$. For a pure product state, $\hat{\rho}_A$ is a pure state projector; for an entangled state, $\hat{\rho}_A$ is a mixed state. The *von Neumann entanglement entropy* $S(\hat{\rho}_A) = -\mathrm{Tr}(\hat{\rho}_A \log \hat{\rho}_A) = -\sum_k \lambda_k^2 \log \lambda_k^2$ is derived from the Schmidt decomposition and vanishes if and only if $|\Psi\rangle$ is a product state.

The *Bell states* for two spin-$\frac{1}{2}$ configurations are the four maximally entangled states of $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$, identified as the $j = 0$ singlet and the three $j = 1$ triplet states of the $\frac{1}{2} \otimes \frac{1}{2}$ Clebsch-Gordan decomposition of QM8. Their Schmidt decompositions, reduced density matrices, and entanglement entropies are derived. The Bell states form a complete orthonormal basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$ (the Bell basis).

The *CHSH inequality* is derived as a consequence of local hidden variable (LHV) theories: any LHV model for correlations of measurements on a bipartite system satisfies $|\mathcal{S}| \leq 2$. Quantum mechanics violates this bound: for the singlet Bell state, the maximum quantum value is $|\mathcal{S}|_{\max} = 2\sqrt{2}$ (Tsirelson's bound), derived from the operator norm of the CHSH operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$.

No new postulates are introduced. All results follow from the QM7 tensor product construction, the QM8 Pauli algebra and Clebsch-Gordan coefficients, and the general spectral theory of QM1.

1 Introduction

1.1 Position Within the QM-Series

The tensor product construction of QM7 produced a two-particle Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ containing all bilinear combinations of single-particle states, including states that cannot be written as $|\Psi_A\rangle \otimes |\Psi_B\rangle$ for any $|\Psi_A\rangle \in \mathcal{H}_A$ and $|\Psi_B\rangle \in \mathcal{H}_B$. The existence of these non-product states was noted in QM7 as a structural consequence of the tensor product — not an additional postulate — and two concrete instances were introduced: the coupled oscillator ground state (QM7 Proposition 7.2), which is entangled for any non-zero coupling $\kappa \neq 0$, and the Bell states (QM7 Remark 8.3 and QM8 Theorem 8.2), identified as the $j = 0$ singlet and $j = 1$ triplet states of the $\frac{1}{2} \otimes \frac{1}{2}$ Clebsch-Gordan decomposition. The present paper, QM9, develops the complete theory of these non-product states within the scalar–conformal NUVO framework: the Schmidt decomposition as their canonical mathematical representation, the reduced density matrix as the description of one subsystem when the joint state is entangled, the von Neumann entropy as the measure of the degree of entanglement, and the Bell inequalities as the operational signature that distinguishes quantum entanglement from all classical correlations.

QM9 depends on the prior series in three structurally specific ways. The tensor product construction of QM7 is the mathematical foundation: the definition of entanglement as the failure to factorize (Sec. 3.1), the Schmidt decomposition derived from the SVD of the coefficient matrix in the product ONB (Sec. 3.2), and the reduced density matrix defined via the partial trace (Sec. 4.2) all operate on the $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ structure of QM7 Definition 3.1. The spectral theorem of QM1 is the analytic input to the Schmidt decomposition: the reduced density matrix $\hat{\rho}_A$ is a positive trace-class operator on $\mathcal{H}_A$, and its spectral decomposition by QM1 Theorem 6.1 gives the Schmidt eigenbasis whose squares are the Schmidt coefficients. The Pauli algebra of QM8 is the computational input to the Bell state analysis and the CHSH inequality: the expectation values $\langle|\Psi^-\rangle|\hat{A}_i \otimes \hat{B}_j||\Psi^-\rangle\rangle$ that determine the CHSH parameter are computed directly from the Pauli anticommutation relations of QM8 Theorem 4.2, and the Tsirelson bound is derived from the operator norm of the CHSH operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$ using the same algebra.

QM9 introduces two qualitatively new structures that have no counterpart in the single-particle papers QM1 through QM6. The first is the *mixed state*. For a single particle described by a pure state $|\Psi\rangle \in \mathcal{H}$, the state space is the set of normalized rays in $\mathcal{H}$; every state is pure. For a subsystem A of a bipartite system in an entangled pure state $|\Psi\rangle \in \mathcal{H}_{AB}$, the subsystem cannot be described by any pure state $|\Psi_A\rangle \in \mathcal{H}_A$: the correct description is the reduced density matrix $\hat{\rho}_A$, a positive trace-class operator with $\mathrm{Tr}(\hat{\rho}_A) = 1$ that is not a pure-state projector when $|\Psi\rangle$ is entangled. Mixed states are not a new physical ingredient but a derived mathematical structure: they arise from the act of restricting attention to one subsystem of a larger entangled pure state, and their statistical properties follow from the Born rule of QM1 applied to the full pure state on $\mathcal{H}_{AB}$. The second new structure is the *Bell inequality violation*. The CHSH inequality $|\mathcal{S}| \leq 2$ is a consequence of probability theory and the assumption that the measurement outcomes of A and B are determined by local pre-existing properties (the local hidden variable assumption); it holds for all classical correlations regardless of their origin. The quantum value $|\mathcal{S}|_{\max} = 2\sqrt{2}$ for the singlet Bell state, derived in Theorem 7.5 from the Pauli algebra of QM8, exceeds this bound. The gap $2\sqrt{2} > 2$ is not a quantitative imprecision: it is a structural incompatibility between quantum mechanical correlations and any local hidden variable description. Its derivation requires no physical postulate beyond the Born rule and the tensor product structure already established; it is a theorem.

QM9 opens the program arc toward quantum information theory, which lies beyond the current QM-series but whose foundational structures are established here. QM10 uses the density matrix formalism of the present paper to describe scattered particles after tracing out the environmental degrees of freedom, making the reduced density matrix a practical computational tool rather than a formal definition. QM11 analyzes the Lorentz transformation of entangled spin states, establishing that entanglement is a Lorentz-invariant property (the von Neumann entropy of the reduced spin density matrix is invariant) even though the Schmidt basis changes under boosts. Beyond the current series, the Bell basis established in Sec. 6 and the operator Schmidt decomposition are the primary tools for quantum teleportation, entanglement swapping, and quantum error correction, whose treatment would require the additional structure of classical communication channels and is deferred.

1.2 Objective of the Present Work

The central objective of the present paper is to develop the complete theory of bipartite entanglement within the scalar–conformal NUVO transport closure framework, from the Schmidt decomposition through the Bell inequality violation. Specifically, the paper establishes six claims.

1. Every pure state $|\Psi\rangle \in \mathcal{H}_{AB}$ has a unique Schmidt decomposition $|\Psi\rangle = \sum_{k=1}^{r} \lambda_k |\phi_k\rangle \otimes |\psi_k\rangle$ with $\lambda_k > 0$, $\{|\phi_k\rangle\}$ orthonormal in $\mathcal{H}_A$, $\{|\psi_k\rangle\}$ orthonormal in $\mathcal{H}_B$, and $\sum_k \lambda_k^2 = 1$. The

Schmidt rank r satisfies $1 \le r \le \min(d_A, d_B)$, and $r = 1$ if and only if $|\Psi\rangle$ is a product state. The decomposition is derived from the singular value decomposition of the coefficient matrix $C_{jk} = \langle \phi_j \otimes \psi_k | \Psi\rangle$ in any product ONB.

2. The reduced density matrix $\hat{\rho}_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$ is the unique positive trace-class operator on $\mathcal{H}_A$ satisfying the Born rule $\langle \hat{A} \otimes \hat{\mathbf{1}}_{\mathcal{H}_B}\rangle_{|\Psi\rangle} = \mathrm{Tr}_{\mathcal{H}_A}(\hat{\rho}_A \hat{A})$ for all observables $\hat{A}$ on $\mathcal{H}_A$. Its eigenvalues are $\{\lambda_k^2\}$ (the squares of the Schmidt coefficients) and its eigenstates are the Schmidt basis $\{|\phi_k\rangle\}$. $\hat{\rho}_A$ is a pure-state projector if and only if $r = 1$.

3. The von Neumann entanglement entropy $S(\hat{\rho}_A) = -\mathrm{Tr}(\hat{\rho}_A \log \hat{\rho}_A) = -\sum_k \lambda_k^2 \log \lambda_k^2$ satisfies: (a) $S \ge 0$ with equality if and only if $|\Psi\rangle$ is a product state; (b) $S \le \log \min(d_A, d_B)$ with equality if and only if all Schmidt coefficients are equal (maximally entangled state); (c) $S(\hat{\rho}_A) = S(\hat{\rho}_B)$ for any pure bipartite state.

4. The four Bell states $|\Phi^+\rangle$, $|\Phi^-\rangle$, $|\Psi^+\rangle$, $|\Psi^-\rangle$ of $\mathbb{C}^2 \otimes \mathbb{C}^2$ each have Schmidt rank $r = 2$, Schmidt coefficients $\lambda_1 = \lambda_2 = 1/\sqrt{2}$, reduced density matrix $\hat{\rho}_A = \frac{1}{2}\sigma_0$ (the maximally mixed state), and entanglement entropy $S = \log 2 = 1$ ebit. They form a complete orthonormal basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$ (the Bell basis). The singlet $|\Psi^-\rangle$ is the $j = 0$ CG state and $|\Psi^+\rangle$ is the $j = 1$, $m_j = 0$ CG state from QM8 Theorem 8.2.

5. Any local hidden variable (LHV) theory satisfies the CHSH inequality $|\mathcal{S}_{\mathrm{LHV}}| \le 2$, where $\mathcal{S} = E(\hat{A}_1, \hat{B}_1) + E(\hat{A}_1, \hat{B}_2) + E(\hat{A}_2, \hat{B}_1) - E(\hat{A}_2, \hat{B}_2)$ and $E(\hat{A}_i, \hat{B}_j)$ is the correlation of dichotomic observables. The bound follows from the arithmetic inequality $|a_1(b_1 + b_2) + a_2(b_1 - b_2)| \le 2$ for $a_i, b_j \in \{+1, -1\}$ and integration over any hidden variable distribution.

6. For the singlet Bell state $|\Psi^-\rangle$ and optimal measurement settings $\hat{A}_1 = \sigma_3$, $\hat{A}_2 = \sigma_1$, $\hat{B}_1 = (\sigma_3 + \sigma_1)/\sqrt{2}$, $\hat{B}_2 = (\sigma_3 - \sigma_1)/\sqrt{2}$, the quantum CHSH parameter is $\mathcal{S}_{\mathrm{Q}} = 2\sqrt{2}$, derived from the Pauli algebra of QM8 Theorem 4.2. The Tsirelson bound $|\mathcal{S}_{\mathrm{Q}}| \le 2\sqrt{2}$ for any quantum state and any dichotomic observables follows from the operator norm bound $\|\hat{\mathcal{C}}\|^2 \le 8\hat{\mathbf{1}}$. Since $2\sqrt{2} > 2$, quantum mechanics violates the CHSH inequality: entangled quantum states produce correlations incompatible with any LHV description.

Claims (1) through (6) are logically ordered. The Schmidt decomposition of claim (1) is the foundational result from which the reduced density matrix of claim (2) is derived (as the spectral decomposition with eigenvalues λ_k^2). The von Neumann entropy of claim (3) is computed from the Schmidt coefficients and inherits its properties from them. The Bell states of claim (4) are the explicit two-qubit realization of claims (1)–(3), with the maximum entropy $\log 2$ confirming their maximally entangled character. The CHSH inequality of claim (5) characterizes what correlations are classically achievable; the quantum violation of claim (6) establishes that the Bell state correlations exceed this classical bound.

1.3 What Is Not Assumed

The present work maintains without modification the interpretive discipline of the prior series. Four exclusions are of particular importance for QM9.

The mixed state density matrix is not introduced as a new axiom of quantum mechanics. In many formulations, the density matrix is presented as the general state of a quantum system, with the pure state as a special case. In the NUVO framework, the density matrix appears exclusively as the reduced density matrix of a subsystem of a bipartite pure state: $\hat{\rho}_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$ is defined by the partial trace and the pure state $|\Psi\rangle$ on $\mathcal{H}_{AB}$. The Born rule for mixed states,

$\mathrm{Tr}(\hat{\rho}_A \hat{A}) = \langle \hat{A} \otimes \mathbf{1} \rangle_{|\Psi\rangle}$, is derived from the pure-state Born rule (established via the transport closure frequency law of QB-series) applied to the full state. No new measurement postulate is introduced for mixed states.

The Schmidt decomposition is not assumed. It is derived as a consequence of the singular value decomposition of the coefficient matrix C_{jk} in the product ONB, which is itself a consequence of the spectral theorem of QM1 applied to the positive operator $C^\dagger C$. The SVD theorem for finite-dimensional matrices is a consequence of the spectral theorem; for the infinite-dimensional case (relevant for the coupled oscillator of Sec. 8), the corresponding result for compact operators is cited from the standard functional analysis literature.

Bell inequality violation is not a physical postulate. The CHSH inequality $|\mathcal{S}_{\mathrm{LHV}}| \leq 2$ is a mathematical theorem about probability distributions and locality assumptions, proved in Theorem 7.2 without reference to quantum mechanics. The quantum value $\mathcal{S}_{\mathrm{Q}} = 2\sqrt{2}$ is a theorem about the Pauli algebra, proved in Theorem 7.5 by direct computation of expectation values and operator norms. The *empirical* violation of the CHSH inequality in experiments (Aspect et al., Zeilinger et al., and others) is a physical fact that the NUVO framework predicts as a theorem; it is not an additional experimental input to the theory.

Quantum information applications are not developed. The Bell basis established in Sec. 6 and the partial trace formalism of Sec. 4 are the mathematical prerequisites for quantum teleportation, dense coding, entanglement swapping, and quantum error correction. These applications require the additional structure of classical communication protocols — the specification of what classical information is transmitted between parties alongside the quantum channel — which is outside the scope of the present paper. The present paper establishes the entanglement theory; the information-theoretic applications are deferred.

1.4 Structure of the Paper

Sec. 2 recalls the tensor product and coefficient matrix from QM7, the spectral theorem from QM1 that underlies the Schmidt decomposition, and the Pauli algebra and Clebsch-Gordan coefficients from QM8 that are used in the Bell state and CHSH calculations. Sec. 3 defines product states and entanglement precisely, derives the Schmidt decomposition from the singular value decomposition of the coefficient matrix in the product ONB, and establishes the Schmidt rank as the canonical measure of entanglement structure. Sec. 4 introduces the density matrix of a pure state, defines the partial trace and the reduced density matrix, derives the Born rule for subsystem observables, and establishes the spectral structure of the reduced density matrix from the Schmidt coefficients. Sec. 5 defines the von Neumann entanglement entropy, derives its three key properties (non-negativity, upper bound by $\log \min(d_A, d_B)$, and equality of subsystem entropies), and computes its value for specific Schmidt coefficient distributions. Sec. 6 defines the four Bell states from the $\frac{1}{2} \otimes \frac{1}{2}$ Clebsch-Gordan decomposition of QM8, computes their Schmidt decompositions, reduced density matrices, and entanglement entropies, verifies their maximally entangled character, and establishes the Bell basis completeness. Sec. 7 derives the CHSH inequality for local hidden variable theories, computes the quantum CHSH parameter for the singlet Bell state and optimal measurement settings, derives Tsirelson's bound from the operator norm of the CHSH operator, and establishes the quantum violation as a theorem. Sec. 8 applies the Schmidt decomposition to the coupled oscillator ground state of QM7, derives the geometric Schmidt coefficient distribution and the squeeze parameter t, and establishes the entanglement entropy as a monotonically increasing function of the coupling constant κ. Sec. 9 records the derivational status of the density matrix and Bell inequality violation, and the scope of the present construction. Sec. 10 summarizes the twelve principal results, records their programmatic significance, and prepares the transition to QM10.

2 Recalled Structure from Prior Papers

The present section collects the results from QM1, QM7, and QM8 that are directly required for the derivations of Secs. 3–8. Nothing in this section is new. The recalled material falls into three categories: the tensor product structure and coefficient matrix representation from QM7, whose singular value decomposition gives the Schmidt decomposition; the spectral theorem from QM1, which underlies both the SVD and the spectral analysis of the reduced density matrix; and the Pauli algebra and Clebsch-Gordan structure from QM8, which are the computational inputs to the Bell state and CHSH analyses.

2.1 The Tensor Product, Product Basis, and Coefficient Matrix

The following results from QM7 are the mathematical setting for the entire paper.

The tensor product Hilbert space (QM7 Definition 3.1 and Proposition 3.2). For Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ with complete orthonormal bases $\{\phi_j\}_{j\geq 1}$ and $\{\psi_k\}_{k\geq 1}$ respectively, the tensor product $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ has complete orthonormal basis $\{\phi_j \otimes \psi_k\}_{j,k\geq 1}$, and every element $|\Psi\rangle \in \mathcal{H}_{AB}$ expands as

$$|\Psi\rangle = \sum_{j,k} C_{jk}\, |\phi_j\rangle \otimes |\psi_k\rangle, \qquad \sum_{j,k} |C_{jk}|^2 = 1, \tag{1}$$

where the *coefficient matrix* $C = (C_{jk})$ encodes the state in the product ONB.

Product states and the rank-one criterion. The state $|\Psi\rangle$ is a product state $|\Psi_A\rangle \otimes |\Psi_B\rangle$ if and only if the coefficient matrix C has rank one: $C_{jk} = a_j b_k$ for sequences (a_j) and (b_k) with $\sum_j |a_j|^2 = \sum_k |b_k|^2 = 1$. An entangled state has $\mathrm{rank}(C) \geq 2$. This characterization makes the Schmidt rank equal to $\mathrm{rank}(C)$, as established in Theorem 3.4.

Observable algebra (QM7 Proposition 3.3). For observables $\hat{A}$ on $\mathcal{H}_A$ and $\hat{B}$ on $\mathcal{H}_B$:

$$[\hat{A} \otimes \hat{\mathbf{1}}_{\mathcal{H}_B},\ \hat{\mathbf{1}}_{\mathcal{H}_A} \otimes \hat{B}] = 0 \quad \text{on } \mathcal{H}_{AB}. \tag{2}$$

This is used in Sec. 4 to derive the Born rule for subsystem observables, and in Sec. 7 to establish that the CHSH operator factorizes as a sum of tensor products.

Non-interacting spectrum (QM7 Proposition 3.4). For a product Hamiltonian $\hat{H} = \hat{H}_A \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes \hat{H}_B$, the energy eigenvalues are pairwise sums $E_n^A + E_m^B$ and the eigenstates are product states $|\phi_n\rangle \otimes |\psi_m\rangle$. The ground state of the *non-interacting* system is a product state; the interaction $\kappa \hat{x}_1 \hat{x}_2$ of QM7 generates entanglement, as analyzed in Sec. 8.

The coupled oscillator ground state (QM7 Proposition 7.2 and Eq. (**??**)). The ground state of the coupled oscillator has position-space representation

$$\Psi_{0,0}(x_1, x_2) = \mathcal{N} \exp\left(-\frac{m\omega_+}{4\Phi_0}(x_1 + x_2)^2 - \frac{m\omega_-}{4\Phi_0}(x_1 - x_2)^2 \right), \tag{3}$$

with normal mode frequencies $\omega_+ = \sqrt{\omega^2 + \kappa/m}$ and $\omega_- = \sqrt{\omega^2 - \kappa/m}$. This state is entangled for $\kappa \neq 0$ (Proposition 7.2) because the exponent contains the cross term $\propto x_1 x_2$, preventing factorization. Its Schmidt decomposition in the normal mode Fock basis is derived in Sec. 8.

Remark 2.1. *The coefficient matrix $C = (C_{jk})$ is the central object in the entanglement analysis. Changing the product ONB from $\{\phi_j \otimes \psi_k\}$ to $\{\phi'_j \otimes \psi'_k\}$ transforms C to UCW for unitary matrices U and W acting on the basis indices. The singular values of C — and hence the Schmidt coefficients — are invariant under such unitary changes of basis, confirming that the Schmidt decomposition is basis-independent even though the coefficient matrix is not. The Schmidt coefficients are the intrinsic entanglement data of the state; the Schmidt bases are the bases that diagonalize C simultaneously from the left and right.*

2.2 The Spectral Theorem and the Singular Value Decomposition

The Schmidt decomposition of Theorem 3.4 is derived from the singular value decomposition (SVD) of the coefficient matrix C, which is itself a consequence of the spectral theorem. The relevant results are recalled here.

Spectral theorem for self-adjoint operators (QM1 Theorem 6.1). Every self-adjoint operator A on a Hilbert space $\mathcal{H}$ has a spectral decomposition $A = \int \lambda \, \mathrm{d}E_\lambda$, with eigenstates (or generalized eigenstates) forming a complete system. For a bounded positive operator $A \geq 0$ with $\mathrm{Tr}(A) < \infty$ (a trace-class operator), the spectral decomposition reduces to a countable sum $A = \sum_k a_k |u_k\rangle\langle u_k|$ with $a_k \geq 0$ and $\{|u_k\rangle\}$ a complete orthonormal set of eigenstates. This result is applied in Sec. 4 to $\hat{\rho}_A = \sum_k \lambda_k^2 |\phi_k\rangle\langle\phi_k|$: the reduced density matrix is a positive trace-class operator with eigenvalues λ_k^2.

Singular value decomposition (SVD). For any $d_A \times d_B$ complex matrix C (finite-dimensional case), there exist unitary matrices $U \in \mathrm{U}(d_A)$ and $V \in \mathrm{U}(d_B)$ and a non-negative diagonal matrix $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0)$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, such that

$$C = U\Sigma V^\dagger. \tag{4}$$

The singular values σ_k are the positive square roots of the eigenvalues of the positive semi-definite matrix $C^\dagger C$, and the columns of U and V are the left and right singular vectors. The SVD follows from the spectral theorem applied to $C^\dagger C$: diagonalizing $C^\dagger C = V\Sigma^2 V^\dagger$ gives V and Σ, and then $U = CV\Sigma^{-1}$ (on the non-zero singular value subspace) completes the factorization.

Extension to infinite-dimensional spaces. For the coupled oscillator of Sec. 8, the coefficient matrix C is an infinite matrix (a Hilbert-Schmidt operator on $\ell^2 \otimes \ell^2$). The SVD generalizes to the polar decomposition of a compact operator: every compact operator C on $\mathcal{H}_A \otimes \mathcal{H}_B$ has a decomposition $C = U\Sigma V^\dagger$ where U and V are partial isometries and Σ is a positive compact operator (the modulus $|C| = (C^\dagger C)^{1/2}$). The resulting Schmidt decomposition has countably many non-zero Schmidt coefficients $\sigma_k \to 0$; the state is normalizable when $\sum_k \sigma_k^2 = 1$, which holds for the coupled oscillator ground state [6].

Remark 2.2. *The connection from SVD to Schmidt decomposition is direct. Given the SVD $C = U\Sigma V^\dagger$, define the new orthonormal families $|\phi'_k\rangle = \sum_j U_{jk}|\phi_j\rangle$ and $|\psi'_k\rangle = \sum_l V_{lk}|\psi_l\rangle$. Then:*

$$|\Psi\rangle = \sum_{j,k} C_{jk} \, |\phi_j\rangle \otimes |\psi_k\rangle = \sum_m (U\Sigma V^\dagger)_{jk} \, |\phi_j\rangle \otimes |\psi_k\rangle = \sum_m \sigma_m |\phi'_m\rangle \otimes |\psi'_m\rangle,$$

which is the Schmidt decomposition with $\lambda_m = \sigma_m$ and Schmidt bases $|\phi'_m\rangle$, $|\psi'_m\rangle$. The Schmidt coefficients σ_m are the singular values of C and are invariant under changes of the original product ONB (since a change of ONB transforms $C \to U'CV'$ with unitaries U', V', which does not change the singular values).

2.3 The Pauli Algebra and the $\frac{1}{2} \otimes \frac{1}{2}$ Clebsch-Gordan Structure from QM8

The following results from QM8 are used in Secs. 6 and 7.

Pauli matrices and spin expectation values (QM8 Theorem 4.2 and Proposition 4.3). On $\mathbb{C}^2 = \mathbb{C}^2$ with basis $\{|\uparrow\rangle, |\downarrow\rangle\}$: the spin operators are $\hat{\mathbf{S}}_j = (\Phi_0/2)\boldsymbol{\sigma}_j$, and the Pauli matrices satisfy the product formula $\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k = \delta_{jk}\sigma_0 + \mathrm{i}\epsilon_{jkl}\boldsymbol{\sigma}_l$. For the Bell state calculations, the key expectation values are those of $\boldsymbol{\sigma}_j \otimes \boldsymbol{\sigma}_k$ in the singlet state $|\Psi^-\rangle$:

$$\langle |\Psi^-\rangle | \boldsymbol{\sigma}_j \otimes \boldsymbol{\sigma}_k | |\Psi^-\rangle \rangle = -\delta_{jk}, \tag{5}$$

established in Sec. 6 as a consequence of the singlet's antisymmetry under exchange and the Pauli algebra.

The $\frac{1}{2} \otimes \frac{1}{2}$ Clebsch-Gordan decomposition (QM8 Theorem 8.1 and the $\ell = \frac{1}{2}$ special case of Theorem 8.2). The tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^2 \otimes \mathbb{C}^2$ decomposes as:

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathcal{H}_1 \oplus \mathcal{H}_0, \tag{6}$$

with the triplet ($j = 1$, three states) and singlet ($j = 0$, one state). The explicit CG states from QM8 Theorem 8.2 with $\ell_1 = \ell_2 = \frac{1}{2}$ are:

$$|j = 1, m_j = +1\rangle = |\uparrow\rangle \otimes |\uparrow\rangle, \tag{7}$$

$$|j = 1, m_j = 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle), \tag{8}$$

$$|j = 1, m_j = -1\rangle = |\downarrow\rangle \otimes |\downarrow\rangle, \tag{9}$$

$$|j = 0, m_j = 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle). \tag{10}$$

The singlet Eq. (10) is the Bell state $|\Psi^-\rangle$ and the $m_j = 0$ triplet Eq. (8) is $|\Psi^+\rangle$; the states $|\Phi^+\rangle$ and $|\Phi^-\rangle$ are linear combinations of the $m_j = +1$ and $m_j = -1$ triplet states.

Remark 2.3. *The singlet $|\Psi^-\rangle = |j = 0, m_j = 0\rangle$ is the antisymmetric combination $(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle)/\sqrt{2}$, i.e., $\hat{P}_{12}||\Psi^-\rangle\rangle = -||\Psi^-\rangle\rangle$ where $\hat{P}_{12}$ is the exchange operator of QM7 Definition 4.1. This antisymmetry is the property $\pi = -1$, confirming that the singlet lies in the antisymmetric sector $\mathcal{H}_{\text{anti}}$ of QM7 Proposition 4.2. The three triplet states are symmetric ($\pi = +1$) and lie in $\mathcal{H}_{\text{sym}}$. In the context of two identical fermionic spin-$\frac{1}{2}$ configurations (for which $\pi = -1$ by QM7 Theorem 5.1), only the singlet is consistent with the fermionic exchange symmetry when both particles occupy the same spatial mode; the triplet states require the two particles to be in different spatial modes (by the Pauli exclusion principle, QM7 Corollary 5.2) to form a fully antisymmetric total state. This remark anticipates the connection between the Bell basis and the fermionic structure that will appear in the atomic physics applications of QM10.*

Remark 2.4. *The identification of two Bell states ($|\Psi^-\rangle$ and $|\Psi^+\rangle$) as CG states of the $\frac{1}{2} \otimes \frac{1}{2}$ decomposition reflects the fact that the CG decomposition diagonalizes the total spin $\hat{J}^2$ and $\hat{J}_3$, while the Bell basis is a complete ONB for the four-dimensional space $\mathbb{C}^2 \otimes \mathbb{C}^2$. The other two Bell states $|\Phi^+\rangle$ and $|\Phi^-\rangle$ are not CG states in the standard labeling because they are superpositions of $m_j = +1$ and $m_j = -1$ triplet states (i.e., they are not eigenstates of $\hat{J}_3$), but they are related to the CG basis by a local unitary transformation on subsystem A alone. The Bell basis and the CG basis are therefore distinct orthonormal bases for $\mathbb{C}^2 \otimes \mathbb{C}^2$, with the Bell basis optimized for the CHSH measurement settings of Sec. 7 and the CG basis optimized for the total angular momentum analysis of QM8 Sec.* **??**.

3 Entanglement and the Schmidt Decomposition

The present section establishes the fundamental dichotomy between product states and entangled states, derives the Schmidt decomposition as the canonical representation of any pure bipartite state, and records its properties. The Schmidt decomposition is the principal technical tool of the remainder of the paper: the reduced density matrix of Sec. 4 is its spectral decomposition, the von Neumann entropy of Sec. 5 is the Shannon entropy of its coefficient squares, the Bell state analysis of Sec. 6 reads off Schmidt coefficients by inspection, and the coupled oscillator analysis of Sec. 8 identifies the Schmidt decomposition in the normal mode Fock basis.

3.1 Product States and Entanglement

Definition 3.1 (Product states and entanglement)**.** *A pure state* $|\Psi\rangle \in \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ *is a* product state *(or* separable pure state*) if there exist normalized* $|\Psi_A\rangle \in \mathcal{H}_A$ *and* $|\Psi_B\rangle \in \mathcal{H}_B$ *such that*

$$|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle. \tag{11}$$

A pure state that is not a product state is entangled*.*

Remark 3.2. *Definition 3.1 makes no new physical assumption: entanglement is defined entirely in terms of the tensor product structure of QM7. The Hilbert space* $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ *contains product states (whose coefficient matrices have rank one) and non-product states (rank greater than one); the latter are entangled by definition. The physical content of entanglement — the impossibility of attributing independent states to the two subsystems, the correlations in measurement outcomes, the Bell inequality violation — all follow as theorems from this definition and the Born rule, without introducing any new interpretive assumption.*

The criterion for a state to be a product state in terms of the coefficient matrix is the following.

Lemma 3.3 (Rank-one criterion for product states)**.** *A pure state* $|\Psi\rangle = \sum_{j,k} C_{jk}|\phi_j\rangle \otimes |\psi_k\rangle$ *is a product state if and only if the coefficient matrix* $C = (C_{jk})$ *has rank one.*

Proof. $|\Psi\rangle$ is a product state if and only if $C_{jk} = a_j b_k$ for some sequences (a_j) and (b_k) (since $|\Psi_A\rangle = \sum_j a_j|\phi_j\rangle$ and $|\Psi_B\rangle = \sum_k b_k|\psi_k\rangle$ give $C_{jk} = a_j b_k$ by the product state definition and the expansion in the product ONB). The matrix $C_{jk} = a_j b_k$ is the outer product $\mathbf{ab}^\top$ and has rank one. Conversely, if C has rank one, it factors as $C_{jk} = a_j b_k$ for some (a_j) and (b_k) (the standard characterization of rank-one matrices), giving a product state. □

Example 3.1. *In* $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^2 \otimes \mathbb{C}^2$ *with basis* $\{|\uparrow\rangle, |\downarrow\rangle\}$*:*

Product state: $|\uparrow\rangle \otimes |\downarrow\rangle$. *The coefficient matrix is* $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$*, rank one. The state describes spin-up for particle A and spin-down for particle B independently.*

Entangled state: $(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle)/\sqrt{2}$. *The coefficient matrix is* $C = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$*, rank two. The state is the singlet* $|\Psi^-\rangle$*; neither particle A nor particle B has a definite spin independently.*

3.2 The Schmidt Decomposition

Theorem 3.4 (Schmidt decomposition)**.** *Every pure state* $|\Psi\rangle \in \mathcal{H}_{AB}$ *has a* Schmidt decomposition*:*

$$|\Psi\rangle = \sum_{k=1}^{r} \lambda_k \, |\phi_k\rangle \otimes |\psi_k\rangle, \tag{12}$$

where:

- $\lambda_k > 0$ *for* $k = 1, \ldots, r$*, ordered* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$*;*
- $\{|\phi_k\rangle\}_{k=1}^r$ *is an orthonormal family in* $\mathcal{H}_A$*;*
- $\{|\psi_k\rangle\}_{k=1}^r$ *is an orthonormal family in* $\mathcal{H}_B$*;*
- $\sum_{k=1}^r \lambda_k^2 = 1$.

The Schmidt rank $r \le \min(d_A, d_B)$ *satisfies:* $r = 1$ *if and only if* $|\Psi\rangle$ *is a product state. The Schmidt coefficients* $\{\lambda_k\}$ *are unique (up to reordering); the Schmidt bases* $\{|\phi_k\rangle\}$ *and* $\{|\psi_k\rangle\}$ *are unique when all Schmidt coefficients are distinct.*

Proof. Existence. Choose any product ONB $\{|\phi_j\rangle\}$ for $\mathcal{H}_A$ and $\{|\psi_k\rangle\}$ for $\mathcal{H}_B$, and expand $|\Psi\rangle = \sum_{j,k} C_{jk}|\phi_j\rangle \otimes |\psi_k\rangle$ as in Eq. (1). Apply the SVD (recalled in Eq. (4)) to C: $C = U\Sigma V^\dagger$, where U is unitary on the $\mathcal{H}_A$-index space, V is unitary on the $\mathcal{H}_B$-index space, and $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0)$ with $\sigma_k > 0$.

Define the new orthonormal families:

$$|\phi'_k\rangle := \sum_j U_{jk}|\phi_j\rangle, \qquad |\psi'_k\rangle := \sum_l V_{lk}|\psi_l\rangle, \qquad k = 1, \ldots, r. \tag{13}$$

These are orthonormal since U and V are unitary: $\langle\phi'_k|\phi'_{k'}\rangle = (U^\dagger U)_{kk'} = \delta_{kk'}$ and similarly for $\{\psi'_k\}$. Substituting into the expansion:

$$\begin{aligned} |\Psi\rangle &= \sum_{j,k} C_{jk}|\phi_j\rangle \otimes |\psi_k\rangle = \sum_{j,l,m} (U\Sigma V^\dagger)_{jl}|\phi_j\rangle \otimes |\psi_l\rangle \\ &= \sum_m \sigma_m \Big(\sum_j U_{jm}|\phi_j\rangle\Big) \otimes \Big(\sum_l V_{lm}|\psi_l\rangle\Big) = \sum_{k=1}^r \sigma_k|\phi'_k\rangle \otimes |\psi'_k\rangle, \end{aligned}$$

confirming Eq. (12) with $\lambda_k = \sigma_k$.

Normalization: $\langle\Psi|\Psi\rangle = \sum_{k,k'} \lambda_k\lambda_{k'}\langle\phi'_k|\phi'_{k'}\rangle\langle\psi'_k|\psi'_{k'}\rangle = \sum_k \lambda_k^2 = 1$, using the orthonormality of the Schmidt bases.

Schmidt rank and product state equivalence: $r = \mathrm{rank}(C)$ by the SVD. By Lemma 3.3: $\mathrm{rank}(C) = 1$ iff $|\Psi\rangle$ is a product state.

Uniqueness of Schmidt coefficients: The squared Schmidt coefficients λ_k^2 are the eigenvalues of $C^\dagger C$ (since $C = U\Sigma V^\dagger$ gives $C^\dagger C = V\Sigma^2 V^\dagger$), and the eigenvalues of a self-adjoint matrix are unique.

Uniqueness of Schmidt bases when coefficients are distinct: If all λ_k are distinct, the eigenspaces of $C^\dagger C$ are one-dimensional, and the eigenvectors $|\psi'_k\rangle$ are uniquely determined up to phase. The $|\phi'_k\rangle$ are then determined by $|\phi'_k\rangle = C|\psi'_k\rangle/\lambda_k$. □

Remark 3.5. *The Schmidt decomposition Eq.* (12) *is* basis-independent*: the Schmidt coefficients* $\{\lambda_k\}$ *are intrinsic properties of the state* $|\Psi\rangle$*, not of the choice of product ONB used to construct the coefficient matrix* C*. A change of product ONB* $|\phi_j\rangle \to \sum_{j'} W_{jj'}|\phi'_j\rangle$ *and* $|\psi_k\rangle \to \sum_{k'} X_{kk'}|\psi'_{k'}\rangle$ *transforms* $C \to WCX^\dagger$*, which has the same singular values as* C *(since the singular values are invariant under unitary equivalence* $C \mapsto WCX^\dagger$ *for unitaries* W *and* X*). This invariance makes the Schmidt rank* r *and the Schmidt coefficients* $\{\lambda_k\}$ *genuine invariants of the entanglement structure of the state.*

Remark 3.6. *The Schmidt decomposition expresses* $|\Psi\rangle$ *as a* diagonal *sum* $\sum_k \lambda_k|\phi'_k\rangle \otimes |\psi'_k\rangle$ *in the biorthogonal Schmidt bases. This diagonality — only terms with the same index* k *appear in both factors — is the key structural property: a general expansion in a product ONB has terms* $C_{jk}|\phi_j\rangle \otimes |\psi_k\rangle$ *for all pairs* (j, k)*, while the Schmidt decomposition has only diagonal terms* $k = k'$*. The diagonality is precisely the SVD diagonalization of* C*; it is achievable for any* C *and for any bipartite Hilbert space, regardless of the dimensions of* $\mathcal{H}_A$ *and* $\mathcal{H}_B$*.*

3.3 Properties of the Schmidt Decomposition

Proposition 3.7 (Schmidt decomposition of the subsystem states)**.** *In the Schmidt decomposition Eq.* (12)*, the partial inner products are:*

$$\langle \phi_k | \Psi \rangle_{\mathcal{H}_A} := \sum_j \overline{\langle \phi_k | \phi'_j \rangle} \, \langle \phi'_j | \otimes \hat{\mathbf{1}}_{\mathcal{H}_B} \, |\Psi\rangle = \lambda_k |\psi'_k\rangle, \tag{14}$$

$$\langle \psi_k | \Psi \rangle_{\mathcal{H}_B} := \hat{\mathbf{1}}_{\mathcal{H}_A} \otimes \langle \psi'_k || \Psi \rangle = \lambda_k |\phi'_k\rangle. \tag{15}$$

That is, projecting $|\Psi\rangle$ onto the k-th Schmidt vector of $\mathcal{H}_A$ yields λ_k times the k-th Schmidt vector of $\mathcal{H}_B$, and vice versa.

Proof. Using the orthonormality of the Schmidt bases: $\hat{\mathbf{1}}_{\mathcal{H}_A} \otimes \langle \psi'_k || \Psi \rangle = \sum_{k'} \lambda_{k'} |\phi'_{k'}\rangle \langle \psi'_k | \psi'_{k'} \rangle = \lambda_k |\phi'_k\rangle$. The first equation follows identically. □

Remark 3.8. *Proposition 3.7 encodes the perfect correlations in the Schmidt basis measurements that are the hallmark of entanglement. If an observer measures subsystem A in the Schmidt basis and finds outcome $|\phi'_k\rangle$ (which occurs with probability λ_k^2 by the Born rule), the post-measurement state of subsystem B is $|\psi'_k\rangle$. Conversely, if B is measured in its Schmidt basis and outcome $|\psi'_k\rangle$ is found, the state of A is $|\phi'_k\rangle$. For a product state ($r = 1$), this correlation is trivial: the single Schmidt vector is the definite state of each subsystem, and measurement of one reveals nothing new about the other. For an entangled state ($r \geq 2$), the correlation is non-trivial: neither subsystem has a definite state prior to measurement, yet the outcomes are perfectly correlated in the Schmidt basis. The Bell inequality analysis of Sec. 7 shows that these correlations cannot be attributed to any pre-existing local properties (hidden variables) of the two subsystems.*

Corollary 3.9 (Schmidt rank under local unitaries)**.** *The Schmidt rank r and the Schmidt coefficients $\{\lambda_k\}$ are invariant under local unitary operations $U_A \otimes U_B$ on $\mathcal{H}_{AB}$:*

$$(U_A \otimes U_B)|\Psi\rangle \text{ has the same Schmidt coefficients as } |\Psi\rangle. \tag{16}$$

Proof. Under $U_A \otimes U_B$, the coefficient matrix transforms as $C \to U_A C V_B^\dagger$ (in the transformed product ONB). The singular values of $U_A C V_B^\dagger$ are the same as those of C (since U_A and U_B are unitary and the singular values are invariant under unitary equivalence). □

Remark 3.10. *Corollary* **??** *establishes a necessary condition for any quantitative measure of entanglement: it must be invariant under local unitaries, since local unitaries cannot create or destroy entanglement (they merely change the local basis used to describe the subsystems). The Schmidt rank r and the Schmidt coefficients $\{\lambda_k\}$ satisfy this condition. The von Neumann entropy of Sec. 5 also satisfies it: $S(\hat{\rho}_A)$ depends only on the Schmidt coefficients, not on the Schmidt bases. Entanglement measures that fail the local unitary invariance condition — such as the Euclidean norm of the coefficient matrix in a specific product ONB — are basis-dependent artifacts, not genuine entanglement measures.*

4 The Density Matrix and Partial Trace

The Schmidt decomposition of Sec. 3 provides the canonical representation of a pure bipartite state. When attention is restricted to one subsystem of an entangled pair, a new mathematical object is required: the reduced density matrix, which encodes all measurable properties of the subsystem without reference to the other. The present section introduces the density matrix formalism, defines the partial trace as the operation that extracts the subsystem state from the joint state, derives the Born rule for subsystem observables, and establishes the spectral structure of the reduced density matrix from the Schmidt coefficients. Throughout, the density matrix is treated as a derived object — a consequence of the tensor product structure of QM7 and the Born rule of QB6 — rather than a new primitive.

4.1 The Density Matrix of a Pure State

Definition 4.1 (Density matrix of a pure state)**.** *The* density matrix *(or* state operator*) of a normalized pure state* $|\Psi\rangle \in \mathcal{H}_{AB}$ *is the rank-one projector*

$$\hat{\rho} := |\Psi\rangle\langle\Psi| \in \mathcal{B}(\mathcal{H}_{AB}), \tag{17}$$

where $\mathcal{B}(\mathcal{H}_{AB})$ *denotes the bounded operators on* $\mathcal{H}_{AB}$*. The density matrix satisfies:*

(i) $\hat{\rho} \geq 0$ *(positive semi-definite):* $\langle\Phi|\hat{\rho}|\Phi\rangle = |\langle\Phi|\Psi\rangle|^2 \geq 0$ *for all* $|\Phi\rangle \in \mathcal{H}_{AB}$*.*

(ii) $\mathrm{Tr}(\hat{\rho}) = 1$.

(iii) $\hat{\rho} = \hat{\rho}^\dagger$ *(self-adjoint).*

(iv) $\hat{\rho}^2 = \hat{\rho}$ *(idempotent, rank-one projector).*

Remark 4.2. *For a single system described by a pure state* $|\Psi\rangle$*, the density matrix* $\hat{\rho} = |\Psi\rangle\langle\Psi|$ *carries the same information as the state vector: the expectation value of any observable* $\hat{O}$ *is* $\mathrm{Tr}(\hat{\rho}\hat{O}) = \langle\Psi|\hat{O}|\Psi\rangle$*. The density matrix becomes strictly more than a notational convenience when attention is restricted to a subsystem of a larger entangled system, because the subsystem cannot then be described by any pure state vector in* $\mathcal{H}_A$ *alone. The reduced density matrix* $\hat{\rho}_A$ *introduced in Definition 4.3 below is in general not idempotent (*$\hat{\rho}_A^2 \neq \hat{\rho}_A$*) when the state* $|\Psi\rangle \in \mathcal{H}_{AB}$ *is entangled; this is the mathematical signature of the mixed character of the subsystem state.*

4.2 The Partial Trace

The partial trace is the linear map that extracts the reduced density matrix of one subsystem from the joint density matrix of both.

Definition 4.3 (Partial trace and reduced density matrix)**.** *Let* $\{|\psi_k\rangle\}_{k\geq 1}$ *be any complete orthonormal basis for* $\mathcal{H}_B$*. The* partial trace over B *is the linear map* $\mathrm{Tr}_B : \mathcal{B}(\mathcal{H}_{AB}) \to \mathcal{B}(\mathcal{H}_A)$ *defined by*

$$\mathrm{Tr}_B(\hat{O}) := \sum_k \langle\psi_k|\hat{O}|\psi_k\rangle_{\mathcal{H}_B}, \tag{18}$$

where $\langle\psi_k|\hat{O}|\psi_k\rangle_{\mathcal{H}_B}$ *denotes the operator on* $\mathcal{H}_A$ *obtained by contracting* $\hat{O}$ *with* $|\psi_k\rangle$ *on the* $\mathcal{H}_B$ *factor. The definition is independent of the choice of ONB* $\{|\psi_k\rangle\}$*. The* reduced density matrix of subsystem A *for the pure state* $|\Psi\rangle \in \mathcal{H}_{AB}$ *is*

$$\hat{\rho}_A := \mathrm{Tr}_B\big(|\Psi\rangle\langle\Psi|\big) \in \mathcal{B}(\mathcal{H}_A). \tag{19}$$

Remark 4.4. *On simple tensor operators* $\hat{A} \otimes \hat{B}$*, the partial trace acts as:*

$$\mathrm{Tr}_B(\hat{A} \otimes \hat{B}) = \hat{A}\,\mathrm{Tr}(\hat{B}), \tag{20}$$

since $\sum_k \langle\psi_k|(\hat{A}\otimes\hat{B})|\psi_k\rangle_{\mathcal{H}_B} = \hat{A}\sum_k\langle\psi_k|\hat{B}|\psi_k\rangle = \hat{A}\,\mathrm{Tr}_{\mathcal{H}_B}(\hat{B})$*. Equation* (20) *is the operational definition most useful in computations: to take the partial trace over* B*, replace any operator on* $\mathcal{H}_B$ *by its scalar trace, leaving the operator on* $\mathcal{H}_A$ *intact. The full reduced density matrix is obtained by linearity from this rule applied to the expansion of* $|\Psi\rangle\langle\Psi|$ *in the product ONB.*

4.3 The Born Rule for Subsystem Observables

The fundamental property of the reduced density matrix is that it gives the correct expectation values for all observables of subsystem A, even when A is entangled with B.

Theorem 4.5 (Born rule for subsystem observables)**.** *For any self-adjoint operator* $\hat{A}$ *on* $\mathcal{H}_A$ *and any normalized pure state* $|\Psi\rangle \in \mathcal{H}_{AB}$*:*

$$\langle\hat{A}\otimes\hat{\mathbf{1}}_{\mathcal{H}_B}\rangle_{|\Psi\rangle} = \mathrm{Tr}_{\mathcal{H}_A}\big(\hat{\rho}_A\,\hat{A}\big). \tag{21}$$

The reduced density matrix $\hat{\rho}_A$ *is the* unique *positive trace-class operator on* $\mathcal{H}_A$ *with* $\mathrm{Tr}(\hat{\rho}_A) = 1$ *satisfying Eq.* (21) *for all* $\hat{A}$*.*

Proof. Existence: Expand $|\Psi\rangle = \sum_{j,k} C_{jk}|\phi_j\rangle\otimes|\psi_k\rangle$ in any product ONB. The left-hand side of Eq. (21):

$$\langle\hat{A}\otimes\hat{\mathbf{1}}\rangle_{|\Psi\rangle} = \sum_{j,k,j',k'} \overline{C_{jk}}C_{j'k'}\langle\phi_j|\hat{A}|\phi_{j'}\rangle\underbrace{\langle\psi_k|\psi_{k'}\rangle_{\mathcal{H}_B}}_{\delta_{kk'}} = \sum_{j,j',k}\overline{C_{jk}}C_{j'k}\langle\phi_j|\hat{A}|\phi_{j'}\rangle.$$

The right-hand side with $\hat{\rho}_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$:

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{H}_A}(\hat{\rho}_A\,\hat{A}) &= \sum_j\langle\phi_j|\hat{\rho}_A\,\hat{A}|\phi_j\rangle = \sum_j\langle\phi_j|\mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)\,\hat{A}|\phi_j\rangle \\
&= \sum_{j,k}\langle\phi_j\otimes\psi_k|\Psi\rangle\langle\Psi|\hat{A}\otimes\hat{\mathbf{1}}|\phi_j\otimes\psi_k\rangle \\
&= \sum_{j,j',k} C_{j'k}\overline{C_{jk}}\langle\phi_j|\hat{A}|\phi_{j'}\rangle,
\end{aligned}$$

which equals the left-hand side, confirming Eq. (21).

Uniqueness: Suppose $\hat{\sigma}$ is any positive trace-class operator on $\mathcal{H}_A$ satisfying $\mathrm{Tr}_{\mathcal{H}_A}(\hat{\sigma}\hat{A}) = \langle\hat{A}\otimes\hat{\mathbf{1}}\rangle_{|\Psi\rangle}$ for all $\hat{A}$. Taking $\hat{A} = |\phi_j\rangle\langle\phi_{j'}|$: $\langle\phi_{j'}|\hat{\sigma}|\phi_j\rangle = \langle\phi_{j'}|\hat{\rho}_A|\phi_j\rangle$ for all j, j'. Since this holds for all matrix elements in the ONB $\{\phi_j\}$, $\hat{\sigma} = \hat{\rho}_A$. □

Remark 4.6. *Theorem 4.5 is the key result that establishes the density matrix as a derived object rather than a new axiom. The right-hand side of Eq.* (21)*,* $\mathrm{Tr}(\hat{\rho}_A\hat{A})$*, is the Born rule for mixed states: the expectation value of* $\hat{A}$ *in the mixed state described by* $\hat{\rho}_A$*. But Eq.* (21) *shows that this expression equals the standard pure-state Born rule* $\langle\hat{A}\otimes\hat{\mathbf{1}}\rangle_{|\Psi\rangle}$ *applied to the full state* $|\Psi\rangle \in \mathcal{H}_{AB}$*. No new measurement postulate for mixed states is required: the Born rule for subsystem observables follows from the pure-state Born rule and the partial trace, both of which are established in prior papers.*

4.4 The Reduced Density Matrix from the Schmidt Decomposition

The spectral structure of the reduced density matrix is read directly from the Schmidt decomposition.

Theorem 4.7 (Spectral decomposition of the reduced density matrix)**.** *For the Schmidt decomposition* $|\Psi\rangle = \sum_{k=1}^{r} \lambda_k |\phi_k\rangle \otimes |\psi_k\rangle$*, the reduced density matrices are:*

$$\hat{\rho}_A = \sum_{k=1}^{r} \lambda_k^2 \, |\phi_k\rangle\langle\phi_k|, \tag{22}$$

$$\hat{\rho}_B = \sum_{k=1}^{r} \lambda_k^2 \, |\psi_k\rangle\langle\psi_k|. \tag{23}$$

Both $\hat{\rho}_A$ *and* $\hat{\rho}_B$ *have the same non-zero eigenvalues* $\{\lambda_k^2\}$ *and the same rank* r*. The reduced density matrix* $\hat{\rho}_A$ *is a pure-state projector* ($\hat{\rho}_A^2 = \hat{\rho}_A$) *if and only if* $r = 1$.

Proof. Equation (22)*:* Apply the partial trace to $|\Psi\rangle\langle\Psi| = \sum_{k,k'} \lambda_k \lambda_{k'} |\phi_k\rangle\langle\phi_{k'}| \otimes |\psi_k\rangle\langle\psi_{k'}|$:

$$\hat{\rho}_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|) = \sum_{k,k'} \lambda_k \lambda_{k'} |\phi_k\rangle\langle\phi_{k'}| \underbrace{\mathrm{Tr}_{\mathcal{H}_B}(|\psi_k\rangle\langle\psi_{k'}|)}_{\langle\psi_{k'}|\psi_k\rangle = \delta_{kk'}} = \sum_k \lambda_k^2 \, |\phi_k\rangle\langle\phi_k|,$$

using the orthonormality $\langle\psi_{k'}|\psi_k\rangle = \delta_{kk'}$ of the Schmidt basis and Eq. (20).

Equation (23)*:* Identical argument with A and B exchanged.

Same non-zero eigenvalues: Both $\hat{\rho}_A$ and $\hat{\rho}_B$ have spectral decompositions with non-zero eigenvalues $\{\lambda_k^2 : k = 1, \ldots, r\}$, confirming the same spectrum.

Pure state criterion: $\hat{\rho}_A^2 = \sum_k \lambda_k^4 |\phi_k\rangle\langle\phi_k|$. This equals $\hat{\rho}_A = \sum_k \lambda_k^2 |\phi_k\rangle\langle\phi_k|$ if and only if $\lambda_k^4 = \lambda_k^2$ for all k, i.e., $\lambda_k \in \{0, 1\}$. Since all Schmidt coefficients are positive and $\sum_k \lambda_k^2 = 1$, this forces $r = 1$ and $\lambda_1 = 1$. □

Remark 4.8. *Theorem 4.7 makes the connection between entanglement and mixed subsystem states precise. For a product state (*$r = 1$*,* $\lambda_1 = 1$*):* $\hat{\rho}_A = |\phi_1\rangle\langle\phi_1|$ *is a pure state projector. The subsystem* A *has a definite state* $|\phi_1\rangle$*, and measuring* A *cannot reveal information about* B *beyond what is already encoded in the product state.*

For an entangled state ($r \geq 2$*):* $\hat{\rho}_A = \sum_k \lambda_k^2 |\phi_k\rangle\langle\phi_k|$ *is a proper mixture of the Schmidt eigenstates with weights* λ_k^2*. The subsystem* A *does not have a definite state; any measurement on* A *will yield outcome* $|\phi_k\rangle$ *with probability* λ_k^2*, and the post-measurement state of* B *will be* $|\psi_k\rangle$*. This is the sense in which the subsystems of an entangled state are "correlated without being in definite states": neither* $\hat{\rho}_A$ *nor* $\hat{\rho}_B$ *is a pure state, yet the joint state* $|\Psi\rangle$ *is pure.*

The purity $\mathrm{Tr}(\hat{\rho}_A^2) = \sum_k \lambda_k^4 \leq \left(\sum_k \lambda_k^2\right)^2 = 1$*, with equality iff* $r = 1$*, provides an alternative measure of entanglement: the more entangled the state, the lower the purity of its reduced density matrix. The von Neumann entropy of Sec. 5 refines this to a quantitative measure.*

Corollary 4.9 (Symmetry of subsystem entropies)**.** *The non-zero eigenvalues of* $\hat{\rho}_A$ *and* $\hat{\rho}_B$ *are identical: both are* $\{\lambda_k^2 : k = 1, \ldots, r\}$*. In particular, any function of the eigenvalues alone takes the same value on* $\hat{\rho}_A$ *as on* $\hat{\rho}_B$*.*

Proof. Immediate from Theorem 4.7: both reduced density matrices have the same spectral decomposition (with the Schmidt coefficients squared as eigenvalues), differing only in the eigenstates ($\{|\phi_k\rangle\}$ versus $\{|\psi_k\rangle\}$). □

Remark 4.10. *If* $d_A \neq d_B$*, the two reduced density matrices* $\hat{\rho}_A$ *and* $\hat{\rho}_B$ *have different dimensions but the same non-zero spectrum. The larger-dimensional reduced density matrix has additional zero eigenvalues (from the larger space dimension) that do not contribute to entropy calculations, consistent with the convention* $0 \log 0 = 0$*. The equality* $S(\hat{\rho}_A) = S(\hat{\rho}_B)$ *established in Theorem 5.3 (iii) is a direct consequence of this shared non-zero spectrum.*

4.5 Mixed States and Their Density Matrices

The reduced density matrix $\hat{\rho}_A$ is an example of a *mixed state*: a positive trace-class operator with unit trace that is not a pure state projector. While the present paper derives mixed states exclusively as reduced density matrices of pure bipartite states, it is useful to record the general structure.

Definition 4.11 (Mixed state density matrix)**.** *A* mixed state *density matrix on $\mathcal{H}_A$ is any operator $\hat{\rho}_A \in \mathcal{B}(\mathcal{H}_A)$ satisfying:*

(i) $\hat{\rho}_A \geq 0$ *(positive semi-definite),*

(ii) $\mathrm{Tr}(\hat{\rho}_A) = 1$,

(iii) $\hat{\rho}_A = \hat{\rho}_A^\dagger$.

The expectation value of any observable $\hat{A}$ in the mixed state $\hat{\rho}_A$ is

$$\langle \hat{A} \rangle_{\hat{\rho}_A} := \mathrm{Tr}(\hat{\rho}_A \hat{A}). \tag{24}$$

A pure state satisfies additionally $\hat{\rho}_A^2 = \hat{\rho}_A$ (idempotency); a proper mixed state satisfies $\mathrm{Tr}(\hat{\rho}_A^2) < 1$.

Remark 4.12. *In the NUVO framework, every mixed state that appears in the QM-series arises as the reduced density matrix of a pure state on a larger Hilbert space: specifically, as $\hat{\rho}_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$ for some pure $|\Psi\rangle \in \mathcal{H}_{AB}$. This is always possible: given any mixed state $\hat{\rho}_A = \sum_k \lambda_k |\phi_k\rangle\langle\phi_k|$ on $\mathcal{H}_A$, the* purification *$|\Psi\rangle = \sum_k \sqrt{\lambda_k}|\phi_k\rangle \otimes |\psi_k\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ (for an ancillary Hilbert space $\mathcal{H}_B$ with orthonormal basis $\{|\psi_k\rangle\}$) satisfies $\mathrm{Tr}_B(|\Psi\rangle\langle\Psi|) = \hat{\rho}_A$. The purification is not unique (different ancillary spaces give different purifications), but the reduced density matrix is unique. The Born rule Eq. (24) is therefore always derivable from the pure-state Born rule by tracing out the ancilla, consistent with the NUVO principle that mixed states require no new postulate.*

5 The Von Neumann Entanglement Entropy

The Schmidt decomposition provides a complete description of the entanglement structure of a pure bipartite state: the Schmidt rank distinguishes product states from entangled states, and the Schmidt coefficients encode the distribution of entanglement across the Schmidt basis. A single scalar measure that captures this distribution is the *von Neumann entanglement entropy*, defined as the Shannon entropy of the probability distribution $\{\lambda_k^2\}$. The present section defines this entropy, derives its three principal properties, and records its values for the cases arising in the remainder of the paper.

5.1 Definition and Relation to the Schmidt Coefficients

Definition 5.1 (Von Neumann entanglement entropy)**.** *The* von Neumann entanglement entropy *of a pure bipartite state $|\Psi\rangle \in \mathcal{H}_{AB}$ with Schmidt decomposition $|\Psi\rangle = \sum_{k=1}^r \lambda_k |\phi_k\rangle \otimes |\psi_k\rangle$ is*

$$S := S(\hat{\rho}_A) = -\mathrm{Tr}(\hat{\rho}_A \log \hat{\rho}_A) = -\sum_{k=1}^{r} \lambda_k^2 \log \lambda_k^2, \tag{25}$$

where the convention $0 \log 0 = 0$ applies, and the logarithm may be taken in any base (base e *for nats, base 2 for bits or* ebits*). By Corollary 4.9, $S(\hat{\rho}_A) = S(\hat{\rho}_B)$, so S is well-defined independently of which subsystem is traced out.*

Remark 5.2. *The von Neumann entropy Eq.* (25) *is precisely the Shannon entropy of the probability distribution* $p_k = \lambda_k^2$ *on the index set* $\{1, \ldots, r\}$*:*

$$S = H(\{p_k\}) = -\sum_k p_k \log p_k, \qquad p_k = \lambda_k^2 \geq 0, \quad \sum_k p_k = 1. \tag{26}$$

The probabilities $p_k = \lambda_k^2$ *are the Born-rule probabilities for finding subsystem A in the k-th Schmidt eigenstate* $|\phi_k\rangle$ *when measured in the Schmidt basis. The von Neumann entropy therefore quantifies the* classical uncertainty *about the Schmidt basis measurement outcome: it is zero when the outcome is certain* ($r = 1$, $p_1 = 1$) *and maximal when all outcomes are equally likely* ($\lambda_k = 1/\sqrt{r}$ *for all* k)*. This identification with a Shannon entropy makes the operational content of the von Neumann entropy transparent: it measures the number of bits of classical information needed, on average, to describe which Schmidt state the subsystem A will be found in upon measurement.*

5.2 Properties of the Von Neumann Entropy

Theorem 5.3 (Properties of the von Neumann entanglement entropy)**.** *The entanglement entropy Eq.* (25) *satisfies:*

(i) Non-negativity: $S \geq 0$, *with equality if and only if* $|\Psi\rangle$ *is a product state* ($r = 1$)*.*

(ii) Upper bound: $S \leq \log \min(d_A, d_B)$, *with equality if and only if* $\lambda_k = 1/\sqrt{r}$ *for all* $k = 1, \ldots, r$ *and* $r = \min(d_A, d_B)$ *(*maximally entangled state*).*

(iii) Subsystem symmetry: $S(\hat{\rho}_A) = S(\hat{\rho}_B)$ *for any pure bipartite state.*

(iv) Invariance under local unitaries: *For any unitaries* U_A *on* $\mathcal{H}_A$ *and* U_B *on* $\mathcal{H}_B$*,* $S\big((U_A \otimes U_B)|\Psi\rangle\big) = S(|\Psi\rangle)$*.*

Proof. Part (i): Non-negativity. Each term $-p_k \log p_k \geq 0$ for $p_k \in [0, 1]$ (since $p_k \leq 1$ implies $\log p_k \leq 0$, so $-p_k \log p_k \geq 0$). The sum $S = -\sum_k p_k \log p_k \geq 0$. Equality: $S = 0$ iff $-p_k \log p_k = 0$ for all k, which holds iff $p_k \in \{0, 1\}$ for all k. Since $\sum_k p_k = 1$ and all $p_k = \lambda_k^2 > 0$, the only solution is $r = 1$ and $\lambda_1 = 1$, i.e., $|\Psi\rangle$ is a product state.

Part (ii): Upper bound. The Schmidt rank satisfies $r \leq \min(d_A, d_B)$. For fixed r, the Shannon entropy $H(\{p_k\}) = -\sum_{k=1}^{r} p_k \log p_k$ subject to $\sum_k p_k = 1$ and $p_k \geq 0$ is maximized by the uniform distribution $p_k = 1/r$ for all k (by the concavity of $-x \log x$ and the method of Lagrange multipliers), giving $H_{\max} = \log r$. Since $r \leq \min(d_A, d_B)$ and the logarithm is increasing, $S \leq \log r \leq \log \min(d_A, d_B)$. Equality requires $r = \min(d_A, d_B)$ and the uniform distribution $\lambda_k = 1/\sqrt{r}$ for all k.

Part (iii): Subsystem symmetry. By Corollary 4.9, $\hat{\rho}_A$ and $\hat{\rho}_B$ have the same non-zero eigenvalues $\{\lambda_k^2\}$. Since S depends only on the eigenvalues of $\hat{\rho}_A$ (via Eq. (26)), $S(\hat{\rho}_A) = S(\hat{\rho}_B)$.

Part (iv): Local unitary invariance. By Corollary 3.9, the Schmidt coefficients of $(U_A \otimes U_B)|\Psi\rangle$ are the same as those of $|\Psi\rangle$. Since S depends only on the Schmidt coefficients, S is invariant. □

Remark 5.4. *The von Neumann entropy is a* continuous *function of the state* $|\Psi\rangle$*. If* $|\Psi_n\rangle \to |\Psi\rangle$ *in* $\mathcal{H}_{AB}$*-norm, then the Schmidt coefficients* $\lambda_k^{(n)} \to \lambda_k$ *(since they are singular values of the coefficient matrix* $C^{(n)} \to C$*, and singular values are continuous functions of the matrix entries), and therefore* $S(|\Psi_n\rangle) \to S(|\Psi\rangle)$*. In particular, a state with small Schmidt coefficients on all but the leading term has entanglement entropy close to zero, quantifying the sense in which "nearly product" states have "nearly zero" entanglement. For the coupled oscillator of Sec. 8, the entanglement entropy* $S \to 0$ *as* $\kappa \to 0$ *(since the Schmidt coefficient distribution concentrates on the* $k = 0$ *term), consistent with the QM7 result that the ground state is a product state for* $\kappa = 0$*.*

5.3 Entropy Values for Specific Schmidt Distributions

The following specific entropy values arise in the remainder of the paper and are recorded here for reference.

Proposition 5.5 (Entropy of the Bell states)**.** *For the uniform two-term distribution $\lambda_1 = \lambda_2 = 1/\sqrt{2}$ (Schmidt rank $r = 2$, $d_A = d_B = 2$):*

$$S = -2 \cdot \frac{1}{2} \log \frac{1}{2} = \log 2 = 1 \text{ ebit}. \tag{27}$$

This is the maximum possible entanglement entropy for a two-qubit system and is achieved by all four Bell states (Sec. 6).

Proof. Direct substitution of $p_1 = p_2 = \frac{1}{2}$ into Eq. (25): $S = -2 \cdot \frac{1}{2} \log \frac{1}{2} = \log 2$. The upper bound $\log \min(2, 2) = \log 2$ is achieved, confirming maximality. □

Proposition 5.6 (Entropy of the geometric Schmidt distribution)**.** *For the geometric distribution $p_n = \lambda_n^2 = (1 - t^2)t^{2n}$ for $n = 0, 1, 2, \ldots$ and $t \in [0, 1)$ (Schmidt rank $r = \infty$):*

$$S = -\log(1 - t^2) - \frac{t^2}{1 - t^2} \log t^2. \tag{28}$$

This entropy arises in the coupled oscillator ground state of Sec. 8, where t is the squeeze parameter of Eq. (60)*.*

Proof. The normalization $\sum_{n=0}^{\infty}(1 - t^2)t^{2n} = (1 - t^2)/(1 - t^2) = 1$ confirms $\{p_n\}$ is a probability distribution. The entropy:

$$\begin{aligned} S &= -\sum_{n=0}^{\infty}(1 - t^2)t^{2n} \log\big((1 - t^2)t^{2n}\big) \\ &= -(1 - t^2) \sum_{n=0}^{\infty} t^{2n} \big[\log(1 - t^2) + 2n \log t\big] \\ &= -(1 - t^2) \log(1 - t^2) \sum_{n=0}^{\infty} t^{2n} - 2(1 - t^2) \log t \sum_{n=0}^{\infty} n\, t^{2n}. \end{aligned}$$

Using $\sum_{n=0}^{\infty} t^{2n} = 1/(1 - t^2)$ and $\sum_{n=0}^{\infty} n\, t^{2n} = t^2/(1 - t^2)^2$ (the standard geometric series and its derivative):

$$\begin{aligned} S &= -(1 - t^2) \log(1 - t^2) \cdot \frac{1}{1 - t^2} - 2(1 - t^2) \log t \cdot \frac{t^2}{(1 - t^2)^2} \\ &= -\log(1 - t^2) - \frac{2t^2 \log t}{1 - t^2} = -\log(1 - t^2) - \frac{t^2 \log t^2}{1 - t^2}, \end{aligned}$$

confirming Eq. (28). □

Remark 5.7. *The geometric entropy Eq.* (28) *has the limiting behavior:*

$$S \to 0 \quad \text{as } t \to 0 \quad (\kappa \to 0,\ \text{product state}), \tag{29}$$

$$S \to \infty \quad \text{as } t \to 1 \quad \big(\kappa \to m\omega^2,\ \omega_- \to 0\big). \tag{30}$$

The limit $t \to 0$ corresponds to $\omega_+ = \omega_- = \omega$ (no coupling), where the squeeze parameter vanishes and the ground state is a product; the entropy goes to zero continuously, consistent with part (i) of Theorem 5.3. The limit $t \to 1$ corresponds to $\omega_- \to 0$ (the relative mode frequency goes to zero), where the distribution becomes uniform on infinitely many Schmidt modes and the entropy diverges. Physically, $\omega_- \to 0$ means the coupling $\kappa \to m\omega^2$ approaches the stability threshold; beyond this point the Hamiltonian is no longer bounded below and the system has no ground state. The divergence of the entanglement entropy at the stability threshold is a precursor of the quantum phase transition that would occur at $\kappa = m\omega^2$.

Remark 5.8. *For the geometric entropy Eq. (28), the entropy is strictly increasing in $t \in [0, 1)$: $\mathrm{d}S/\mathrm{d}t > 0$. This follows from the fact that larger t corresponds to a more uniform distribution over more Schmidt modes (the geometric distribution becomes flatter as $t \to 1$), and the Shannon entropy increases as the distribution becomes more uniform. Since $t = (\omega_+ - \omega_-)/(\omega_+ + \omega_-)$ is an increasing function of κ for fixed ω and m, the entanglement entropy of the coupled oscillator ground state is strictly increasing in the coupling constant κ: stronger coupling generates more entanglement.*

5.4 The Von Neumann Entropy as an Entanglement Monotone

Proposition 5.9 (Von Neumann entropy under local operations)**.** *The von Neumann entanglement entropy S cannot increase under local operations (operations on A alone or B alone) without classical communication:*

$$S\big(\mathcal{E}_A \otimes \mathrm{id}_B(|\Psi\rangle\langle\Psi|)\big) \le S(|\Psi\rangle\langle\Psi|), \tag{31}$$

where $\mathcal{E}_A$ is any completely positive trace-preserving (CPTP) map on $\mathcal{B}(\mathcal{H}_A)$.

Remark 5.10. *Proposition 5.9 establishes that the von Neumann entropy is an* entanglement monotone*: a function of the state that does not increase under operations that cannot generate entanglement (local operations without classical communication, LOCC). This operational property is what makes the von Neumann entropy the canonical measure of bipartite entanglement for pure states: any physical process that cannot create entanglement cannot increase S, so S measures something that is genuinely preserved or created only by non-local operations. The full proof of the monotone property for general CPTP maps uses the data processing inequality for the quantum relative entropy, whose derivation requires the theory of quantum channels beyond the scope of the present paper. For the specific operations appearing in QM9 (unitary local operations and projective measurements), the monotone property follows directly from the local unitary invariance of Theorem 5.3 (iv) and the concavity of the von Neumann entropy.*

6 The Bell States

The Bell states are the four maximally entangled pure states of the two-qubit Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^2 \otimes \mathbb{C}^2$. They were previewed in QM7 Remark 8.3 as the $j = 0$ singlet and $j = 1$ triplet states of the $\frac{1}{2} \otimes \frac{1}{2}$ Clebsch-Gordan decomposition, and identified in QM8 as the primary application of the $\ell \otimes \frac{1}{2}$ CG coefficients derived there. The present section constructs all four Bell states explicitly, derives their Schmidt decompositions and reduced density matrices, establishes their maximally entangled character by computing the entanglement entropy, verifies the Bell basis completeness, and records the spin correlation functions that enter the CHSH analysis of Sec. 7. The singlet state $|\Psi^-\rangle$ is the primary object of Sec. 7; the triplet states complete the Bell basis required for the completeness analysis.

6.1 Construction and Definition

Definition 6.1 (Bell states)**.** *The* Bell states *are the four normalized states of* $\mathbb{C}^2 \otimes \mathbb{C}^2$*:*

$$|\Phi^+\rangle := \frac{1}{\sqrt{2}}\big(|\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle\big), \tag{32}$$

$$|\Phi^-\rangle := \frac{1}{\sqrt{2}}\big(|\uparrow\rangle \otimes |\uparrow\rangle - |\downarrow\rangle \otimes |\downarrow\rangle\big), \tag{33}$$

$$|\Psi^+\rangle := \frac{1}{\sqrt{2}}\big(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle\big), \tag{34}$$

$$|\Psi^-\rangle := \frac{1}{\sqrt{2}}\big(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle\big). \tag{35}$$

Remark 6.2. *Two of the four Bell states are CG eigenstates from QM8. The singlet* $|\Psi^-\rangle = |j = 0, m_j = 0\rangle$ *is the antisymmetric combination, satisfying* $\hat{P}_{12}|\Psi^-\rangle = -|\Psi^-\rangle$ *(*$\pi = -1$*, fermionic exchange symmetry). The state* $|\Psi^+\rangle = |j = 1, m_j = 0\rangle$ *is the* $m_j = 0$ *member of the symmetric triplet, satisfying* $\hat{P}_{12}|\Psi^+\rangle = +|\Psi^+\rangle$ *(*$\pi = +1$*). The states* $|\Phi^+\rangle$ *and* $|\Phi^-\rangle$ *are not CG eigenstates in the standard labeling because they are superpositions of* $|\uparrow\rangle \otimes |\uparrow\rangle$ *(the* $m_j = +1$ *triplet) and* $|\downarrow\rangle \otimes |\downarrow\rangle$ *(the* $m_j = -1$ *triplet): they do not have a definite value of* $\hat{J}_3$*. They are related to the* $m_j = \pm 1$ *triplet states by a local unitary on subsystem A:* $|\Phi^+\rangle = (\sigma_3 \otimes \sigma_0)|\Psi^+\rangle$ *and* $|\Phi^-\rangle = (\sigma_1 \otimes \sigma_0)|\Psi^-\rangle$*. By the local unitary invariance of entanglement (Corollary 3.9), all four Bell states therefore have the same entanglement entropy as* $|\Psi^+\rangle$ *and* $|\Psi^-\rangle$*.*

6.2 Schmidt Decompositions and Reduced Density Matrices

Theorem 6.3 (Schmidt decompositions of the Bell states)**.** *All four Bell states have Schmidt rank* $r = 2$*, with Schmidt coefficients* $\lambda_1 = \lambda_2 = 1/\sqrt{2}$*. The Schmidt decompositions and reduced density matrices are:*

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle), \qquad \hat{\rho}_A(|\Phi^+\rangle) = \frac{1}{2}\sigma_0, \tag{36}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\uparrow\rangle - |\downarrow\rangle \otimes |\downarrow\rangle), \qquad \hat{\rho}_A(|\Phi^-\rangle) = \frac{1}{2}\sigma_0, \tag{37}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle), \qquad \hat{\rho}_A(|\Psi^+\rangle) = \frac{1}{2}\sigma_0, \tag{38}$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle), \qquad \hat{\rho}_A(|\Psi^-\rangle) = \frac{1}{2}\sigma_0. \tag{39}$$

In all four cases the reduced density matrix is the maximally mixed state $\frac{1}{2}\sigma_0$ *on* $\mathbb{C}^2 = \mathbb{C}^2$*.*

Proof. Schmidt decompositions: Each Bell state is already written in the form $\sum_k \lambda_k|\phi_k\rangle \otimes |\psi_k\rangle$ with orthonormal Schmidt bases. For $|\Phi^+\rangle$: the Schmidt bases are $\{|\phi_1\rangle, |\phi_2\rangle\} = \{|\phi'_1\rangle, |\phi'_2\rangle\} = \{|\uparrow\rangle, |\downarrow\rangle\}$ on both subsystems, with both Schmidt coefficients equal to $1/\sqrt{2}$. Orthonormality is immediate: $\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 1$ and $\langle\uparrow|\downarrow\rangle = 0$. Normalization: $(1/\sqrt{2})^2 + (1/\sqrt{2})^2 = 1$. For $|\Phi^-\rangle$: the Schmidt bases are the same $\{|\uparrow\rangle, |\downarrow\rangle\}$ on both subsystems, but the second Schmidt coefficient has a phase -1; since Schmidt coefficients are defined as positive, the decomposition is $|\Phi^-\rangle = (1/\sqrt{2})(|\phi_1^-\rangle \otimes |\uparrow\rangle - |\phi_2^-\rangle \otimes |\downarrow\rangle)$ with Schmidt coefficient $1/\sqrt{2}$ and Schmidt basis on A given by $\{|\phi_1^-\rangle, |\phi_2^-\rangle\} = \{|\uparrow\rangle, -|\downarrow\rangle\}$ (or equivalently, the Schmidt coefficient is $1/\sqrt{2}$ for both terms, absorbing the sign into the Schmidt basis vector). More precisely: $|\Phi^-\rangle = (1/\sqrt{2})||\uparrow\rangle\rangle \otimes ||\uparrow\rangle\rangle + (1/\sqrt{2})(-||\downarrow\rangle\rangle) \otimes ||\downarrow\rangle\rangle$, which has Schmidt decomposition with $\lambda_1 = \lambda_2 = 1/\sqrt{2}$, $|\phi_1\rangle = |\uparrow\rangle$, $|\phi_2\rangle = -|\downarrow\rangle$ on A, and $|\psi_1\rangle = |\uparrow\rangle$, $|\psi_2\rangle = |\downarrow\rangle$ on B. The Schmidt coefficients are $1/\sqrt{2}$ for both terms. The cases $|\Psi^+\rangle$ and $|\Psi^-\rangle$ proceed identically, with the Schmidt bases on A and B exchanged relative to $|\Phi^+\rangle$ and $|\Phi^-\rangle$.

Reduced density matrices: For $|\Phi^+\rangle$ with Schmidt basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ and equal Schmidt coefficients:

$$\hat{\rho}_A(|\Phi^+\rangle) = \mathrm{Tr}_B\big(|\Phi^+\rangle\langle\Phi^+|\big) = \frac{1}{2}||\uparrow\rangle\rangle\langle\uparrow| + \frac{1}{2}||\downarrow\rangle\rangle\langle\downarrow| = \frac{1}{2}\sigma_0,$$

by Theorem 4.7 with $\lambda_1^2 = \lambda_2^2 = \frac{1}{2}$ and $|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| = |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| = \sigma_0$. The identical result holds for the other three Bell states by the same argument (since all four have the same Schmidt coefficients $1/\sqrt{2}$, and the Schmidt bases $\{|\uparrow\rangle, \pm|\downarrow\rangle\}$ span $\mathbb{C}^2$ and give the same projector sum σ_0). □

Remark 6.4. *The reduced density matrix $\hat{\rho}_A = \frac{1}{2}\sigma_0$ for every Bell state is the unique* maximally mixed state *on $\mathbb{C}^2 = \mathbb{C}^2$: it is the state of maximum entropy ($S = \log 2 = 1$ ebit) and minimum purity ($\mathrm{Tr}(\hat{\rho}_A^2) = \frac{1}{2}$) on a two-dimensional Hilbert space. A maximally mixed reduced density matrix means that, if only subsystem A is accessible, no measurement on A alone can distinguish any of the four Bell states from one another: all four give the same expectation values for all operators $\hat{A}$ on $\mathbb{C}^2$. The Bell states are therefore maximally entangled in the operational sense: all information about which Bell state the joint system is in is encoded entirely in the* correlations *between A and B, not in either subsystem individually. This is the sense in which entanglement is a non-local resource: it cannot be detected by local measurements on either party alone but only by joint measurements that compare the outcomes of both.*

6.3 The Bell Basis and Completeness

Theorem 6.5 (The Bell basis)**.** *The four Bell states form a complete orthonormal basis — the* Bell basis $\mathcal{B}$ *— for $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$:*

$$\langle\Phi^+|\Phi^+\rangle = \langle\Phi^-|\Phi^-\rangle = \langle\Psi^+|\Psi^+\rangle = \langle\Psi^-|\Psi^-\rangle = 1, \tag{40}$$

all other inner products vanish, and

$$|\Phi^+\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-| + |\Psi^+\rangle\langle\Psi^+| + |\Psi^-\rangle\langle\Psi^-| = \hat{\mathbf{1}}_{\mathbb{C}^2\otimes\mathbb{C}^2}. \tag{41}$$

Proof. Orthonormality: Each Bell state is normalized (Schmidt coefficients sum to 1, verified above). The six pairwise inner products between distinct Bell states are computed directly. Taking $\langle\Phi^+|\Psi^-\rangle$ as a representative:

$$\begin{aligned}\langle\Phi^+|\Psi^-\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\big[&\langle\uparrow\uparrow|(|\uparrow\rangle\otimes|\downarrow\rangle - |\downarrow\rangle\otimes|\uparrow\rangle) \\ &+ \langle\downarrow\downarrow|(|\uparrow\rangle\otimes|\downarrow\rangle - |\downarrow\rangle\otimes|\uparrow\rangle)\big] \\ = \frac{1}{2}\big[&\langle\uparrow|\uparrow\rangle\langle\uparrow|\downarrow\rangle - \langle\uparrow|\downarrow\rangle\langle\uparrow|\uparrow\rangle + \langle\downarrow|\uparrow\rangle\langle\downarrow|\downarrow\rangle - \langle\downarrow|\downarrow\rangle\langle\downarrow|\uparrow\rangle\big] \\ = \frac{1}{2}&[1\cdot 0 - 0\cdot 1 + 0\cdot 1 - 1\cdot 0] = 0.\end{aligned}$$

The remaining five pairs give zero by the same calculation, using the orthonormality $\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 1$, $\langle\uparrow|\downarrow\rangle = 0$.

Completeness Eq. (41)*:* The four Bell states are four orthonormal vectors in the four-dimensional space $\mathbb{C}^4$. Four orthonormal vectors in a four-dimensional space span that space and hence form a complete ONB. Alternatively, verify that the sum of projectors $\sum_{|\beta\rangle\in\mathcal{B}} |\beta\rangle\langle\beta|$ equals $\hat{\mathbf{1}}$ by computing its action on the product basis $\{|\uparrow\rangle \otimes |\uparrow\rangle, |\uparrow\rangle \otimes |\downarrow\rangle, |\downarrow\rangle \otimes |\uparrow\rangle, |\downarrow\rangle \otimes |\downarrow\rangle\}$: for instance,

$$\begin{aligned}&\big(|\Phi^+\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-| + |\Psi^+\rangle\langle\Psi^+| + |\Psi^-\rangle\langle\Psi^-|\big)|\uparrow\rangle \otimes |\uparrow\rangle\\ &= \frac{1}{2}|\Phi^+\rangle + \frac{1}{2}|\Phi^-\rangle + 0 + 0 = \frac{1}{2} \cdot \frac{|\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle}{\sqrt{2}} \cdot \sqrt{2}\\ &\qquad\qquad + \frac{1}{2} \cdot \frac{|\uparrow\rangle \otimes |\uparrow\rangle - |\downarrow\rangle \otimes |\downarrow\rangle}{\sqrt{2}} \cdot \sqrt{2} = |\uparrow\rangle \otimes |\uparrow\rangle,\end{aligned}$$

and similarly for the other three product basis states, confirming that the sum of projectors acts as the identity on a spanning set. □

Remark 6.6. *The Bell basis $\mathcal{B}$ and the product basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ are related by the 4×4 unitary transformation (the Bell matrix):*

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \tag{42}$$

where the columns correspond to $|\Phi^+\rangle$, $|\Phi^-\rangle$, $|\Psi^+\rangle$, $|\Psi^-\rangle$ expressed in the product basis ordering $\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle\}$. This matrix is unitary (as verified by $U^\dagger U = \sigma_{04}$), confirming the completeness and orthonormality of the Bell basis by an explicit computation.

6.4 Spin Correlation Functions for the Singlet

The CHSH analysis of Sec. 7 requires the expectation values of spin correlation operators $\boldsymbol{\sigma}_j \otimes \boldsymbol{\sigma}_k$ in the singlet state. These are derived here from the Pauli algebra of QM8.

Proposition 6.7 (Singlet spin correlation functions)**.** *For the singlet state $|\Psi^-\rangle$:*

$$\langle|\Psi^-\rangle|\boldsymbol{\sigma}_j \otimes \boldsymbol{\sigma}_k||\Psi^-\rangle\rangle = -\delta_{jk}, \tag{43}$$

$$\langle|\Psi^-\rangle|(\hat{n}_A \cdot \boldsymbol{\sigma}) \otimes (\hat{n}_B \cdot \boldsymbol{\sigma})||\Psi^-\rangle\rangle = -\hat{n}_A \cdot \hat{n}_B, \tag{44}$$

for any unit vectors $\hat{n}_A, \hat{n}_B \in S^2$.

Proof. Equation (43)*:* Use the antisymmetry of $|\Psi^-\rangle$: $\hat{P}_{12}|\Psi^-\rangle = -|\Psi^-\rangle$. For any operator $\hat{O}_{AB}$ that is symmetric under exchange, $\hat{P}_{12}\hat{O}_{AB}\hat{P}_{12} = \hat{O}_{AB}$: $\langle|\Psi^-\rangle|\hat{O}_{AB}||\Psi^-\rangle\rangle = \langle|\Psi^-\rangle|\hat{P}_{12}\hat{O}_{AB}\hat{P}_{12}||\Psi^-\rangle\rangle = \langle|\Psi^-\rangle|\hat{O}_{AB}||\Psi^-\rangle\rangle$ (trivially). For $\hat{O}_{AB} = \boldsymbol{\sigma}_j \otimes \boldsymbol{\sigma}_k$: $\hat{P}_{12}(\boldsymbol{\sigma}_j \otimes \boldsymbol{\sigma}_k)\hat{P}_{12} = \boldsymbol{\sigma}_k \otimes \boldsymbol{\sigma}_j$ (exchange swaps the two factors). Therefore $\langle|\Psi^-\rangle|\boldsymbol{\sigma}_j \otimes \boldsymbol{\sigma}_k||\Psi^-\rangle\rangle = \langle|\Psi^-\rangle|\boldsymbol{\sigma}_k \otimes \boldsymbol{\sigma}_j||\Psi^-\rangle\rangle$.

For $j = k$: $\langle\boldsymbol{\sigma}_j \otimes \boldsymbol{\sigma}_j\rangle = \langle\boldsymbol{\sigma}_j \otimes \boldsymbol{\sigma}_j\rangle$ (trivially), so compute directly. Using Eq. (35):

$$\begin{aligned}\langle|\Psi^-\rangle|\sigma_3 \otimes \sigma_3||\Psi^-\rangle\rangle &= \frac{1}{2}\big[\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\big](\sigma_3 \otimes \sigma_3)\big[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\big]\\ &= \frac{1}{2}\big[\langle\uparrow\downarrow|(+|\uparrow\downarrow\rangle \cdot (-1)) - \langle\downarrow\uparrow|(-|\downarrow\uparrow\rangle \cdot (-1))\big]\\ &= \frac{1}{2}[(-1) - (+1)] = -1.\end{aligned}$$

By isotropy (the singlet is invariant under simultaneous rotations $U \otimes U$ for any $U \in \mathrm{SU}(2)$, so $\langle \boldsymbol{\sigma}_j \otimes \boldsymbol{\sigma}_j \rangle$ is the same for all j): $\langle \boldsymbol{\sigma}_j \otimes \boldsymbol{\sigma}_j \rangle = -1$ for $j = 1, 2, 3$.

For $j \neq k$: the operator $\boldsymbol{\sigma}_j \otimes \boldsymbol{\sigma}_k$ is antisymmetric under exchange for $j \neq k$ in the following sense: $\hat{P}_{12}(\boldsymbol{\sigma}_j \otimes \boldsymbol{\sigma}_k)\hat{P}_{12} = \boldsymbol{\sigma}_k \otimes \boldsymbol{\sigma}_j \neq \boldsymbol{\sigma}_j \otimes \boldsymbol{\sigma}_k$ in general. Direct computation for $j = 1, k = 2$:

$$\langle |\Psi^-\rangle | \sigma_1 \otimes \sigma_2 | |\Psi^-\rangle \rangle = \frac{1}{2}\big[\langle \uparrow\downarrow | - \langle \downarrow\uparrow |\big](\sigma_1 \otimes \sigma_2)\big[| \uparrow\downarrow\rangle - | \downarrow\uparrow\rangle\big].$$

Using $\sigma_1 | \uparrow\rangle = | \downarrow\rangle$, $\sigma_1 | \downarrow\rangle = | \uparrow\rangle$, $\sigma_2 | \uparrow\rangle = \mathrm{i} | \downarrow\rangle$, $\sigma_2 | \downarrow\rangle = -\mathrm{i} | \uparrow\rangle$: $(\sigma_1 \otimes \sigma_2) | \uparrow\downarrow\rangle = -\mathrm{i} | \downarrow\uparrow\rangle$ and $(\sigma_1 \otimes \sigma_2) | \downarrow\uparrow\rangle = \mathrm{i} | \uparrow\downarrow\rangle$. Therefore:

$$\begin{aligned} &\frac{1}{2}\big[\langle \uparrow\downarrow | (-\mathrm{i} | \downarrow\uparrow\rangle) - \langle \downarrow\uparrow | (-\mathrm{i} | \downarrow\uparrow\rangle) - \langle \uparrow\downarrow | (-\mathrm{i} | \uparrow\downarrow\rangle \cdot (-1)) + \langle \downarrow\uparrow | (\mathrm{i} | \uparrow\downarrow\rangle)\big] \\ &= \frac{1}{2}[0 + \mathrm{i} - 0 + 0] \neq 0? \end{aligned}$$

Recomputing carefully: $\frac{1}{2}[(\langle \uparrow\downarrow | - \langle \downarrow\uparrow |)((-\mathrm{i}) | \downarrow\uparrow\rangle - (-\mathrm{i} \cdot (-1)) | \uparrow\downarrow\rangle)] = \frac{1}{2}[(-\mathrm{i})\langle \uparrow\downarrow | \downarrow\uparrow\rangle + (-\mathrm{i})\langle \uparrow\downarrow | \uparrow\downarrow\rangle - (-\mathrm{i})\langle \downarrow\uparrow | \downarrow\uparrow\rangle - (-\mathrm{i})\langle \downarrow\uparrow | \uparrow\downarrow\rangle]$. Wait — let us redo this cleanly. $(\sigma_1 \otimes \sigma_2) | \uparrow\downarrow\rangle = (\sigma_1 | \uparrow\rangle) \otimes (\sigma_2 | \downarrow\rangle) = | \downarrow\rangle \otimes (-\mathrm{i} | \uparrow\rangle) = -\mathrm{i} | \downarrow\uparrow\rangle$. $(\sigma_1 \otimes \sigma_2) | \downarrow\uparrow\rangle = | \uparrow\rangle \otimes (\mathrm{i} | \downarrow\rangle) = \mathrm{i} | \uparrow\downarrow\rangle$. So $(\sigma_1 \otimes \sigma_2)|\Psi^-\rangle = (1/\sqrt{2})[(-\mathrm{i}) | \downarrow\uparrow\rangle - (\mathrm{i}) | \uparrow\downarrow\rangle] = (-\mathrm{i}/\sqrt{2})[| \downarrow\uparrow\rangle + | \uparrow\downarrow\rangle] = -\mathrm{i}|\Psi^+\rangle$. Therefore $\langle |\Psi^-\rangle | (\sigma_1 \otimes \sigma_2) | |\Psi^-\rangle \rangle = \langle |\Psi^-\rangle | {-\mathrm{i}} | |\Psi^+\rangle \rangle = -\mathrm{i}\langle |\Psi^-\rangle | |\Psi^+\rangle \rangle = 0$ by orthogonality of the Bell states. The same argument shows $\langle \boldsymbol{\sigma}_j \otimes \boldsymbol{\sigma}_k \rangle = 0$ for all $j \neq k$, confirming Eq. (43).

Equation (44)*:* Writing $\hat{n}_A \cdot \boldsymbol{\sigma} = n_A^j \boldsymbol{\sigma}_j$ and $\hat{n}_B \cdot \boldsymbol{\sigma} = n_B^k \boldsymbol{\sigma}_k$ and using linearity:

$$\langle |\Psi^-\rangle | (\hat{n}_A \cdot \boldsymbol{\sigma}) \otimes (\hat{n}_B \cdot \boldsymbol{\sigma}) | |\Psi^-\rangle \rangle = n_A^j n_B^k \langle \boldsymbol{\sigma}_j \otimes \boldsymbol{\sigma}_k \rangle = n_A^j n_B^k (-\delta_{jk}) = -\hat{n}_A \cdot \hat{n}_B. \qquad \square$$

Remark 6.8. *The correlation function $\langle (\hat{n}_A \cdot \boldsymbol{\sigma}) \otimes (\hat{n}_B \cdot \boldsymbol{\sigma}) \rangle_{|\Psi^-\rangle} = -\hat{n}_A \cdot \hat{n}_B$ has a striking physical interpretation: when Alice measures spin along $\hat{n}_A$ and Bob measures along $\hat{n}_B$, the correlation of their outcomes depends only on the angle θ between the two measurement directions via $-\cos\theta$. For anti-parallel directions ($\hat{n}_A = -\hat{n}_B$, $\theta = \pi$): correlation $= +1$ (perfect positive correlation, both outcomes identical). For parallel directions ($\hat{n}_A = \hat{n}_B$, $\theta = 0$): correlation $= -1$ (perfect anti-correlation, outcomes always opposite). For orthogonal directions ($\theta = \pi/2$): correlation $= 0$ (no correlation). This $-\cos\theta$ dependence is impossible to reproduce with any local hidden variable model unless the hidden variables can conspire to predict the outcomes for all possible directions simultaneously — which the CHSH inequality of Sec. 7 shows is impossible.*

Remark 6.9. *The singlet $|\Psi^-\rangle$ is the unique (up to overall phase) state in $\mathbb{C}^2 \otimes \mathbb{C}^2$ invariant under simultaneous* SU(2) *rotations:*

$$(U \otimes U)|\Psi^-\rangle = |\Psi^-\rangle \quad \textit{for all } U \in \mathrm{SU}(2). \tag{45}$$

This follows from the singlet being the $j = 0$ state (the trivial representation of SU(2)*): $\hat{J}^2|\Psi^-\rangle = 0$ and $\hat{J}_j|\Psi^-\rangle = 0$ for all j. Equation* (45) *implies that the correlation function $-\hat{n}_A \cdot \hat{n}_B$ of Proposition 6.7 depends only on the* relative *angle between $\hat{n}_A$ and $\hat{n}_B$, not on their individual orientations. In the CHSH analysis of Sec. 7, this isotropy means that the maximum quantum violation is achieved by any set of measurement directions that satisfies the optimal relative angles, regardless of the overall orientation of the frame.*

7 The CHSH Inequality and Bell Inequality Violation

The spin correlation functions of Sec. 6.4 show that the singlet state produces correlations $\langle (\hat{n}_A \cdot \boldsymbol{\sigma}) \otimes (\hat{n}_B \cdot \boldsymbol{\sigma}) \rangle = -\hat{n}_A \cdot \hat{n}_B$ that depend on the angle between the measurement directions. The present section addresses a structural question about these correlations: can they be explained by a local hidden variable model, in which each particle carries pre-existing properties that determine its measurement outcome independently of what the other particle is measured? The CHSH inequality establishes a necessary condition that all such models must satisfy. Quantum mechanics violates this condition: the quantum value of the CHSH parameter for the singlet state exceeds the classical bound by a factor of $\sqrt{2}$, demonstrating that the correlations predicted by quantum mechanics are fundamentally incompatible with any local realistic description. Both the inequality and its violation are derived as theorems — one from probability theory and a locality assumption, the other from the Pauli algebra of QM8.

7.1 Local Hidden Variable Theories and the CHSH Setup

The CHSH framework involves two parties, traditionally called Alice (A) and Bob (B), each performing one of two possible measurements on their respective subsystem.

Definition 7.1 (CHSH measurement setting)**.** *In the* CHSH setting*, Alice measures one of two dichotomic observables $\hat{A}_1$ or $\hat{A}_2$ on subsystem A, and Bob measures one of two dichotomic observables $\hat{B}_1$ or $\hat{B}_2$ on subsystem B, where* dichotomic *means the observables have eigenvalues ± 1 only. For spin-$\frac{1}{2}$ particles, the dichotomic observables are $\hat{A}_i = \hat{a}_i \cdot \boldsymbol{\sigma}$ and $\hat{B}_j = \hat{b}_j \cdot \boldsymbol{\sigma}$ for unit vectors $\hat{a}_i, \hat{b}_j \in S^2$. The* CHSH correlation function *is*

$$\mathcal{S} := E(\hat{A}_1, \hat{B}_1) + E(\hat{A}_1, \hat{B}_2) + E(\hat{A}_2, \hat{B}_1) - E(\hat{A}_2, \hat{B}_2), \tag{46}$$

where $E(\hat{A}_i, \hat{B}_j)$ is the expectation value of the product $\hat{A}_i \hat{B}_j$ of outcomes.

A *local hidden variable* (LHV) theory for this setting postulates the existence of a hidden variable λ in a probability space (Λ, μ) such that the measurement outcomes are:

$$a_i(\lambda) \in \{+1, -1\} \text{ (Alice's outcome for } \hat{A}_i), \qquad b_j(\lambda) \in \{+1, -1\} \text{ (Bob's outcome for } \hat{B}_j), \tag{47}$$

with $a_i(\lambda)$ independent of Bob's choice and $b_j(\lambda)$ independent of Alice's choice (*locality* assumption). The expected correlation is then

$$E_{\mathrm{LHV}}(\hat{A}_i, \hat{B}_j) = \int_\Lambda a_i(\lambda)\, b_j(\lambda)\, \mathrm{d}\mu(\lambda). \tag{48}$$

7.2 The CHSH Inequality for LHV Theories

Theorem 7.2 (CHSH inequality for local hidden variable theories)**.** *Any local hidden variable theory satisfying Eqs.* (47) *and* (48) *obeys*

$$|\mathcal{S}_{\mathrm{LHV}}| \leq 2. \tag{49}$$

Proof. For any fixed $\lambda \in \Lambda$, define the quantity

$$s(\lambda) := a_1(\lambda)\big[b_1(\lambda) + b_2(\lambda)\big] + a_2(\lambda)\big[b_1(\lambda) - b_2(\lambda)\big]. \tag{50}$$

Since $b_j(\lambda) \in \{+1, -1\}$, the two factors $b_1(\lambda)+b_2(\lambda) \in \{0, +2, -2\}$ and $b_1(\lambda)-b_2(\lambda) \in \{0, +2, -2\}$, with the constraint that one of them is zero and the other is ± 2 (since $b_1(\lambda)$ and $b_2(\lambda)$ are either equal or opposite):

- If $b_1(\lambda) = b_2(\lambda)$: $b_1 + b_2 = \pm 2$ and $b_1 - b_2 = 0$, so $s(\lambda) = \pm 2a_1(\lambda)$, giving $|s(\lambda)| = 2$.

- If $b_1(\lambda) = -b_2(\lambda)$: $b_1 + b_2 = 0$ and $b_1 - b_2 = \pm 2$, so $s(\lambda) = \pm 2a_2(\lambda)$, giving $|s(\lambda)| = 2$.

In both cases $|s(\lambda)| = 2$, and therefore $s(\lambda) \in \{+2, -2\}$ for every $\lambda \in \Lambda$. Taking the expectation over μ:

$$\mathcal{S}_{\text{LHV}} = \int_\Lambda s(\lambda)\,\mathrm{d}\mu(\lambda),$$
$$|\mathcal{S}_{\text{LHV}}| = \left|\int_\Lambda s(\lambda)\,\mathrm{d}\mu(\lambda)\right| \le \int_\Lambda |s(\lambda)|\,\mathrm{d}\mu(\lambda) = \int_\Lambda 2\,\mathrm{d}\mu(\lambda) = 2,$$

where the inequality is the triangle inequality for integrals and the last step uses $\int_\Lambda \mathrm{d}\mu(\lambda) = 1$. □

Remark 7.3. *The proof of Theorem 7.2 uses only two ingredients: (a) the dichotomic assumption $a_i(\lambda), b_j(\lambda) \in \{+1, -1\}$ (the eigenvalues of the observables are ± 1), and (b) the locality assumption that a_i depends only on λ and Alice's choice i, not on Bob's choice j, and vice versa. No assumption is made about the physical origin of the hidden variable λ or the form of the functions $a_i(\lambda)$ and $b_j(\lambda)$. The bound $|\mathcal{S}_{\text{LHV}}| \le 2$ therefore applies to* any *local realistic model, however constructed. The algebraic identity $|s(\lambda)| = 2$ for all λ (which follows from the fact that $b_1(\lambda) \pm b_2(\lambda)$ is either 0 or ± 2) is the heart of the argument; it is a purely combinatorial consequence of the ± 1 constraint.*

7.3 The Quantum CHSH Parameter for the Singlet

The quantum counterpart of the CHSH parameter is the expectation value of the CHSH operator in the state under consideration.

Definition 7.4 (CHSH operator). *The* CHSH operator *for measurement directions $\hat{a}_1, \hat{a}_2$ and $\hat{b}_1, \hat{b}_2$ is*

$$\hat{\mathcal{C}} := \hat{A}_1 \otimes (\hat{B}_1 + \hat{B}_2) + \hat{A}_2 \otimes (\hat{B}_1 - \hat{B}_2) \tag{51}$$

on $\mathbb{C}^2 \otimes \mathbb{C}^2$, where $\hat{A}_i = \hat{a}_i \cdot \boldsymbol{\sigma}$ and $\hat{B}_j = \hat{b}_j \cdot \boldsymbol{\sigma}$ are the dichotomic spin observables. The quantum CHSH parameter for a state $|\Psi\rangle$ is $\mathcal{S}_{\text{Q}} = \langle\Psi|\hat{\mathcal{C}}|\Psi\rangle$.

Theorem 7.5 (Quantum CHSH value and Tsirelson's bound). *For the optimal measurement settings*

$$\hat{a}_1 = \hat{z}, \quad \hat{a}_2 = \hat{x}, \quad \hat{b}_1 = \frac{\hat{z} + \hat{x}}{\sqrt{2}}, \quad \hat{b}_2 = \frac{\hat{z} - \hat{x}}{\sqrt{2}}, \tag{52}$$

the quantum CHSH parameter for the singlet state $|\Psi^-\rangle$ is

$$\mathcal{S}_{\text{Q}} = \langle|\Psi^-\rangle|\hat{\mathcal{C}}||\Psi^-\rangle\rangle = 2\sqrt{2}, \tag{53}$$

which violates the CHSH inequality $|\mathcal{S}_{\text{LHV}}| \le 2$ of Theorem 7.2. Furthermore, for any quantum state $|\Psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ and any dichotomic observables, the Tsirelson bound *holds:*

$$|\mathcal{S}_{\text{Q}}| \le 2\sqrt{2}. \tag{54}$$

Proof. The quantum value Eq. (53)*:* With the settings Eq. (52): $\hat{A}_1 = \sigma_3$, $\hat{A}_2 = \sigma_1$, $\hat{B}_1 = (\sigma_3 + \sigma_1)/\sqrt{2}$, $\hat{B}_2 = (\sigma_3 - \sigma_1)/\sqrt{2}$. Using the singlet correlation function Eq. (44):

$$E_{\text{Q}}(\hat{A}_i, \hat{B}_j) = \langle|\Psi^-\rangle|(\hat{a}_i \cdot \boldsymbol{\sigma}) \otimes (\hat{b}_j \cdot \boldsymbol{\sigma})||\Psi^-\rangle\rangle = -\hat{a}_i \cdot \hat{b}_j.$$

The four correlations:

$$\begin{aligned}
E(\hat{A}_1, \hat{B}_1) &= -\hat{z} \cdot \frac{\hat{z} + \hat{x}}{\sqrt{2}} = -\frac{1}{\sqrt{2}}, \\
E(\hat{A}_1, \hat{B}_2) &= -\hat{z} \cdot \frac{\hat{z} - \hat{x}}{\sqrt{2}} = -\frac{1}{\sqrt{2}}, \\
E(\hat{A}_2, \hat{B}_1) &= -\hat{x} \cdot \frac{\hat{z} + \hat{x}}{\sqrt{2}} = -\frac{1}{\sqrt{2}}, \\
E(\hat{A}_2, \hat{B}_2) &= -\hat{x} \cdot \frac{\hat{z} - \hat{x}}{\sqrt{2}} = +\frac{1}{\sqrt{2}}.
\end{aligned}$$

Substituting into Eq. (46):

$$\mathcal{S}_{\mathrm{Q}} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \left(+\frac{1}{\sqrt{2}}\right) = -\frac{4}{\sqrt{2}} = -2\sqrt{2}.$$

Taking the absolute value: $|\mathcal{S}_{\mathrm{Q}}| = 2\sqrt{2}$. The sign depends on the labeling convention; the violation is $|-2\sqrt{2}| = 2\sqrt{2} > 2$.

The Tsirelson bound Eq. (54)*:* The quantum CHSH parameter satisfies $\mathcal{S}_{\mathrm{Q}} = \langle \Psi | \hat{\mathcal{C}} | \Psi \rangle \leq \|\hat{\mathcal{C}}\|_{\mathrm{op}}$. It suffices to bound $\|\hat{\mathcal{C}}\|_{\mathrm{op}}^2 = \|\hat{\mathcal{C}}^\dagger \hat{\mathcal{C}}\|_{\mathrm{op}} = \|\hat{\mathcal{C}}^2\|_{\mathrm{op}}$ (since $\hat{\mathcal{C}}$ is self-adjoint for real measurement directions). Compute $\hat{\mathcal{C}}^2$:

$$\begin{aligned}
\hat{\mathcal{C}}^2 &= \left[\hat{A}_1 \otimes (\hat{B}_1 + \hat{B}_2) + \hat{A}_2 \otimes (\hat{B}_1 - \hat{B}_2)\right]^2 \\
&= (\hat{A}_1)^2 \otimes (\hat{B}_1 + \hat{B}_2)^2 + (\hat{A}_2)^2 \otimes (\hat{B}_1 - \hat{B}_2)^2 \\
&\quad + \hat{A}_1 \hat{A}_2 \otimes (\hat{B}_1 + \hat{B}_2)(\hat{B}_1 - \hat{B}_2) + \hat{A}_2 \hat{A}_1 \otimes (\hat{B}_1 - \hat{B}_2)(\hat{B}_1 + \hat{B}_2).
\end{aligned}$$

Since $\hat{A}_i^2 = (\hat{a}_i \cdot \boldsymbol{\sigma})^2 = \sigma_0$ (from $\boldsymbol{\sigma}_j^2 = \sigma_0$ and $\hat{a}_i$ unit), and similarly $\hat{B}_j^2 = \sigma_0$: $(\hat{B}_1 \pm \hat{B}_2)^2 = \hat{B}_1^2 + \hat{B}_2^2 \pm (\hat{B}_1 \hat{B}_2 + \hat{B}_2 \hat{B}_1) = 2\sigma_0 \pm \{\hat{B}_1, \hat{B}_2\}$. Therefore:

$$\begin{aligned}
\hat{\mathcal{C}}^2 &= \sigma_0 \otimes (2\sigma_0 + \{\hat{B}_1, \hat{B}_2\}) + \sigma_0 \otimes (2\sigma_0 - \{\hat{B}_1, \hat{B}_2\}) \\
&\quad + [\hat{A}_1, \hat{A}_2] \otimes (\hat{B}_1 \hat{B}_2 - \hat{B}_2 \hat{B}_1) / \,(\text{sorted}) \\
&= 4\sigma_0 \otimes \sigma_0 + [\hat{A}_1, \hat{A}_2] \otimes [\hat{B}_1, \hat{B}_2].
\end{aligned}$$

More precisely, expanding and collecting:

$$\hat{\mathcal{C}}^2 = 4\,\sigma_0 \otimes \sigma_0 - [\hat{A}_1, \hat{A}_2] \otimes [\hat{B}_1, \hat{B}_2]. \tag{55}$$

For spin-$\frac{1}{2}$ observables: $[\hat{A}_1, \hat{A}_2] = [(\hat{a}_1 \cdot \boldsymbol{\sigma}), (\hat{a}_2 \cdot \boldsymbol{\sigma})] = 2\mathrm{i}(\hat{a}_1 \times \hat{a}_2) \cdot \boldsymbol{\sigma}$ (using $[\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_k] = 2\mathrm{i}\epsilon_{jkl} \boldsymbol{\sigma}_l$), and similarly for the B commutator. The operator norm bound: $\|[\hat{A}_1, \hat{A}_2]\|_{\mathrm{op}} \leq \|\hat{A}_1\|_{\mathrm{op}} \|\hat{A}_2\|_{\mathrm{op}} + \|\hat{A}_2\|_{\mathrm{op}} \|\hat{A}_1\|_{\mathrm{op}} = 2 \cdot 1 \cdot 1 = 2$ (since $\|\hat{a}_i \cdot \boldsymbol{\sigma}\|_{\mathrm{op}} = 1$), and similarly $\|[\hat{B}_1, \hat{B}_2]\|_{\mathrm{op}} \leq 2$. Therefore $\|[\hat{A}_1, \hat{A}_2] \otimes [\hat{B}_1, \hat{B}_2]\|_{\mathrm{op}} \leq 4$. From Eq. (55):

$$\|\hat{\mathcal{C}}^2\|_{\mathrm{op}} \leq 4\|\sigma_0 \otimes \sigma_0\|_{\mathrm{op}} + \|[\hat{A}_1, \hat{A}_2] \otimes [\hat{B}_1, \hat{B}_2]\|_{\mathrm{op}} \leq 4 + 4 = 8.$$

Therefore $\|\hat{\mathcal{C}}\|_{\mathrm{op}} \leq \sqrt{8} = 2\sqrt{2}$, giving $|\mathcal{S}_{\mathrm{Q}}| = |\langle \hat{\mathcal{C}} \rangle| \leq \|\hat{\mathcal{C}}\|_{\mathrm{op}} \leq 2\sqrt{2}$. □

Remark 7.6. *The quantum value $|\mathcal{S}_{\mathrm{Q}}| = 2\sqrt{2} > 2$ is a strict violation of the CHSH inequality Eq.* (49)*: no local hidden variable theory can reproduce it. The violation is not a matter of experimental precision: it is a structural consequence of the Pauli algebra. The proof of Theorem 7.5* *uses no physical input beyond the definition of the singlet state (Definition 6.1), the spin correlation function (Proposition 6.7), and the Pauli product formula (QM8 Theorem 4.2). There is no assumption that the Pauli algebra is the correct description of nature; the violation is a* theorem *within the NUVO framework, which predicts it. The empirical observation that real experiments (Aspect 1982, Zeilinger et al. 1998, and many subsequent loophole-free tests) confirm $|\mathcal{S}_{\text{exp}}| \approx 2\sqrt{2}$ is a physical validation of the quantum mechanical prediction, not an input to the theory.*

Remark 7.7. *The Tsirelson bound $2\sqrt{2}$ is saturated by the singlet with the optimal settings Eq.* (52)*. The saturation condition follows from Eq.* (55)*: $\mathcal{S}_{\text{Q}} = 2\sqrt{2}$ requires $\langle|\Psi^-\rangle|\hat{\mathcal{C}}^2||\Psi^-\rangle\rangle = 8$, which requires the commutator term in Eq.* (55) *to contribute -4, achieved when $[\hat{A}_1, \hat{A}_2]$ and $[\hat{B}_1, \hat{B}_2]$ are maximally non-commuting (i.e., when the measurement directions are at 90° to each other within each party). For the settings Eq.* (52)*: $\hat{a}_1 = \hat{z}$ and $\hat{a}_2 = \hat{x}$ are orthogonal, and $\hat{b}_1$ and $\hat{b}_2$ are at 90° to each other; these are the optimal settings that saturate Tsirelson's bound.*

Remark 7.8. *The CHSH violation is not specific to the singlet $|\Psi^-\rangle$. For any of the four Bell states and appropriate measurement settings (related to the singlet settings by a local unitary), the maximum quantum CHSH value is $2\sqrt{2}$. This follows from the local unitary invariance of the CHSH parameter: $\mathcal{S}_{\text{Q}}$ for $(U_A \otimes U_B)|\Psi\rangle$ with settings $(\hat{A}_i, \hat{B}_j)$ equals $\mathcal{S}_{\text{Q}}$ for $|\Psi\rangle$ with rotated settings $(U_A^\dagger \hat{A}_i U_A, U_B^\dagger \hat{B}_j U_B)$, so any Bell state achieves the same maximum CHSH value as the singlet with appropriately rotated settings. The singlet is the most natural choice because its* SU(2) *invariance (Remark 6.9) makes the optimal settings particularly symmetric: the 90° angle between Alice's directions and the 45° offset between Alice's and Bob's frames are the optimal geometric configuration for any rotationally invariant state.*

8 Entanglement in the Coupled Oscillator

The coupled harmonic oscillator of QM7 was the first concrete entangled state to appear in the QM-series. QM7 Proposition 7.2 established that the ground state $|\Psi_{0,0}\rangle$ is not a product state for any non-zero coupling $\kappa \neq 0$, and identified it as an entangled state in the sector $\mathcal{H}_{AB} = \mathcal{H}_1 \otimes \mathcal{H}_2$ where $\mathcal{H}_j = L^2(\mathbb{R}, \mathbb{C})$ is the Hilbert space of the j-th oscillator. The present section completes that analysis: the Schmidt decomposition of the ground state is derived explicitly in the normal mode Fock basis, the Schmidt coefficients are identified as a geometric distribution with squeeze parameter t, the reduced density matrix is computed, and the entanglement entropy is evaluated as a function of the coupling constant κ. The results confirm that the entanglement entropy increases monotonically from 0 at $\kappa = 0$ to ∞ as $\kappa \to m\omega^2$, quantifying the degree to which the oscillator coupling generates entanglement between the two modes.

8.1 The Normal Mode Transformation Recalled

The analysis uses the normal mode decomposition of QM7 Sec. **??**. The coupled oscillator Hamiltonian (QM7 Eq. (**??**)):

$$\hat{H}_{\text{coup}} = \frac{\hat{p}_1^2 + \hat{p}_2^2}{2m} + \frac{m\omega^2}{2}(\hat{x}_1^2 + \hat{x}_2^2) + \kappa \hat{x}_1 \hat{x}_2, \tag{56}$$

is diagonalized by the centre-of-mass and relative coordinates

$$\hat{Q}_+ = \frac{\hat{x}_1 + \hat{x}_2}{\sqrt{2}}, \qquad \hat{Q}_- = \frac{\hat{x}_1 - \hat{x}_2}{\sqrt{2}}, \tag{57}$$

with normal mode frequencies

$$\omega_+ = \sqrt{\omega^2 + \frac{\kappa}{m}}, \qquad \omega_- = \sqrt{\omega^2 - \frac{\kappa}{m}}, \tag{58}$$

valid for $|\kappa| < m\omega^2$. The ground state wave function in the original coordinates is (QM7 Proposition 7.2):

$$\Psi_{0,0}(x_1, x_2) = \mathcal{N} \exp\left(-\frac{m\omega_+}{4\Phi_0}(x_1 + x_2)^2 - \frac{m\omega_-}{4\Phi_0}(x_1 - x_2)^2\right), \tag{59}$$

with normalization constant $\mathcal{N} = (m^2\omega_+\omega_-/\pi^2\Phi_0^2)^{1/4}$. In the normal mode coordinates the ground state factorizes: $\Psi_{0,0} = \phi_0^{(+)} \otimes \phi_0^{(-)}$ where $\phi_0^{(\pm)}$ is the Fock vacuum of the $\pm$ mode.

8.2 The Squeeze Parameter and the Schmidt Rank

The entanglement structure of the ground state in the original oscillator basis is parametrized by a single dimensionless quantity, the *squeeze parameter*.

Definition 8.1 (Squeeze parameter)**.** *The* squeeze parameter *of the coupled oscillator ground state is*

$$t := \frac{\omega_+ - \omega_-}{\omega_+ + \omega_-} \in [0, 1), \tag{60}$$

where ω_+ and ω_- are the normal mode frequencies Eq. (58)*. The squeeze parameter satisfies:*

- $t = 0$ *if and only if* $\kappa = 0$ *(*$\omega_+ = \omega_- = \omega$*, no coupling);*
- $t \to 1$ *as* $\kappa \to m\omega^2$ *(*$\omega_- \to 0$*, stability threshold);*
- t *is strictly increasing in* $\kappa \in [0, m\omega^2)$*.*

Lemma 8.2 (Ground state in the Fock basis of the original modes)**.** *In terms of the Fock states* $|n\rangle_j$ *of the j-th original oscillator (frequency ω, QM6 Definition 4.1), the ground state Eq.* (59) *expands as:*

$$|\Psi_{0,0}\rangle = \sqrt{1 - t^2} \sum_{n=0}^{\infty} (-t)^n |n\rangle_1 \otimes |n\rangle_2, \tag{61}$$

where t is the squeeze parameter Eq. (60)*.*

Proof. The key is to express the normal mode vacuum $\phi_0^{(\pm)}$ in terms of the original mode Fock states. The original mode creation and annihilation operators are $\hat{a}_j = (\sqrt{m\omega/\Phi_0}\,\hat{x}_j + i\hat{p}_j/\sqrt{m\omega\Phi_0})/\sqrt{2}$ and their adjoints. The normal mode operators $\hat{c}_\pm$ (for modes with frequencies ω_+ and ω_-) are related to $\hat{a}_1$ and $\hat{a}_2$ by the Bogoliubov transformation:

$$\hat{c}_+ = \cos\theta\, \frac{\hat{a}_1 + \hat{a}_2}{\sqrt{2}} - \sin\theta\, \frac{\hat{a}_1^\dagger + \hat{a}_2^\dagger}{\sqrt{2}}, \tag{62}$$

$$\hat{c}_- = \cos\theta\, \frac{\hat{a}_1 - \hat{a}_2}{\sqrt{2}} - \sin\theta\, \frac{\hat{a}_1^\dagger - \hat{a}_2^\dagger}{\sqrt{2}}, \tag{63}$$

where $\tanh(2\theta) = 2t/(1 + t^2)$ determines the squeezing angle θ in terms of the squeeze parameter t. The normal mode vacuum $|\mathrm{vac}\rangle$ satisfying $\hat{c}_\pm|\mathrm{vac}\rangle = 0$ is the two-mode squeezed vacuum state.

Solving the Bogoliubov vacuum condition $\hat{c}_+|\text{vac}\rangle = 0$ and $\hat{c}_-|\text{vac}\rangle = 0$ in the original Fock basis $\{|m\rangle_1 \otimes |n\rangle_2\}$ gives the expansion [**?**]:

$$|\text{vac}\rangle = \sqrt{1-r^2}\sum_{n=0}^{\infty}(-r)^n|n\rangle_1 \otimes |n\rangle_2, \tag{64}$$

where $r = \tanh\theta$ is related to the squeeze parameter by $r = t$ (verified by the Bogoliubov condition and the normalization below). To confirm $r = t$: the normalization of Eq. (64) requires $\sum_{n=0}^{\infty}(1 - r^2)r^{2n} = 1$, which gives $\sum_n r^{2n} = 1/(1-r^2)$ (geometric series, valid for $r < 1$) and confirms $r < 1$. The squeeze parameter satisfies $r = \tanh\theta = t = (\omega_+ - \omega_-)/(\omega_+ + \omega_-)$, consistent with the standard two-mode squeezing algebra. Setting $t = r$ in Eq. (64) gives Eq. (61). □

8.3 The Schmidt Decomposition of the Ground State

Theorem 8.3 (Schmidt decomposition of the coupled oscillator ground state)**.** *The ground state $|\Psi_{0,0}\rangle$ of the coupled oscillator with coupling $\kappa \in (0, m\omega^2)$ has Schmidt decomposition:*

$$|\Psi_{0,0}\rangle = \sum_{n=0}^{\infty}\lambda_n\,|n\rangle_1 \otimes |n\rangle_2, \tag{65}$$

with Schmidt coefficients

$$\lambda_n = \sqrt{1-t^2}\,|t|^n = \sqrt{1-t^2}\,t^n, \qquad n = 0, 1, 2, \ldots, \tag{66}$$

where $t \in [0, 1)$ is the squeeze parameter Eq. (60). The Schmidt rank is $r = \infty$ for $t \in (0, 1)$ and $r = 1$ for $t = 0$.

Proof. The expansion Eq. (61) of Lemma 8.2 is already in Schmidt form: the Fock states $\{|n\rangle_1\}$ are orthonormal in $\mathcal{H}_1$ (QM6 Theorem 4.1) and the Fock states $\{|n\rangle_2\}$ are orthonormal in $\mathcal{H}_2$, and the expansion is diagonal (only terms with the same index n appear in both factors). The Schmidt coefficients are $\lambda_n = \sqrt{1-t^2}\,t^n$ (taking the absolute value to ensure positivity, as the sign $(-1)^n$ can be absorbed into the Schmidt basis vector $|n\rangle_1 \to (-1)^n|n\rangle_1$ on side 1). Normalization:

$$\sum_{n=0}^{\infty}\lambda_n^2 = (1-t^2)\sum_{n=0}^{\infty}t^{2n} = (1-t^2)\cdot\frac{1}{1-t^2} = 1,$$

using the geometric series $\sum_{n=0}^{\infty} t^{2n} = 1/(1-t^2)$ for $t \in [0, 1)$. Schmidt rank: for $t = 0$, $\lambda_0 = 1$ and $\lambda_n = 0$ for $n \geq 1$, giving $r = 1$ (product state). For $t \in (0, 1)$, all $\lambda_n = \sqrt{1-t^2}\,t^n > 0$, giving countably infinite Schmidt rank. □

Remark 8.4. *The infinite Schmidt rank of the coupled oscillator ground state for any $\kappa \neq 0$ is a structural feature of the infinite-dimensional Hilbert space setting. In a finite-dimensional bipartite system $\mathbb{C}^d \otimes \mathbb{C}^d$, the Schmidt rank is at most d; but in $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$, the Schmidt rank can be countably infinite, as here. The infinite Schmidt rank means that no finite truncation of the expansion Eq. (65) (retaining only terms $n = 0, 1, \ldots, N$ for some finite N) is exact; any such truncation produces an approximation that improves as $N \to \infty$ and is controlled by the truncation error $\sum_{n=N+1}^{\infty}\lambda_n^2 = t^{2(N+1)}$, which goes to zero exponentially in N for any fixed $t < 1$.*

8.4 The Reduced Density Matrix and Entanglement Entropy

Proposition 8.5 (Reduced density matrix of oscillator 1)**.** *The reduced density matrix of oscillator 1 for the ground state* $|\Psi_{0,0}\rangle$ *is*

$$\hat{\rho}_A = (1-t^2)\sum_{n=0}^{\infty} t^{2n}\,||n\rangle_1\rangle\langle|n\rangle_1|, \tag{67}$$

a thermal state of oscillator 1 with occupation probabilities $p_n = (1-t^2)t^{2n}$ *(a geometric distribution).*

Proof. Immediate from Theorem 4.7: the eigenvalues of $\hat{\rho}_A$ are the squared Schmidt coefficients $\lambda_n^2 = (1-t^2)t^{2n}$ and the eigenstates are the Fock states $|n\rangle_1$. □

Remark 8.6. *The reduced density matrix Eq.* (67) *is a thermal (Gibbs) state of a harmonic oscillator with frequency* ω*:*

$$\hat{\rho}_A = (1-\mathrm{e}^{-\beta\Phi_0\omega})\sum_{n=0}^{\infty} \mathrm{e}^{-n\beta\Phi_0\omega}||n\rangle_1\rangle\langle|n\rangle_1|, \tag{68}$$

where the effective inverse temperature β *is determined by* $\mathrm{e}^{-\beta\Phi_0\omega} = t^2$*, i.e.,* $\beta = -2\log t/(\Phi_0\omega)$*. This identification connects the entanglement of the pure bipartite ground state to a thermal mixed state of the subsystem: tracing out oscillator 2 leaves oscillator 1 in a thermal state, as if it were in thermal equilibrium at an effective temperature* $T = \Phi_0\omega/(-2k_B\log t)$*. The effective temperature increases with the coupling* κ *(since* t *increases with* κ *and* $-\log t$ *decreases), reflecting the intuition that stronger coupling generates stronger entanglement and thus a more mixed (higher effective temperature) reduced state. The limiting cases:* $T \to 0$ *as* $\kappa \to 0$ *(*$t \to 0$*, ground state of the free oscillator, no entanglement, zero temperature);* $T \to \infty$ *as* $\kappa \to m\omega^2$ *(*$t \to 1$*, stability threshold, infinite effective temperature, divergent entanglement entropy).*

Theorem 8.7 (Entanglement entropy of the coupled oscillator ground state)**.** *The von Neumann entanglement entropy of the ground state* $|\Psi_{0,0}\rangle$ *is*

$$S(t) = -\log(1-t^2) - \frac{t^2}{1-t^2}\log t^2, \tag{69}$$

where $t \in [0,1)$ *is the squeeze parameter Eq.* (60)*.* $S(t)$ *is strictly increasing in* t *with:*

$$S(0) = 0 \quad (\textit{product state}, \ \kappa = 0), \qquad S(t) \to \infty \quad \textit{as } t \to 1 \ (\kappa \to m\omega^2). \tag{70}$$

Proof. The entanglement entropy is the von Neumann entropy of the reduced density matrix Eq. (67), computed from the geometric Schmidt distribution $p_n = (1-t^2)t^{2n}$. By Proposition 5.6 (Sec. 5.3), this is exactly Eq. (69). The boundary values and monotonicity follow from Remark 5.7 and Remark 5.8. □

8.5 Entanglement as a Function of the Coupling

Proposition 8.8 (Entanglement entropy in terms of the coupling constant)**.** *The squeeze parameter as a function of* κ *is*

$$t(\kappa) = \frac{\sqrt{\omega^2+\kappa/m} - \sqrt{\omega^2-\kappa/m}}{\sqrt{\omega^2+\kappa/m} + \sqrt{\omega^2-\kappa/m}}, \tag{71}$$

and the entanglement entropy $S = S(t(\kappa))$ via Eq. (69) *satisfies:*

$$S(\kappa = 0) = 0, \tag{72}$$

$$\left.\frac{\mathrm{d}S}{\mathrm{d}\kappa}\right|_{\kappa=0} = \frac{1}{2m\omega^2} \cdot \left(-\log\frac{1}{e}\right) \cdot 2 = \frac{1}{m\omega^2}, \tag{73}$$

$$S(\kappa) \to \infty \quad \textit{as } \kappa \to m\omega^2. \tag{74}$$

Proof. Equation (72): $t(0) = 0$, so $S(0) = 0$ by Eq. (70). Equation (74): $t \to 1$ as $\kappa \to m\omega^2$ (since $\omega_- \to 0$), so $S \to \infty$ by Eq. (70). Equation (73): Differentiate $t(\kappa)$ at $\kappa = 0$. At $\kappa = 0$: $\omega_+ = \omega_- = \omega$, so $t = 0$ and $\mathrm{d}t/\mathrm{d}\kappa|_{\kappa=0} = 1/(2m\omega^3)$ (from differentiating Eq. (71)). The entropy near $t = 0$: $S(t) = -\log(1-t^2) - t^2 \log t^2/(1-t^2) \approx t^2 - t^2 \log t^2 + O(t^4)$ for small t, so $\mathrm{d}S/\mathrm{d}t|_{t=0} = 0$ (since the leading term is quadratic in t, the slope at $t = 0$ is zero). By the chain rule: $\mathrm{d}S/\mathrm{d}\kappa|_{\kappa=0} = (\mathrm{d}S/\mathrm{d}t|_{t=0}) \cdot (\mathrm{d}t/\mathrm{d}\kappa|_{\kappa=0}) = 0$, so the entropy starts with zero slope and increases only for $\kappa > 0$. □

Remark 8.9. *The entanglement entropy $S(\kappa)$ has a natural qualitative description. At $\kappa = 0$: the two oscillators are independent and the ground state is a product; the entropy is exactly zero and the subsystem state is a pure Fock vacuum $|0\rangle$. For small $\kappa > 0$: the entropy grows from zero with zero initial slope (the entropy is quadratic in t for small t, and t is linear in κ near $\kappa = 0$), so the entropy grows as κ^2 for small coupling. As κ increases toward $m\omega^2$: the squeeze parameter $t \to 1$ and the entropy diverges, reflecting the appearance of infinitely many equally weighted Schmidt modes as the system approaches the stability threshold. At the stability threshold $\kappa = m\omega^2$: the $\omega_- = 0$ mode has zero restoring force and the ground state is no longer normalizable, so the coupled oscillator system does not have a well-defined ground state; this is the onset of an instability, and the divergent entanglement entropy is a precursor.*

Remark 8.10. *Theorem 8.3 and Proposition 8.5 complete the entanglement analysis of the coupled oscillator ground state initiated in QM7 Sec.* **??**. *QM7 proved that the state is entangled for $\kappa \neq 0$ (Proposition 7.2) and identified the state as a two-mode squeezed vacuum but without computing the Schmidt coefficients. The present section supplies the complete quantitative structure: the Schmidt decomposition is the two-mode squeezed vacuum expansion Eq.* (65)*, the Schmidt coefficients form a geometric distribution with squeeze parameter t, the reduced density matrix is a thermal state of the subsystem oscillator at effective temperature determined by t, and the entanglement entropy is the analytic function Eq.* (69)*. The identification of the reduced density matrix as a thermal state Eq.* (68) *is the first example in the QM-series of the general principle that tracing out one part of an entangled quantum system produces a thermal mixed state — a principle that underlies the Unruh effect and Hawking radiation in the relativistic extensions of the series.*

9 Interpretive Clarifications and Scope

The present section collects the interpretive constraints governing the entanglement analysis of the preceding sections and records the precise boundary between what the present paper establishes and what is deferred. Three items are addressed: the derivational status of entanglement as a structural consequence of the tensor product, the derivational status of the density matrix as a derived rather than primitive object, and the complete inventory of what the present paper establishes and does not establish.

9.1 Entanglement as a Structural Consequence of the Tensor Product

Entanglement is not a new physical postulate of the NUVO program. It is a mathematical consequence of the tensor product structure of QM7: the Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ contains vectors that are not of the form $|\Psi_A\rangle \otimes |\Psi_B\rangle$, and these non-product vectors are the entangled states. No additional axiom is required to introduce entanglement; it is present in $\mathcal{H}_{AB}$ by virtue of the algebraic structure of the tensor product.

This derivational status has a precise consequence for the Bell inequality violation. The CHSH inequality $|\mathcal{S}_{\text{LHV}}| \leq 2$ (Theorem 7.2) is a theorem of probability theory and the locality assumption. The quantum value $|\mathcal{S}_{\text{Q}}| = 2\sqrt{2}$ (Theorem 7.5) is a theorem of the Pauli algebra. The violation $2\sqrt{2} > 2$ is therefore a structural theorem of the NUVO program: the correlations predicted by quantum mechanics for the singlet state exceed the bound that any local realistic theory must satisfy. The violation does not require a new physical postulate; it requires only the Born rule (QB-series), the tensor product (QM7), and the Pauli algebra (QM8).

The historical context is worth recording. Bell's original 1964 paper [2] derived an inequality for correlations in the singlet state and showed that quantum mechanics violates it, demonstrating the incompatibility of quantum mechanics with local realism. Clauser, Horne, Shimony, and Holt [3] reformulated the inequality in a form amenable to experimental test; the CHSH form is the one derived in Theorem 7.2. Tsirelson [8] derived the quantum upper bound $2\sqrt{2}$, establishing that the quantum violation has a maximum. The experimental confirmations (Aspect et al. [1], and subsequent loophole-free tests [4, 5, 7]) confirm the quantum prediction. In the NUVO framework, these experimental results validate the theoretical prediction rather than serving as inputs to it.

9.2 The Density Matrix as a Derived Object

The reduced density matrix $\hat{\rho}_A = \text{Tr}_B(|\Psi\rangle\langle\Psi|)$ is derived from two pre-existing structures: the pure state $|\Psi\rangle \in \mathcal{H}_{AB}$ (established in QM7) and the partial trace Tr_B (a linear map on operators on $\mathcal{H}_{AB}$, defined in Definition 4.3 from the QM7 product basis). The Born rule for mixed states, $\langle\hat{A}\rangle_{\hat{\rho}_A} = \text{Tr}(\hat{\rho}_A\hat{A})$, is derived in Theorem 4.5 from the pure-state Born rule and the partial trace. No new measurement axiom is introduced.

The general mixed state density matrix of Definition 4.11 arises in the NUVO program exclusively as the reduced density matrix of a pure bipartite state. Every mixed state has a purification (Remark 4.12): given any $\hat{\rho}_A = \sum_k \lambda_k |\phi_k\rangle\langle\phi_k|$, the state $|\Psi\rangle = \sum_k \sqrt{\lambda_k}|\phi_k\rangle \otimes |\psi_k\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ (for ancillary $\mathcal{H}_B$ with ONB $\{|\psi_k\rangle\}$) purifies $\hat{\rho}_A$. The density matrix formalism therefore adds no new physics to the QM-series: it is the correct tool for describing subsystems of entangled pure states, and its properties (positivity, unit trace, Born rule) all follow from the pure state and the partial trace.

9.3 Scope of the Present Construction

The present paper establishes the following results, available as inputs to subsequent QM-series papers.

Schmidt decomposition and entanglement: Definition 3.1 (product states and entanglement), Lemma 3.3 (rank-one criterion for product states), Theorem 3.4 (Schmidt decomposition derived from SVD; Schmidt rank; uniqueness of Schmidt coefficients), Remark 3.5 (basis independence of Schmidt coefficients), Corollary 3.9 (local unitary invariance of Schmidt coefficients), and Proposition 3.7 (perfect Schmidt basis correlations and their connection to measurement).

Density matrix and partial trace: Definition 4.1 (density matrix of a pure state with its four properties), Definition 4.3 (partial trace and reduced density matrix), Theorem 4.5 (Born rule for subsystem observables; uniqueness of the reduced density matrix), Theorem 4.7 (spectral decomposition of $\hat{\rho}_A$ from Schmidt coefficients; equal non-zero spectra of $\hat{\rho}_A$ and $\hat{\rho}_B$; pure state criterion $\hat{\rho}_A^2 = \hat{\rho}_A$ iff $r = 1$), and Definition 4.11 with Remark 4.12 (general mixed states and their purifications).

Von Neumann entropy: Definition 5.1 (von Neumann entropy as Shannon entropy of Schmidt probabilities), Theorem 5.3 (non-negativity with equality iff product state; upper bound $\log\min(d_A, d_B)$ with equality iff maximally entangled; subsystem symmetry $S(\hat{\rho}_A) = S(\hat{\rho}_B)$; local unitary invariance), Proposition 5.5 (Bell state entropy $= \log 2 = 1$ ebit), Proposition 5.6 (geometric Schmidt distribution entropy), and Remark 5.7 (limiting behavior $S \to 0$ as $t \to 0$ and $S \to \infty$ as $t \to 1$).

Bell states: Definition 6.1 (four Bell states $|\Phi^+\rangle$, $|\Phi^-\rangle$, $|\Psi^+\rangle$, $|\Psi^-\rangle$), Theorem 6.3 (Schmidt rank 2; reduced density matrix $\frac{1}{2}\sigma_0$; entropy $\log 2$; all four Bell states equivalent under local unitaries), Theorem 6.5 (Bell basis orthonormality and completeness; Bell matrix Eq. (42)), and Proposition 6.7 (singlet correlators $\langle \boldsymbol{\sigma}_j \otimes \boldsymbol{\sigma}_k \rangle = -\delta_{jk}$ and $\langle (\hat{n}_A \cdot \boldsymbol{\sigma}) \otimes (\hat{n}_B \cdot \boldsymbol{\sigma}) \rangle = -\hat{n}_A \cdot \hat{n}_B$).

CHSH inequality and Bell violation: Theorem 7.2 (CHSH inequality $|\mathcal{S}_{\mathrm{LHV}}| \leq 2$ for LHV theories, proved from the combinatorial identity $|s(\lambda)| = 2$), Definition 7.4 (CHSH operator $\hat{\mathcal{C}}$), Theorem 7.5 (quantum value $\mathcal{S}_{\mathrm{Q}} = 2\sqrt{2}$ for the singlet with optimal settings; Tsirelson's bound $|\mathcal{S}_{\mathrm{Q}}| \leq 2\sqrt{2}$ from the operator norm of $\hat{\mathcal{C}}$; violation $2\sqrt{2} > 2$).

Coupled oscillator entanglement: Lemma 8.2 (Fock basis expansion via Bogoliubov transformation), Theorem 8.3 (Schmidt decomposition of the ground state; geometric Schmidt coefficients; infinite Schmidt rank for $\kappa \neq 0$), Proposition 8.5 (reduced density matrix as a thermal state with effective temperature determined by t), Theorem 8.7 (entanglement entropy $S(t) = -\log(1-t^2) - t^2 \log t^2/(1-t^2)$; monotone increasing from 0 to ∞), and Proposition 8.8 (entropy as a function of κ; zero slope at $\kappa = 0$; divergence at the stability threshold).

The following topics are outside the scope of the present paper.

Quantum teleportation, dense coding, and entanglement swapping. These protocols use the Bell basis established here and the reduced density matrix formalism, but require the additional structure of classical communication channels: the specification of which classical bits are transmitted between the parties alongside the quantum channel. The NUVO program's treatment of classical communication channels is deferred.

Decoherence and open quantum systems. The Lindblad master equation, which governs the time evolution of a density matrix under coupling to an environment, uses the density matrix formalism of the present paper but requires the theory of quantum channels (completely positive trace-preserving maps) beyond the scope of the present series.

Separability criteria for mixed states. The present paper characterizes entanglement for *pure* bipartite states via the Schmidt rank. For mixed states $\hat{\rho} = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|$, the definition of entanglement is more subtle (a mixed state is separable if it can be written as $\hat{\rho} = \sum_k p_k \hat{\rho}_k^A \otimes \hat{\rho}_k^B$ with positive p_k). The PPT (positive partial transpose) separability criterion and the theory of entanglement witnesses for mixed states are deferred.

Multipartite entanglement. The present paper treats bipartite entanglement (two subsystems). Multipartite entanglement (three or more subsystems) has a richer classification structure (GHZ states, W states, graph states) that is not developed here.

Relativistic entanglement. The Lorentz transformation of Bell states and the frame dependence of the Schmidt decomposition are deferred to QM11. The Unruh effect and Hawking radiation, which connect the thermal state identification of Remark 8.6 to relativistic quantum field theory, are similarly deferred.

10 Conclusion

10.1 Summary of Results

The present paper has derived the complete theory of bipartite quantum entanglement within the scalar–conformal NUVO transport closure framework, from the Schmidt decomposition through the Bell inequality violation and the coupled oscillator entanglement analysis. The twelve principal results are as follows.

Product states and entanglement (Definition 3.1 and Lemma 3.3). A pure state $|\Psi\rangle \in \mathcal{H}_{AB}$ is entangled if and only if its coefficient matrix C in any product ONB has rank greater than one. Entanglement is not a postulate but a structural consequence of the tensor product $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ of QM7.

Schmidt decomposition (Theorem 3.4). Every pure bipartite state has a unique Schmidt decomposition $|\Psi\rangle = \sum_{k=1}^{r} \lambda_k |\phi_k\rangle \otimes |\psi_k\rangle$ derived from the SVD of the coefficient matrix. The Schmidt coefficients $\{\lambda_k\}$ are basis-independent invariants; the Schmidt rank $r = 1$ iff the state is a product state.

Reduced density matrix (Theorem 4.5 and Theorem 4.7). The reduced density matrix $\hat{\rho}_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$ is the unique positive trace-class operator satisfying the subsystem Born rule $\mathrm{Tr}(\hat{\rho}_A \hat{A}) = \langle \hat{A} \otimes \hat{\mathbf{1}} \rangle_{|\Psi\rangle}$. Its eigenvalues are $\{\lambda_k^2\}$; it is pure iff $r = 1$. The reduced density matrices $\hat{\rho}_A$ and $\hat{\rho}_B$ have the same non-zero spectrum.

Von Neumann entropy: properties (Theorem 5.3). $S = -\sum_k \lambda_k^2 \log \lambda_k^2$ satisfies: $S \geq 0$ (equality iff product state); $S \leq \log \min(d_A, d_B)$ (equality iff maximally entangled); $S(\hat{\rho}_A) = S(\hat{\rho}_B)$; invariance under local unitaries.

Von Neumann entropy: values (Propositions 5.5 and 5.6). The uniform two-term distribution gives $S = \log 2 = 1$ ebit (Bell states, maximum for a two-qubit system). The geometric distribution $p_n = (1-t^2)t^{2n}$ gives $S = -\log(1-t^2) - t^2 \log t^2/(1-t^2)$ (coupled oscillator ground state).

Bell states: construction and maximality (Definition 6.1 and Theorem 6.3). The four Bell states $\{|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle, |\Psi^-\rangle\}$ each have Schmidt rank $r = 2$, reduced density matrix $\hat{\rho}_A = \frac{1}{2}\sigma_0$ (maximally mixed), and entropy $S = \log 2 = 1$ ebit (maximum for two qubits). The singlet $|\Psi^-\rangle = |j = 0\rangle$ and triplet $|\Psi^+\rangle = |j = 1, m_j = 0\rangle$ are identified as CG states from QM8.

Bell basis completeness (Theorem 6.5). The four Bell states form a complete orthonormal basis for $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$, related to the product basis by the unitary Bell matrix Eq. (42).

Singlet correlations (Proposition 6.7). $\langle |\Psi^-\rangle | (\hat{n}_A \cdot \boldsymbol{\sigma}) \otimes (\hat{n}_B \cdot \boldsymbol{\sigma}) | |\Psi^-\rangle \rangle = -\hat{n}_A \cdot \hat{n}_B$ for all unit vectors $\hat{n}_A$, $\hat{n}_B$; in particular $\langle \boldsymbol{\sigma}_j \otimes \boldsymbol{\sigma}_k \rangle = -\delta_{jk}$.

CHSH inequality (Theorem 7.2). Any local hidden variable theory satisfies $|\mathcal{S}_{\mathrm{LHV}}| \leq 2$, proved from the combinatorial identity $|s(\lambda)| = 2$ for all λ.

Quantum violation and Tsirelson's bound (Theorem 7.5). The singlet with optimal settings achieves $|\mathcal{S}_{\mathrm{Q}}| = 2\sqrt{2} > 2$, violating the CHSH inequality. The general bound $|\mathcal{S}_{\mathrm{Q}}| \leq 2\sqrt{2}$ follows from $\|\hat{\mathcal{C}}^2\|_{\mathrm{op}} \leq 8$. The violation is a theorem, not a physical postulate.

Coupled oscillator Schmidt decomposition (Theorem 8.3). The ground state $|\Psi_{0,0}\rangle$ has Schmidt decomposition $\sum_{n=0}^{\infty} \sqrt{1-t^2}\, t^n |n\rangle_1 \otimes |n\rangle_2$ with geometric Schmidt coefficients and infinite Schmidt rank for $\kappa \neq 0$.

Coupled oscillator entropy and thermal identification (Theorem 8.7 and Proposition 8.5). The entanglement entropy $S(t) = -\log(1-t^2) - t^2 \log t^2/(1-t^2)$ increases monotonically from 0 to ∞ as κ goes from 0 to $m\omega^2$. The reduced density matrix is a thermal state of the subsystem oscillator at effective temperature $T = \Phi_0 \omega/(-2k_B \log t)$.

10.2 Programmatic Significance

The results of the present paper are of broad programmatic significance on three grounds.

The first is the completion of the entanglement theory initiated in QM7. QM7 Proposition 7.2 established that the coupled oscillator ground state is entangled for $\kappa \neq 0$ and identified the Bell states as the $\frac{1}{2} \otimes \frac{1}{2}$ CG states. QM8 Theorem 8.2 provided the explicit CG coefficients that give the Bell states in the product basis. QM9 completes the program: the Schmidt decomposition identifies the structure of entanglement (the Schmidt rank and coefficients), the reduced density matrix provides the subsystem description, the von Neumann entropy quantifies the degree of entanglement, and the Bell inequality violation establishes that this entanglement has observable consequences that are structurally incompatible with any local classical description. The three papers QM7, QM8, and QM9 together constitute the full entanglement theory of the non-relativistic NUVO program.

The second ground of significance is the Bell inequality violation as a theorem. In the standard formulation of quantum mechanics, the Bell inequality violation is typically presented as an experimental fact (confirmed by many experiments) or as a consequence of the quantum mechanical formalism without derivation. In the NUVO program, the violation is a theorem derived from three ingredients: the tensor product (QM7), the Pauli algebra (QM8), and the Born rule (QB-series). No new postulate is required. The derivational chain — from the SU(2) double cover holonomy of QM8 through the Pauli matrices, the spin correlator $\langle \boldsymbol{\sigma}_j \otimes \boldsymbol{\sigma}_k \rangle = -\delta_{jk}$, and the CHSH operator norm bound — makes the logical structure of the violation completely explicit: it is a consequence of the algebraic structure of the spin operators, not of any physical assumption about the behavior of entangled particles.

The third ground is the thermal state identification for the coupled oscillator reduced density matrix. Remark 8.6 establishes that tracing out one oscillator from the entangled ground state produces a thermal mixed state of the other oscillator. This is the first instance in the QM-series of a general principle: tracing out one part of an entangled bipartite system produces a thermal (or more generally, a mixed) state of the remaining part, even when the full system is in a pure ground state at zero temperature. The effective temperature is determined by the entanglement structure (the squeeze parameter t), not by any external heat bath. This principle — entanglement as the origin of apparent thermality for subsystems — is the quantum mechanical precursor of the Unruh effect (where tracing out the modes beyond the Rindler horizon of an accelerating observer produces a thermal state at the Unruh temperature) and Hawking radiation (where tracing out the interior of a black hole produces a thermal state for the exterior observer). Establishing it here as a derived consequence of the NUVO tensor product structure positions QM9 as the direct precursor to the relativistic entanglement theory of QM11.

10.3 Transition to QM10

QM10 develops the theory of quantum scattering within the two-particle framework of QM7 and the density matrix formalism of QM9. The primary objects are the scattering states — non-normalizable continuum eigenstates $|\mathbf{p}\rangle$ of the free Hamiltonian $\hat{H}_0$ at positive energy $E = \Phi_0^2\mathbf{p}^2/(2m) > 0$ — and the S-matrix, which encodes the probability amplitudes for scattering from an initial state $|\mathbf{p}_{\text{in}}\rangle$ to a final state $|\mathbf{p}_{\text{out}}\rangle$. The central result is the Lippmann-Schwinger equation:

$$|\mathbf{p}^{\pm}\rangle = |\mathbf{p}\rangle + \frac{1}{E - \hat{H}_0 \pm \mathrm{i}\epsilon}\hat{V}|\mathbf{p}^{\pm}\rangle, \tag{75}$$

which defines the out-going $(+)$ and in-coming $(-)$ scattering states as perturbative corrections to the free states by the interaction potential $\hat{V}$. The S-matrix $S = \langle \mathbf{p}^{-}_{\text{out}} | \mathbf{p}^{+}_{\text{in}} \rangle$ is unitary as a consequence of the conservation laws derived in QM4. The density matrix formalism of QM9 enters when spin-dependent scattering is analyzed: if the incident particle is in a spin superposition, the post-scattering state is an entangled spatial-spin state in $\mathcal{H}_{\text{full}} = \mathcal{H} \otimes \mathbb{C}^2$, and the spin-dependent differential cross-section is computed from the reduced density matrix of the spin degree of freedom. This connection makes QM10 the first paper in the series where the full Hilbert space $\mathcal{H}_{\text{full}}$ of QM8 and the density matrix formalism of QM9 are used together in a single physical calculation.

References

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[2] John S. Bell. On the Einstein-Podolsky-Rosen paradox. *Physics*, 1(3):195–200, 1964. Original paper deriving an inequality that any local hidden variable theory must satisfy and showing that quantum mechanics violates it for the singlet state; the inequality derived here is a predecessor of the CHSH form proved in QM9 Theorem 7.1; the physical setting is identical to QM9 §7 (two spin-$\frac{1}{2}$ particles, correlation of spin measurements along different axes) but Bell's original inequality uses three measurement directions rather than the CHSH's four; the CHSH form [3] is preferred for its direct experimental testability.

[3] John F. Clauser, Michael A. Horne, Abner Shimony, and Richard A. Holt. Proposed experiment to test local hidden-variable theories. *Physical Review Letters*, 23(15):880–884, 1969. Derives the CHSH inequality $|\mathcal{S}| \leq 2$ in the form used throughout QM9 §7; proposes the four-setting measurement protocol whose quantum violation $|\mathcal{S}|_{\max} = 2\sqrt{2}$ is derived in QM9 Theorem 7.2; establishes the framework for the first experimental tests of Bell inequalities; the CHSH parameter Eq. (7.1) and CHSH operator Definition 7.2 of QM9 are the NUVO derivations of the structures introduced in this paper.

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[6] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics, Vol. I: Functional Analysis*. Academic Press, New York, 1972. Reference for: the singular value decomposition of compact operators (the infinite-dimensional generalization of the SVD needed for the coupled oscillator Schmidt decomposition of QM9 Theorem 8.1, Chapter VI); the trace-class operators

and the partial trace (Chapter VI, §VI.6, supporting QM9 Definition 4.2); the spectral theorem for positive compact operators (Theorem VI.15, underlying the Schmidt decomposition proof).

[7] Lynden K. Shalm, Evan Meyer-Scott, Bradley G. Christensen, et al. Strong loophole-free test of local realism. *Physical Review Letters*, 115:250402, 2015. Third simultaneous loophole-free Bell test (2015), using photons with independent random measurement settings generated by quantum random number generators; confirms the CHSH violation with $|\mathcal{S}| > 2$; together with [5] and [4] establishes the experimental refutation of all local realistic theories as a robust result.

[8] Boris S. Tsirelson. Quantum generalizations of Bell's inequality. *Letters in Mathematical Physics*, 4(2):93–100, 1980. Derives the quantum upper bound $|\mathcal{S}|_{\mathrm{Q}} \leq 2\sqrt{2}$ (Tsirelson's bound) for any quantum state and any dichotomic observables; the bound is proved in QM9 Theorem 7.2 via the operator norm identity $\|\hat{\mathcal{S}}^2\|_{\mathrm{op}} \leq 8$, which is the NUVO derivation of the Tsirelson argument; the bound is shown to be saturated by the singlet with the optimal settings Eq. (7.4) of QM9.