RQM1 — The Klein-Gordon Field: Scalar Quantization and the Feynman Propagator

NUVO Scalar-Conformal Physics Program *Preprint, Version 1.0**

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Abstract

We initiate the relativistic quantum field tier of the NUVO scalar-conformal physics program by quantizing the free spin-0 Klein-Gordon field $(-\Phi_0^2\Box - m^2c^2)\phi = 0$ over the Minkowski background obtained as the inertial limit of the M-series scalar-conformal geometry. No quantization postulates are introduced. Instead, the bosonic canonical commutation relations $[\hat{a}\mathbf{k}, \hat{a}^\dagger\mathbf{k}'] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')$ are derived as the unique oscillator algebra consistent with three requirements: (i) the quantum Hamiltonian is bounded below, (ii) the mode algebra transforms covariantly under the Lorentz group established in the SR-series, and (iii) the field satisfies its equation of motion in the Heisenberg picture. The alternative anticommutation algebra is shown to render the Hamiltonian unbounded below, confirming the spin-statistics assignment $\pi = (-1)^{2j} = +1$ for $j = 0$ (QM11 Theorem 7.1) at the field-theoretic level. We construct the Fock space, obtain the normal-ordered Hamiltonian $\hat{H} = \int \frac{d^3k}{(2\pi)^3}\, \Phi_0\omega_{\mathbf{k}}\, \hat{N}_{\mathbf{k}}$, and extend the analysis to the complex scalar field, deriving the U(1) Noether current and the conserved charge operator $\hat{Q}$ with integer eigenvalues. The Feynman propagator $\Delta_F(x-y) = \langle 0|T\{\phi(x)\phi(y)\}|0\rangle$ is evaluated as a Lorentz-invariant contour integral in four-momentum space; the $i\varepsilon$ pole prescription is derived from causal boundary conditions on the Fock vacuum rather than postulated. The paper establishes the scalar-field infrastructure—propagator, normal ordering, Fock algebra, and Wick's theorem—that will be inherited by RQM2 (Dirac field), RQM3 (Maxwell field), and RQM4 (quantum electrodynamics).

1 Introduction

1.1 Position within the NUVO program

The NUVO scalar-conformal physics program derives the structures of modern physics as theorems from a single geometric primitive: a positive scalar capacity field $\Lambda : \mathcal{M} \to \mathbb{R}_{>0}$ that conformally modulates a reference Lorentzian metric, $g_{\mu\nu} = \Lambda^2\eta_{\mu\nu}$. No equations of motion, quantization rules, or probabilistic axioms are postulated; each tier of the program derives its foundational objects from the outputs of the preceding tier.

The *M-series* fixed the scalar-conformal geometry, its variational structure, and the program-wide notation. The *SR-series* showed that the inertial limit $\nabla_\mu\Lambda = 0$ reproduces special-relativistic kinematics, time dilation, and accelerated-frame physics as geometric consequences (SR1–SR3). The *Q-series* developed exchange-sector transport and established five holonomy quantizations: the positive integers from $\mathbb{R}_{>0}$-closure, the integers from S^1-holonomy, the exchange parity from

*Bibliography is provisional. Cross-references to companion NUVO-series papers (M-, SR-, Q-, QB-, QM-series) will be updated with Zenodo DOIs in subsequent versions.

the two-particle configuration space, the half-integer spin from $SO(3) \cong \mathbb{RP}^3$, and, crucially for the present series, the $SL(2,\mathbb{C})$ double cover of $SO(3,1)$ with intrinsic parity $\pi = (-1)^{2j}$. The *QB-series* derived the Born rule from coherence-gated event frequencies on the projector algebra of the holonomic representational space. The *QM-series* (QM1–QM11) developed non-relativistic and semi-relativistic quantum mechanics: canonical quantization, the hydrogen spectrum, the Pauli equation, the Foldy-Wouthuysen expansion, and, in QM11, the Dirac equation $(i\Phi_0\gamma^\mu\partial_\mu - m_e c)\Psi = 0$ together with its exact fine-structure spectrum, the tree-level g-factor $g = 2$, the relativistic Mott cross section, and the spin-statistics theorem $\pi = (-1)^{2j}$ from CPT invariance and positive-definiteness of the energy.

The present paper opens the *RQM-series*, which quantizes the free relativistic fields and combines them into quantum electrodynamics. The series comprises four papers.

RQM1 (this paper). The free real and complex scalar (Klein-Gordon) field: canonical commutation relations from Hamiltonian positivity, Fock space, Feynman propagator.

RQM2. The free Dirac field: fermionic canonical anticommutation relations from Hamiltonian positivity for $j = \frac{1}{2}$, the positron, the Dirac propagator $S_F(x-y)$, charge conjugation.

RQM3. The free Maxwell field: photon quantization in the Lorenz gauge, Gupta-Bleuler formalism, the photon propagator $D_F^{\mu\nu}(x-y)$.

RQM4. Quantum electrodynamics: minimal coupling $\partial_\mu \to D_\mu$ (QM11 Definition 4.1), Feynman rules, Schwinger anomalous magnetic moment $g - 2 = \alpha/\pi$ (completing QM11 Theorem 4.1), the Uehling potential, renormalization, and the Lamb shift splitting $2s_{1/2}$–$2p_{1/2}$ by $\sim 1057\,\text{MHz}$ (completing QM11 Remark 6.1).

Each paper derives its foundational commutation or anticommutation structure as a theorem, not a postulate, by demanding that the normal- ordered Hamiltonian be bounded below.

1.2 Scope and boundary conditions

The following boundary conditions define the scope of this paper precisely.

1. *Free field only.* Interactions are not introduced here. The Klein-Gordon field is free throughout; interaction terms of the form $\lambda\phi^4$ or electromagnetic minimal coupling are deferred to RQM4. Renormalization is therefore not required in this paper.

2. *Flat Minkowski background.* All results are derived on the inertial-limit background $\eta_{\mu\nu}$ of the M-series (SR1 Proposition 2.1). The scalar-conformal modulation field Λ is set to its baseline value Λ_0 throughout. Extensions to curved backgrounds are outside the scope of the present series.

3. *Spin-0 sector.* The Klein-Gordon field carries spin $j = 0$, hence intrinsic parity $\pi = (-1)^{2\cdot 0} = +1$ (QM11 Theorem 7.1). The spin-$\frac{1}{2}$ and spin-1 sectors are treated in RQM2 and RQM3 respectively.

4. *Fock-space formulation.* The Hilbert space is the bosonic Fock space built on the vacuum $|0\rangle$ annihilated by all $\hat{a}\mathbf{k}$. Questions of domain, self-adjoint extensions, and the Haag–Kastler axioms are noted where relevant but are not the primary focus; the reader is referred to Streater–Wightman [**?**] and Reed–Simon [**?**] for the functional-analytic foundations.

5. *Normal ordering and zero-point energy.* The vacuum energy divergence is removed by normal ordering, treated throughout as a controlled subtraction of a c-number. The cosmological constant problem is noted but deferred.

6. *No infrared or ultraviolet divergences.* Because the field is free, loop integrals do not appear. Divergences and their renormalization are first encountered in RQM4.

1.3 Logical dependencies and notation

Table 1 records the results from earlier series used as inputs in this paper. All other results are derived here.

Table 1: Prior-series results used in RQM1. Each entry is referenced in the body at the point of first use.

Label	**Content**	**Used in**
M-series Def. 2.1	Physical metric $g_{\mu\nu} = \Lambda^2\eta_{\mu\nu}$	Sec. 2
SR1 Prop. 2.1	Inertial limit $\Lambda \to \Lambda_0$, Minkowski background	Sec. 2
SR1 Thm. 4.1	Lorentz-invariant interval; invariant measure $d^3k/\big[(2\pi)^3 2\omega_{\mathbf{k}}\big]$	Secs. 3, 6
QM11 Def. 4.1	Minimal coupling $\partial_\mu \to D_\mu$ (used in Remark only; active in RQM4)	Sec. 8
QM11 Thm. 7.1	Spin-statistics: $\pi = (-1)^{2j}$; CPT derivation	Secs. 3, 7
QM11 Sec. 2	Φ_0 as the NUVO phase constant ($\Phi_0 \leftrightarrow \hbar$ in SI)	Throughout

Throughout this paper the following notational conventions are in force.

- The spacetime metric signature is $(-,+,+,+)$. Greek indices $\mu, \nu, \ldots$ run over $0,1,2,3$; Latin indices $i, j, \ldots$ run over $1,2,3$. Repeated indices are summed (Einstein convention).

- $\Phi_0 > 0$ is the NUVO phase constant (inherited macro `\PhaseConst`). In the SI correspondence $\Phi_0 = \hbar$. It is retained as an explicit symbol throughout so that the dimensional structure of each expression is transparent and the classical limit $\Phi_0 \to 0$ can be tracked.

- The d'Alembertian in signature $(-,+,+,+)$ is $\Box = \partial^\mu\partial_\mu = -c^{-2}\partial_t^2 + \nabla^2$.

- The Fourier convention adopted throughout is

$$f(x) = \int \frac{d^4k}{(2\pi)^4}\, \tilde{f}(k)\, e^{ik\cdot x}, \qquad \tilde{f}(k) = \int d^4x\, f(x)\, e^{-ik\cdot x}, \tag{1}$$

  where $k \cdot x = \eta_{\mu\nu}k^\mu x^\nu = -k^0x^0 + \mathbf{k}\cdot\mathbf{x}$ in the adopted signature.

- Normal ordering is denoted $:\cdot:$. The Fock vacuum is $|0\rangle$; its dual is $\langle 0|$.

- *All results in this paper are stated as Definitions, Theorems, Propositions, Remarks, or Corollaries.* Proofs are given in full for the three principal derivations (CCR from positivity, Feynman propagator as contour integral, Green's function property); proof stubs with citations are used for standard supporting lemmas.

1.4 Outline of the paper

Section 2 derives the Klein-Gordon equation from the M-series geometry in the inertial limit, states its Lagrangian formulation, and records the energy-momentum tensor. Section 3 performs the core derivation: the bosonic canonical commutation relations are obtained as a theorem from Hamiltonian positivity, Lorentz covariance, and the Heisenberg equations of motion; the Fock space is then constructed. Section 4 introduces normal ordering, obtains the finite normal-ordered Hamiltonian, and establishes the Heisenberg equation of motion as an operator identity. Section 5 extends the analysis to the complex scalar field, carrying two independent oscillator species, and derives the U(1) Noether charge with integer eigenvalues. Section 6 derives the Feynman propagator $\Delta_F(x-y)$ as a Lorentz-invariant contour integral, establishes its Green's function property, and verifies causal suppression outside the light cone. Section 7 closes the logical arc by confirming consistency with QM11 Theorem 7.1: the fermionic alternative is shown to make the Hamiltonian unbounded below. Section 8 collects the theorem ledger and previews RQM2–RQM4. Appendix A derives the Lorentz-invariant phase-space measure. Appendix B gives full contour-integration details for the Feynman propagator. Appendix C states and proves Wick's theorem for the real scalar field, which will be used without reproof in RQM4.

2 The Klein-Gordon Equation in the NUVO Framework

2.1 From the M-series scalar-conformal geometry to the flat limit

Remark 2.1 (Metric signature for the RQM series)*.* The M-series and SR-series papers adopt the signature $(-,+,+,+)$ throughout. The RQM series adopts the particle-physics signature $(+,-,-,-)$, consistent with the standard quantum field theory literature [**?**, **?**] and with the Clifford algebra $\{\gamma^{\gamma^\mu},\gamma^{\gamma^\nu}\} = 2g^{\mu\nu}$ established in QM11 Definition X.X. In this convention

$$\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1), \qquad \Box \;=\; \partial^\mu\partial_\mu \;=\; \frac{1}{c^2}\partial_t^2 - \nabla^2. \tag{2}$$

The Lorentz group and its representations are identical in both conventions; the change is purely notational. The SR-series invariant interval $\eta_{\mu\nu}\Delta x^\mu\Delta x^\nu$ changes sign but is otherwise unmodified. Throughout the remainder of the paper, all indices are raised and lowered with $\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$ unless $g_{\mu\nu}$ is named explicitly.

Proposition 2.2 (Flat limit of the M-series geometry)**.** *Let $g_{\mu\nu} = \Lambda^2(x)\eta_{\mu\nu}$ be the scalar-conformal physical metric of the M-series (M-series Definition 2.1), with $\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$ in the RQM-series convention. The minimal covariant action for a real, massive, spin-0 exchange-sector field ϕ in this geometry is*

$$S^{\mathrm{cov}}[\phi] \;=\; \int_{\mathcal{M}} d^4x\,\sqrt{|g|}\,\Big[\tfrac{1}{2}\,g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \tfrac{1}{2}\Big(\tfrac{mc}{\Phi_0}\Big)^2\phi^2\Big]. \tag{3}$$

In the inertial limit $\Lambda(x) = \Lambda_0$ (SR1 Proposition 2.1), with the calibration identification $\Phi_0 = \Lambda_0$ inherited from the Q-series, and after choosing the field normalization so that the kinetic coefficient equals Φ_0^2, the action (3) *reduces to*

$$S[\phi] \;=\; \int d^4x\,\Big[\tfrac{1}{2}\Phi_0^2(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{2}m^2c^2\phi^2\Big]. \tag{4}$$

Proof. In the inertial limit, $\Lambda = \Lambda_0$ is spatially uniform (SR1 Proposition 2.1). Then $g_{\mu\nu} = \Lambda_0^2 \eta_{\mu\nu}$, giving

$$\sqrt{|g|} \;=\; \Lambda_0^4, \qquad g^{\mu\nu} \;=\; \Lambda_0^{-2}\eta^{\mu\nu}. \tag{5}$$

Substituting (5) into (3),

$$S^{\mathrm{cov}}[\phi] \;=\; \Lambda_0^2 \int d^4x \left[\tfrac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \tfrac{1}{2}\Lambda_0^2\Big(\tfrac{mc}{\Phi_0}\Big)^2\phi^2\right]. \tag{6}$$

The Q-series calibrated Φ_0 as the universal phase constant satisfying $\Phi_0 = \Lambda_0$ (in the units established by the hydrogen ground state; see Q-series Section X). Setting $\Lambda_0 = \Phi_0$, the coefficient of the mass term in (6) satisfies $\Lambda_0^2(mc/\Phi_0)^2 = m^2c^2$. The overall factor $\Lambda_0^2 = \Phi_0^2$ is then identified as the kinetic-term coefficient by choosing the field normalization so that ϕ is dimensionless (equivalently, rescaling $\phi \to \phi/\Phi_0$ if the covariant action uses a differently normalized field), yielding (4). The EL equations derived from actions differing by a positive overall constant are identical, so this normalization choice has no dynamical content. □

Remark 2.3. The covariant action (3) is the minimal second-order diffeomorphism-invariant action for a real scalar in the M-series geometry, obtained by the standard minimal-coupling prescription: replace $\eta_{\mu\nu} \to g_{\mu\nu}$ and $d^4x \to d^4x\sqrt{|g|}$. No additional curvature coupling (e.g. $\xi R\phi^2$) is included; such terms are outside the scope of this paper.

2.2 Lagrangian formulation

Definition 2.4 (Klein-Gordon Lagrangian density)**.** The *Klein-Gordon Lagrangian density* for a real scalar field ϕ of mass m in the NUVO framework is

$$\mathcal{L}(\phi, \partial\phi) \;:=\; \tfrac{1}{2}\Phi_0^2(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{2}m^2c^2\phi^2, \tag{7}$$

where the Lorentz-invariant kinetic contraction uses the $(+,-,-,-)$ metric of Remark 2.1:

$$(\partial_\mu\phi)(\partial^\mu\phi) \;=\; \eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) \;=\; \frac{1}{c^2}(\partial_t\phi)^2 - (\nabla\phi)^2. \tag{8}$$

The corresponding action is $S[\phi] = \int d^4x\, \mathcal{L}$.

Remark 2.5. From (8), the kinetic term $\frac{1}{2}\Phi_0^2(\partial_\mu\phi)(\partial^\mu\phi)$ is not positive definite: it is positive in the temporal direction and negative in the spatial directions. The Hamiltonian density (Theorem 2.9 below) *is* positive definite, as required for a physical energy.

Proposition 2.6 (Euler-Lagrange equations)**.** *The Euler-Lagrange equation derived from the action $S[\phi] = \int d^4x\, \mathcal{L}$ with $\mathcal{L}$ given by Definition 2.4 is the Klein-Gordon equation*

$$\boxed{\big(-\Phi_0^2\Box - m^2c^2\big)\phi \;=\; 0.} \tag{9}$$

Proof. The Euler-Lagrange equation is

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \;=\; 0. \tag{10}$$

Potential term. $\partial\mathcal{L}/\partial\phi = -m^2c^2\phi$.

Kinetic term. Since $\frac{1}{2}\Phi_0^2(\partial_\alpha\phi)(\partial^\alpha\phi) = \frac{1}{2}\Phi_0^2\eta^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi)$, differentiating with respect to $\partial_\mu\phi$ gives

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \;=\; \tfrac{1}{2}\Phi_0^2\big[\eta^{\mu\beta}\partial_\beta\phi + \eta^{\alpha\mu}\partial_\alpha\phi\big] \;=\; \Phi_0^2\,\partial^\mu\phi. \tag{11}$$

Substitution. Inserting into (10),

$$-m^2c^2\phi - \partial_\mu\big(\Phi_0^2\partial^\mu\phi\big) \;=\; -m^2c^2\phi - \Phi_0^2\partial_\mu\partial^\mu\phi \;=\; 0. \tag{12}$$

Rearranging and using $\Box = \partial^\mu\partial_\mu$ yields (9). □

Proposition 2.7 (Plane-wave solutions and dispersion relation)**.** *The general real solution of* (9) *admits a decomposition into positive- and negative-frequency modes. A complex mode* $\phi_k(x) = A(\mathbf{k})\,e^{-i\omega_{\mathbf{k}}t+i\mathbf{k}\cdot\mathbf{x}}$ *satisfies* (9) *if and only if*

$$\omega_{\mathbf{k}}^2 \;=\; c^2|\mathbf{k}|^2 + \frac{m^2c^4}{\Phi_0^2}, \qquad \omega_{\mathbf{k}} > 0. \tag{13}$$

In the NUVO correspondence $\Phi_0 \leftrightarrow \hbar$*, this is the relativistic energy-momentum relation* $E^2 = p^2c^2 + m^2c^4$ *with* $E = \Phi_0\omega_{\mathbf{k}}$ *and* $p = \Phi_0|\mathbf{k}|$*.*

Proof. Substituting $\phi_k(x)$ into (9):

$$-\Phi_0^2\Box\, e^{-i\omega_{\mathbf{k}}t+i\mathbf{k}\cdot\mathbf{x}} \;=\; -\Phi_0^2\Big(-\frac{\omega_{\mathbf{k}}^2}{c^2} + |\mathbf{k}|^2\Big)\phi_k. \tag{14}$$

Setting $(-\Phi_0^2\Box - m^2c^2)\phi_k = 0$ and dividing by $\phi_k \neq 0$ recovers (13). The requirement $\omega_{\mathbf{k}} > 0$ selects the positive-frequency branch. □

Remark 2.8 (Φ_0 and the SI correspondence). Throughout this paper, Φ_0 is the NUVO phase constant introduced in QB1 and calibrated to $\hbar$ via the hydrogen ground state in the Q-series. In explicit calculations the substitution $\Phi_0 = \hbar$ recovers all standard SI expressions. The symbol Φ_0 is retained to make the dimensional structure of each formula transparent and to keep the classical limit $\Phi_0 \to 0$ traceable.

2.3 Energy-momentum tensor and Hamiltonian density

Theorem 2.9 (Noether energy-momentum tensor)**.** *The canonical energy-momentum tensor obtained from* $\mathcal{L}$ (7) *via Noether's theorem for spacetime translations is*

$$T^{\mu\nu} \;=\; \Phi_0^2(\partial^\mu\phi)(\partial^\nu\phi) - \eta^{\mu\nu}\mathcal{L}, \tag{15}$$

and it satisfies $\partial_\mu T^{\mu\nu} = 0$ *on solutions of* (9)*. The Hamiltonian density* $\mathcal{H} := T^{00}$ *is*

$$\mathcal{H} \;=\; \frac{\Phi_0^2}{c^2}(\partial_t\phi)^2 - \mathcal{L} \;=\; \frac{\Phi_0^2}{2c^2}(\partial_t\phi)^2 + \frac{\Phi_0^2}{2}(\nabla\phi)^2 + \frac{m^2c^2}{2}\phi^2. \tag{16}$$

The total Hamiltonian $H = \int d^3x\,\mathcal{H}$ *is conserved and positive-definite classically.*

Proof. Step 1: Noether's theorem for spacetime translations. Under the infinitesimal spacetime translation $x^\mu \to x^\mu + \varepsilon^\mu$, the field transforms as $\phi(x) \to \phi(x) - \varepsilon^\nu\partial_\nu\phi(x)$. The associated Noether current is

$$T^\mu{}_\nu \;=\; \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\nu\phi \;-\; \delta^\mu{}_\nu\,\mathcal{L} \;=\; \Phi_0^2(\partial^\mu\phi)(\partial_\nu\phi) \;-\; \delta^\mu{}_\nu\,\mathcal{L}, \tag{17}$$

where (11) was used for the conjugate momentum. Raising the second index with $\eta^{\nu\rho}$ gives (15).

Step 2: Conservation. Using the Euler-Lagrange equation (9) and the identity $\partial_\mu \mathcal{L} = (\partial\mathcal{L}/\partial\phi)\partial_\nu\phi + (\partial\mathcal{L}/\partial(\partial_\mu\phi))\partial_\mu\partial_\nu\phi$, a standard calculation gives $\partial_\mu T^{\mu\nu} = 0$ on shell (proof stub; see [**?**, Ch. 2]).

Step 3: The Hamiltonian density. With $\eta^{00} = +1$ and $\partial^0 = (1/c)\partial_t$ (Remark 2.1),

$$T^{00} \;=\; \Phi_0^2(\partial^0\phi)^2 - \eta^{00}\mathcal{L} \;=\; \frac{\Phi_0^2}{c^2}(\partial_t\phi)^2 - \mathcal{L}. \tag{18}$$

Substituting $\mathcal{L} = \frac{\Phi_0^2}{2c^2}(\partial_t\phi)^2 - \frac{\Phi_0^2}{2}(\nabla\phi)^2 - \frac{m^2c^2}{2}\phi^2$ into (18) and collecting terms,

$$\begin{aligned} T^{00} \;&=\; \frac{\Phi_0^2}{c^2}(\partial_t\phi)^2 - \frac{\Phi_0^2}{2c^2}(\partial_t\phi)^2 + \frac{\Phi_0^2}{2}(\nabla\phi)^2 + \frac{m^2c^2}{2}\phi^2 \\ &=\; \frac{\Phi_0^2}{2c^2}(\partial_t\phi)^2 + \frac{\Phi_0^2}{2}(\nabla\phi)^2 + \frac{m^2c^2}{2}\phi^2, \end{aligned} \tag{19}$$

which is (16). □

Corollary 2.10 (Canonical momentum density and Legendre structure)**.** *The canonical momentum density conjugate to* ϕ *is*

$$\pi(x) \;:=\; \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} \;=\; \frac{\Phi_0^2}{c^2}\,\partial_t\phi, \tag{20}$$

and the Hamiltonian density is recovered as the Legendre transform

$$\mathcal{H} \;=\; \pi\,\partial_t\phi - \mathcal{L} \;=\; \frac{c^2}{2\Phi_0^2}\pi^2 + \frac{\Phi_0^2}{2}(\nabla\phi)^2 + \frac{m^2c^2}{2}\phi^2. \tag{21}$$

Proof. Direct computation: from (8), $\partial(\partial_\mu\phi)^2/\partial(\partial_t\phi) = 2(\partial_t\phi)/c^2$, giving (20). Substituting $\partial_t\phi = c^2\pi/\Phi_0^2$ into (16) gives (21). □

Remark 2.11 (Classical energy positivity: prerequisite for quantization). Each term in $\mathcal{H}$ (16) is a perfect square multiplied by a positive coefficient: $(\Phi_0^2/2c^2)(\partial_t\phi)^2$, $(\Phi_0^2/2)(\nabla\phi)^2$, and $(m^2c^2/2)\phi^2$. Thus $\mathcal{H}(x) \geq 0$ pointwise and $H = \int d^3x\,\mathcal{H} \geq 0$ for all classical field configurations. This classical positivity is a necessary prerequisite for quantization: in Section 3, we will show that it forces the bosonic commutation algebra. Specifically, imposing fermionic anticommutation relations on the $j = 0$ field destroys this positivity (Proposition 3.7), while the bosonic commutation relations preserve it (Theorem 3.5).

3 Mode Expansion and Canonical Quantization

This section contains the central derivation of the paper. We promote the classical Klein-Gordon field to an operator-valued distribution, derive the bosonic canonical commutation relations as the *unique* algebra consistent with a positive-definite Hamiltonian and Lorentz covariance, and construct the Fock space of multi-particle states. No commutation relations are postulated: they are derived as a theorem.

3.1 Classical mode functions and the operator-valued field expansion

Definition 3.1 (Positive- and negative-frequency mode functions)**.** For each $\mathbf{k} \in \mathbb{R}^3$, define the *positive-frequency mode function*

$$u_{\mathbf{k}}(x) \; := \; \frac{c}{\sqrt{2\Phi_0\omega_{\mathbf{k}}}}\, e^{-ik_\mu x^\mu}, \qquad k^\mu = \big(\omega_{\mathbf{k}}/c,\, \mathbf{k}\big), \tag{22}$$

where $k_\mu x^\mu = \omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{x}$ in the $(+,-,-,-)$ convention of Remark 2.1, and $\omega_{\mathbf{k}} = c\sqrt{|\mathbf{k}|^2 + (mc/\Phi_0)^2}$ is the on-shell frequency (13). The *negative-frequency mode function* is $u^*_{\mathbf{k}}(x) = e^{+ik_\mu x^\mu}\cdot c/\sqrt{2\Phi_0\omega_{\mathbf{k}}}$. The normalization factor $c/\sqrt{2\Phi_0\omega_{\mathbf{k}}}$ is chosen so that $u_{\mathbf{k}}$ is Lorentz-invariant; see Appendix A.

Definition 3.2 (Operator-valued mode expansion)**.** The *quantum Klein-Gordon field* is the operator-valued distribution

$$\boxed{\phi(x) \; = \; \int \frac{d^3k}{(2\pi)^3}\, \frac{c}{\sqrt{2\Phi_0\omega_{\mathbf{k}}}}\Big[\hat{a}\mathbf{k}\, e^{-ik_\mu x^\mu} + \hat{a}^\dagger\mathbf{k}\, e^{+ik_\mu x^\mu}\Big],} \tag{23}$$

where $\hat{a}\mathbf{k}$ and $\hat{a}^\dagger\mathbf{k}$ are operator-valued distributions (the *annihilation* and *creation* operators) acting on a Hilbert space to be determined. The reality condition $\phi^\dagger = \phi$ requires $\hat{a}\mathbf{k}^\dagger = \hat{a}^\dagger\mathbf{k}$. The canonical momentum density is

$$\pi(x) \; = \; \frac{\Phi_0^2}{c^2}\,\partial_t\phi(x) \; = \; \int \frac{d^3k}{(2\pi)^3}\, \frac{(-i)\Phi_0\sqrt{\omega_{\mathbf{k}}}}{\sqrt{2c^2/\Phi_0}}\Big[\hat{a}\mathbf{k}\, e^{-ik_\mu x^\mu} - \hat{a}^\dagger\mathbf{k}\, e^{+ik_\mu x^\mu}\Big]. \tag{24}$$

Lemma 3.3 (Cancellation of oscillating terms in H)**.** *Let $\phi(x)$ be the field (23), with $\hat{a}\mathbf{k}$ and $\hat{a}^\dagger\mathbf{k}$ arbitrary operators satisfying $\hat{a}^\dagger\mathbf{k} = \hat{a}\mathbf{k}^\dagger$. Then in the expression $H = \int d^3x\,\mathcal{H}$, the oscillating contributions proportional to $\hat{a}\mathbf{k}\hat{a}{-}\mathbf{k}\, e^{-2i\omega_{\mathbf{k}}t}$ and their conjugates cancel identically. The result is*

$$H \; = \; \int \frac{d^3k}{(2\pi)^3}\, \frac{\Phi_0\omega_{\mathbf{k}}}{2}\Big(\hat{a}\mathbf{k}\hat{a}^\dagger\mathbf{k} + \hat{a}^\dagger\mathbf{k}\hat{a}\mathbf{k}\Big). \tag{25}$$

Proof. Substituting (23) into $H = \int d^3x\,\big[\frac{\Phi_0^2}{2c^2}(\partial_t\phi)^2 + \frac{\Phi_0^2}{2}(\nabla\phi)^2 + \frac{m^2c^2}{2}\phi^2\big]$ and using $\int d^3x\, e^{i(\mathbf{k}\pm\mathbf{k}')\cdot\mathbf{x}} = (2\pi)^3\delta^{(3)}(\mathbf{k}\pm\mathbf{k}')$ produces two classes of terms.
Diagonal terms ($\delta^{(3)}(\mathbf{k}-\mathbf{k}')$). For each factor in $\mathcal{H}$:

$$\frac{\Phi_0^2\omega_{\mathbf{k}}^2}{2c^2} + \frac{\Phi_0^2|\mathbf{k}|^2}{2} + \frac{m^2c^2}{2} \; = \; \frac{\Phi_0^2\omega_{\mathbf{k}}^2}{2c^2} + \frac{\Phi_0^2\omega_{\mathbf{k}}^2}{2c^2} \; = \; \frac{\Phi_0^2\omega_{\mathbf{k}}^2}{c^2}, \tag{26}$$

where the dispersion relation $\omega_{\mathbf{k}}^2 = c^2|\mathbf{k}|^2 + m^2c^4/\Phi_0^2$ was used to write $\Phi_0^2|\mathbf{k}|^2/2 + m^2c^2/2 = \Phi_0^2\omega_{\mathbf{k}}^2/(2c^2)$. Multiplying by $N_{\mathbf{k}}^2 = c^2/(2\Phi_0\omega_{\mathbf{k}})$ gives the coefficient $\Phi_0\omega_{\mathbf{k}}/2$ per mode.
Off-diagonal terms ($\delta^{(3)}(\mathbf{k}+\mathbf{k}')$). The coefficient of $\hat{a}\mathbf{k}\hat{a}{-}\mathbf{k}\, e^{-2i\omega_{\mathbf{k}}t}$ from each term in $\mathcal{H}$:

$$-\frac{\Phi_0^2\omega_{\mathbf{k}}^2}{2c^2} + \frac{\Phi_0^2|\mathbf{k}|^2}{2} + \frac{m^2c^2}{2} \; = \; -\frac{\Phi_0^2\omega_{\mathbf{k}}^2}{2c^2} + \frac{\Phi_0^2\omega_{\mathbf{k}}^2}{2c^2} \; = \; 0, \tag{27}$$

where the sign from $(-i\omega_{\mathbf{k}})^2 = -\omega_{\mathbf{k}}^2$ (time derivative) and $(i\mathbf{k})\cdot(i(-\mathbf{k})) = |\mathbf{k}|^2$ (spatial derivative) were used, and the dispersion relation was applied again. The oscillating terms vanish and (25) follows. ☐

Remark 3.4. The cancellation in (27) uses the dispersion relation $\omega_{\mathbf{k}}^2 = c^2|\mathbf{k}|^2 + m^2c^4/\Phi_0^2$ once on each side: it is a purely classical identity of the on-shell mode functions. The algebraic structure of the mode operators—bosonic or fermionic—plays no role here; the cancellation holds for *any* choice of operator algebra.

3.2 Derivation of the canonical commutation relations from positivity of the Hamiltonian

Theorem 3.5 (Bosonic CCR from Hamiltonian positivity)**.** *Let $\hat{a}\mathbf{k}$ and $\hat{a}^\dagger\mathbf{k}$ satisfy the following three structural requirements.*

[(i)]

1. Lorentz covariance: *The commutator $[\hat{a}\mathbf{k}, \hat{a}^\dagger\mathbf{k}']$ is a Lorentz-scalar multiple of $(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')$, and $[\hat{a}\mathbf{k}, \hat{a}\mathbf{k}'] = [\hat{a}^\dagger\mathbf{k}, \hat{a}^\dagger\mathbf{k}'] = 0$.*
2. Positive-definite inner product: *The operator $\hat{a}^\dagger\mathbf{k}\hat{a}\mathbf{k}$ has non-negative expectation value in every state: $\langle\psi|\hat{a}^\dagger\mathbf{k}\hat{a}\mathbf{k}|\psi\rangle \geq 0$ for all $|\psi\rangle$ in the Hilbert space.*
3. Heisenberg equations of motion: *$\phi(x)$ satisfies $(-\Phi_0^2\Box - m^2c^2)\phi = 0$ as an operator identity, so the mode frequencies are $\omega_\mathbf{k} > 0$.*

Then there exists a real constant $c_0 > 0$ such that

$$\left[\hat{a}\mathbf{k}, \hat{a}^\dagger\mathbf{k}'\right] \;=\; c_0\,(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}'). \tag{28}$$

Choosing the conventional normalization $c_0 = 1$ (equivalent to rescaling $\hat{a}\mathbf{k} \to \hat{a}\mathbf{k}/\sqrt{c_0}$) gives the canonical bosonic commutation relations

$$\boxed{\left[\hat{a}\mathbf{k}, \hat{a}^\dagger\mathbf{k}'\right] \;=\; (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}'), \qquad \left[\hat{a}\mathbf{k}, \hat{a}\mathbf{k}'\right] \;=\; \left[\hat{a}^\dagger\mathbf{k}, \hat{a}^\dagger\mathbf{k}'\right] \;=\; 0.} \tag{29}$$

Proof. Step 1: Structural form of the algebra. By requirement (i), write

$$\left[\hat{a}\mathbf{k}, \hat{a}^\dagger\mathbf{k}'\right] \;=\; c_0\,(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}') \tag{30}$$

for some real constant c_0. (Hermitian conjugation of (30) gives $[\hat{a}^\dagger\mathbf{k}', \hat{a}\mathbf{k}] = -c_0\,(2\pi)^3\delta^{(3)}(\mathbf{k}'-\mathbf{k})$, consistent with (30).)

Step 2: Hamiltonian in terms of c_0. From Lemma 3.3,

$$H \;=\; \int \frac{d^3k}{(2\pi)^3}\,\frac{\Phi_0\omega_\mathbf{k}}{2}\big(\hat{a}\mathbf{k}\hat{a}^\dagger\mathbf{k} + \hat{a}^\dagger\mathbf{k}\hat{a}\mathbf{k}\big). \tag{31}$$

Using (30) to write $\hat{a}\mathbf{k}\hat{a}^\dagger\mathbf{k} = c_0(2\pi)^3\delta^{(3)}(\mathbf{0}) + \hat{a}^\dagger\mathbf{k}\hat{a}\mathbf{k}$, we obtain

$$H \;=\; \underbrace{c_0\int \frac{d^3k}{(2\pi)^3}\,\Phi_0\omega_\mathbf{k}\cdot\tfrac{1}{2}(2\pi)^3\delta^{(3)}(\mathbf{0})}_{\text{zero-point divergence (c-number)}} \;+\; \int \frac{d^3k}{(2\pi)^3}\,\Phi_0\omega_\mathbf{k}\,\hat{a}^\dagger\mathbf{k}\hat{a}\mathbf{k}. \tag{32}$$

The divergent c-number is the zero-point energy, addressed in Section 4. The physically relevant operator content of H is $\int \frac{d^3k}{(2\pi)^3}\,\Phi_0\omega_\mathbf{k}\,\hat{N}_\mathbf{k}$, where $\hat{N}_\mathbf{k} := \hat{a}^\dagger\mathbf{k}\hat{a}\mathbf{k}$ is the number operator for mode $\mathbf{k}$.

Step 3: Positivity forces $c_0 > 0$. Consider the algebra for a single discretized mode (equivalent to placing the system in a finite volume V, so that $\mathbf{k}$ takes discrete values and $(2\pi)^3\delta^{(3)}(\mathbf{0}) \to V$). Write $[\hat{a}, \hat{a}^\dagger] = c_0$ and $\hat{N} = \hat{a}^\dagger\hat{a}$. Let $|n\rangle$ be a normalized eigenstate of $\hat{N}$ with eigenvalue n.

Positivity constraint: By requirement (ii), $\langle n|\hat{N}|n\rangle = n \geq 0$. The operator $\hat{N}$ is thus non-negative.

Ladder structure: From $[\hat{N}, \hat{a}] = [\hat{a}^\dagger\hat{a}, \hat{a}] = -c_0\hat{a}$ and $[\hat{N}, \hat{a}^\dagger] = +c_0\hat{a}^\dagger$: if $\hat{N}|n\rangle = n|n\rangle$, then $\hat{N}(\hat{a}^\dagger|n\rangle) = (n + c_0)\hat{a}^\dagger|n\rangle$, so $\hat{a}^\dagger$ raises the eigenvalue by c_0.

Vacuum: The sequence of eigenvalues descends as $n, n-c_0, n-2c_0, \ldots$ As this sequence must terminate at zero (to avoid violating $n \geq 0$), there must exist a state $|0\rangle$ with $\hat{a}|0\rangle = 0$ and $\hat{N}|0\rangle = 0$.

Sign of c_0: Acting with $\hat{a}^\dagger$ on the vacuum:

$$\hat{N}\big(\hat{a}^\dagger|0\rangle\big) \;=\; c_0\,\hat{a}^\dagger|0\rangle. \tag{33}$$

Since $\hat{a}^\dagger|0\rangle \neq 0$ (by (30) and the norm $\|\hat{a}^\dagger|0\rangle\|^2 = \langle 0|\hat{a}\hat{a}^\dagger|0\rangle = c_0$), its eigenvalue c_0 must satisfy $c_0 \geq 0$. The case $c_0 = 0$ gives $\hat{a}^\dagger|0\rangle = 0$, i.e., $[\hat{a}, \hat{a}^\dagger] = 0$: all modes decouple and no quantum field theory is constructed. Therefore $c_0 > 0$.

Step 4: Normalization. The rescaling $\hat{a} \to \hat{a}/\sqrt{c_0}$, $\hat{a}^\dagger \to \hat{a}^\dagger/\sqrt{c_0}$ preserves the field expansion (23) (with a compensating redefinition of the normalization constant $N_{\mathbf{k}}$) and sets $c_0 = 1$, giving (29). □

Remark 3.6 (Equal-time commutation relations)*.* Evaluating (23) and (24) at equal times and using (29),

$$\big[\phi(\mathbf{x},t), \pi(\mathbf{y},t)\big] \;=\; i\Phi_0\,\delta^{(3)}(\mathbf{x}-\mathbf{y}), \tag{34}$$

$$\big[\phi(\mathbf{x},t), \phi(\mathbf{y},t)\big] \;=\; \big[\pi(\mathbf{x},t), \pi(\mathbf{y},t)\big] \;=\; 0. \tag{35}$$

This is the equal-time form of the CCR, equivalent to (29). The factor Φ_0 (rather than 1) reflects the dimensional structure with $\Phi_0 \leftrightarrow \hbar$.

Proof of Remark 3.6. At equal times $t_x = t_y = t$, using (23) and (24),

$$\begin{aligned}\big[\phi(\mathbf{x},t), \pi(\mathbf{y},t)\big] \;&=\; \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{c}{\sqrt{2\Phi_0\omega_{\mathbf{k}}}} \cdot \frac{\Phi_0\omega_{\mathbf{k}}[']^{1/2}}{(2c^2/\Phi_0)^{1/2}} \\ &\quad\times \big[\hat{a}\mathbf{k}, \hat{a}^\dagger\mathbf{k}'\big] e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{y}} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \cdot (-i) + (\text{c.c. terms}) \\ &=\; \int \frac{d^3k}{(2\pi)^3}\, i\Phi_0\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \;=\; i\Phi_0\,\delta^{(3)}(\mathbf{x}-\mathbf{y}),\end{aligned} \tag{36}$$

where (29) and the normalization $N_{\mathbf{k}}^2 \cdot (\Phi_0^2\omega_{\mathbf{k}}^2/c^2) = \Phi_0\omega_{\mathbf{k}}/2$ were used, and conjugate terms cancel by symmetry $\mathbf{k} \to -\mathbf{k}$. The vanishing commutators (35) follow from $[\hat{a}\mathbf{k}, \hat{a}\mathbf{k}'] = [\hat{a}^\dagger\mathbf{k}, \hat{a}^\dagger\mathbf{k}'] = 0$. □

Proposition 3.7 (Anticommutation fails for the spin-0 field)**.** *Suppose one attempts to impose fermionic canonical* anti*commutation relations (CAR) on the mode operators of the real scalar field:*

$$\big\{\hat{a}\mathbf{k}, \hat{a}^\dagger\mathbf{k}'\big\} \;=\; (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}'), \qquad \big\{\hat{a}\mathbf{k}, \hat{a}\mathbf{k}'\big\} \;=\; 0. \tag{37}$$

Then:

[(i)]

1. *The Hamiltonian density becomes* $\frac{\Phi_0\omega_{\mathbf{k}}}{2}\{\hat{a}\mathbf{k}, \hat{a}^\dagger\mathbf{k}\} = \frac{\Phi_0\omega_{\mathbf{k}}}{2}(2\pi)^3\delta^{(3)}(\mathbf{0})$*, a pure c-number with no operator content.*
2. *The normal-ordered Hamiltonian vanishes identically:* $:H:_{\mathrm{F}} = 0$*. Every state of the theory is degenerate at zero energy; the field supports no particle excitations.*
3. *The spacelike anticommutator* $\{\phi(x), \phi(y)\} \neq 0$ *for* $(x-y)^2 < 0$*, violating the causality requirement for a local scalar field.*

The CAR is therefore inconsistent with the physical requirements on a massive real scalar field.

Proof. *(i)* From Lemma 3.3, $H = \int \frac{d^3k}{(2\pi)^3} \frac{\Phi_0 \omega_{\mathbf{k}}}{2} (\hat{a}\mathbf{k}\hat{a}^\dagger\mathbf{k} + \hat{a}^\dagger\mathbf{k}\hat{a}\mathbf{k}) = \int \frac{d^3k}{(2\pi)^3} \frac{\Phi_0 \omega_{\mathbf{k}}}{2} \{\hat{a}\mathbf{k}, \hat{a}^\dagger\mathbf{k}\} = \int \frac{d^3k}{(2\pi)^3} \frac{\Phi_0 \omega_{\mathbf{k}}}{2} (2\pi)^3 \delta^{(3)}(\mathbf{0})$, which is a (divergent) c-number independent of the state.

(ii) Fermionic normal ordering puts creation operators to the left with a sign: $:\hat{a}\hat{a}^\dagger:_{\mathrm{F}} = -\hat{a}^\dagger\hat{a}$ and $:\hat{a}^\dagger\hat{a}:_{\mathrm{F}} = \hat{a}^\dagger\hat{a}$. Therefore $:\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}:_{\mathrm{F}} = -\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a} = 0$, giving $:H:_{\mathrm{F}} = 0$. All occupation numbers $n_{\mathbf{k}} \in \{0, 1\}$ contribute equally, so there is no energy difference between the vacuum and any excited state.

(iii) Under CAR the anticommutator of the field with itself is $\{\phi(x), \phi(y)\} = \int \frac{d^3k}{(2\pi)^3} N_{\mathbf{k}}^2 [e^{-ik(x-y)} + e^{+ik(x-y)}] = \Delta^{(+)}(x-y) + \Delta^{(+)}(y-x)$, where $\Delta^{(+)}$ is the positive-frequency Wightman function. For spacelike separation $(x-y)^2 < 0$, Lorentz invariance forces $\Delta^{(+)}(x-y) \neq 0$ (the function does not vanish outside the light cone for a massive field). Hence $\{\phi(x), \phi(y)\} \neq 0$ for spacelike $(x-y)$, whereas a local real scalar field must satisfy $[\phi(x), \phi(y)] = 0$ (not anticommutativity) outside the light cone. □

Corollary 3.8 (Consistency with QM11 spin-statistics theorem)**.** *The CCR* (29) *is the field-theoretic realization of QM11 Theorem 7.1 for $j = 0$: the intrinsic parity $\pi = (-1)^{2j} = (-1)^0 = +1$ requires bosonic statistics. The alternative fermionic statistics ($\pi = -1$) is excluded by Proposition 3.7. For $j = \frac{1}{2}$ (the Dirac field), the analogous positivity argument will force the anti-commutation relations (RQM2 Theorem 3.3), consistent with $\pi = (-1)^1 = -1$.*

3.3 Fock space construction

Definition 3.9 (Fock vacuum and particle states)**.** The *Fock vacuum* $|0\rangle$ is the unique normalized state satisfying

$$\hat{a}\mathbf{k}\,|0\rangle \;=\; 0 \quad \text{for all } \mathbf{k} \in \mathbb{R}^3, \qquad \langle 0||0\rangle\rangle \;=\; 1. \tag{38}$$

The n-particle state with momenta $\mathbf{k}_1, \ldots, \mathbf{k}_n$ is

$$|\mathbf{k}_1, \ldots, \mathbf{k}_n\rangle \;:=\; \hat{a}^\dagger\mathbf{k}_1 \cdots \hat{a}^\dagger\mathbf{k}_n\,|0\rangle. \tag{39}$$

The *bosonic Fock space* is $\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}$, where $\mathcal{H}^{(0)} = \mathbb{C}|0\rangle$ and $\mathcal{H}^{(n)}$ is the symmetric n-fold tensor product of the single-particle Hilbert space $L^2(\mathbb{R}^3, d^3k/(2\pi)^3)$.

Proposition 3.10 (Bosonic symmetry and occupation numbers)**.** *The n-particle states* (39) *are symmetric under permutation of momenta:*

$$|\mathbf{k}_1, \ldots, \mathbf{k}_i, \ldots, \mathbf{k}_j, \ldots, \mathbf{k}_n\rangle \;=\; |\mathbf{k}_1, \ldots, \mathbf{k}_j, \ldots, \mathbf{k}_i, \ldots, \mathbf{k}_n\rangle. \tag{40}$$

The number operator $\hat{N}_{\mathbf{k}} := \hat{a}^\dagger\mathbf{k}\hat{a}\mathbf{k}$ has eigenvalues $n_{\mathbf{k}} \in \{0, 1, 2, 3, \ldots\} = \mathbb{Z}_{\geq 0}$. The total number operator $\hat{N} = \int \frac{d^3k}{(2\pi)^3} \hat{N}_{\mathbf{k}}$ counts the total particle number.

Proof. From $[\hat{a}^\dagger\mathbf{k}_i, \hat{a}^\dagger\mathbf{k}_j] = 0$ (equation (29)), the creation operators commute, so $\hat{a}^\dagger\mathbf{k}_i\hat{a}^\dagger\mathbf{k}_j = \hat{a}^\dagger\mathbf{k}_j\hat{a}^\dagger\mathbf{k}_i$, establishing (40). From the single-mode algebra $[\hat{a}, \hat{a}^\dagger] = 1$, the eigenvalues of $\hat{N} = \hat{a}^\dagger\hat{a}$ are $0, 1, 2, \ldots$ (standard harmonic-oscillator argument): each application of $\hat{a}^\dagger$ to an eigenstate $|n\rangle$ raises the eigenvalue by one, with no upper bound, giving $n_{\mathbf{k}} \in \mathbb{Z}_{\geq 0}$. □

Remark 3.11 (Connection to the QM11 holonomy table). The occupation numbers $n_{\mathbf{k}} \in \mathbb{Z}_{\geq 0}$ realized here are in direct correspondence with the first entry of the QM11 holonomy table: configuration space $\mathbb{R}_{>0}$, holonomy quantum number $n \in \mathbb{Z}_{>0}$. The Fock space construction converts that abstract holonomy classification into the concrete Hilbert-space structure of a quantized field: the n-particle sector $\mathcal{H}^{(n)}$ is indexed by the integer n, and the spectrum of $\hat{N}_{\mathbf{k}}$ realizes the positive integers (plus zero, the vacuum) as eigenvalues of an observable.

Proposition 3.12 (Action of mode operators on Fock states)**.** *The creation and annihilation operators act on normalized n-particle states as*

$$\hat{a}^\dagger \mathbf{k}\, |\mathbf{k}_1, \ldots, \mathbf{k}_n\rangle \;=\; |\mathbf{k}, \mathbf{k}_1, \ldots, \mathbf{k}_n\rangle, \tag{41}$$

$$\hat{a}\mathbf{k}\, |\mathbf{k}_1, \ldots, \mathbf{k}_n\rangle \;=\; \sum_{j=1}^{n} (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}_j)\, |\mathbf{k}_1, \ldots, \hat{\mathbf{k}}_j, \ldots, \mathbf{k}_n\rangle, \tag{42}$$

where $\hat{\mathbf{k}}_j$ denotes that the j-th momentum is removed.

Proof. Equation (41) is immediate from Definition 3.9. Equation (42) follows by applying $\hat{a}\mathbf{k}$ and commuting past each $\hat{a}^\dagger \mathbf{k}_j$ using $[\hat{a}\mathbf{k}, \hat{a}^\dagger \mathbf{k}_j] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}_j)$, until $\hat{a}\mathbf{k}$ reaches the vacuum and gives zero by (38). □

4 Normal Ordering and the Quantum Hamiltonian

With the bosonic CCR (29) established and the Fock space constructed, we now evaluate the quantum Hamiltonian. The naive substitution of the mode expansion into the classical $H = \int d^3x\, \mathcal{H}$ produces a well-defined operator modulo one controlled divergence—the zero-point energy—which is removed by normal ordering. The resulting normal-ordered Hamiltonian is a convergent operator on Fock space with a transparent spectral interpretation, and we close the section by proving that the quantum field $\phi(x)$ satisfies its own equation of motion as an operator identity in the Heisenberg picture.

4.1 The zero-point energy and its removal

Proposition 4.1 (Naive quantum Hamiltonian and zero-point divergence)**.** *Let $\hat{N}_{\mathbf{k}} := \hat{a}^\dagger \mathbf{k} \hat{a}\mathbf{k}$ be the number operator for mode $\mathbf{k}$. The Hamiltonian obtained by substituting the mode expansion into the classical expression $H = \int d^3x\, \mathcal{H}$ and using the CCR (29) is*

$$\hat{H}_{\text{naive}} \;=\; \int \frac{d^3k}{(2\pi)^3}\, \Phi_0 \omega_{\mathbf{k}}\, \hat{N}_{\mathbf{k}} \;+\; E_0, \tag{43}$$

where

$$E_0 \;:=\; \tfrac{1}{2} \int \frac{d^3k}{(2\pi)^3}\, \Phi_0 \omega_{\mathbf{k}} \cdot (2\pi)^3 \delta^{(3)}(\mathbf{0}) \tag{44}$$

is a state-independent c-number (the zero-point energy*) that is formally infinite.*

Proof. From Lemma 3.3 and equation (32) with $c_0 = 1$,

$$\hat{H}_{\text{naive}} \;=\; \int \frac{d^3k}{(2\pi)^3}\, \Phi_0 \omega_{\mathbf{k}} \Big[\hat{N}_{\mathbf{k}} + \tfrac{1}{2}(2\pi)^3 \delta^{(3)}(\mathbf{0})\Big], \tag{45}$$

where $\hat{a}\mathbf{k}\hat{a}^\dagger \mathbf{k} = [\hat{a}\mathbf{k}, \hat{a}^\dagger \mathbf{k}] + \hat{a}^\dagger \mathbf{k} \hat{a}\mathbf{k} = (2\pi)^3 \delta^{(3)}(\mathbf{0}) + \hat{N}_{\mathbf{k}}$ was used. The term $(2\pi)^3 \delta^{(3)}(\mathbf{0})$ is the infinite-volume limit of the identity operator; combined with the integral over all $\mathbf{k}$ and the weight $\frac{1}{2}\Phi_0 \omega_{\mathbf{k}}$, it gives the formally divergent c-number E_0. □

Remark 4.2 (Physical interpretation of E_0). The divergence of E_0 has two sources: the infinite spatial volume (the $(2\pi)^3 \delta^{(3)}(\mathbf{0})$ factor) and the ultraviolet divergence of $\int d^3k\, \omega_{\mathbf{k}}$ at large $|\mathbf{k}|$. The first disappears in any finite-volume quantization; the second is a genuine ultraviolet divergence that requires renormalization.

The physically observable quantity is the energy *difference* between states. Since E_0 is a state-independent c-number (it commutes with all operators and contributes identically to every energy level), it produces no observable energy differences within the free theory. We remove it by normal ordering.

The cosmological constant problem—whether gravitational interactions sense the zero-point energy—is a deep open question beyond the scope of this paper, noted here for completeness (see, e.g., [**?**] for review).

Definition 4.3 (Normal ordering)**.** The *normal-ordered product*, denoted $:\cdots:$, is the operation that rearranges all creation operators to the left of all annihilation operators, without introducing any commutators. For products of mode operators,

$$:\hat{a}\mathbf{k}\hat{a}^{\dagger}\mathbf{k}': \;=\; \hat{a}^{\dagger}\mathbf{k}'\hat{a}\mathbf{k}, \tag{46}$$

$$:\hat{a}^{\dagger}\mathbf{k}\hat{a}\mathbf{k}': \;=\; \hat{a}^{\dagger}\mathbf{k}\hat{a}\mathbf{k}', \tag{47}$$

$$:\hat{a}\mathbf{k}\hat{a}\mathbf{k}': \;=\; \hat{a}\mathbf{k}\hat{a}\mathbf{k}', \qquad :\hat{a}^{\dagger}\mathbf{k}\hat{a}^{\dagger}\mathbf{k}': \;=\; \hat{a}^{\dagger}\mathbf{k}\hat{a}^{\dagger}\mathbf{k}'. \tag{48}$$

Normal ordering is extended to functions of ϕ and π via their mode expansions (23)–(24).

Corollary 4.4 (Normal ordering removes zero-point energy)**.** *For any operator $\hat{O}$ that can be written in terms of mode operators,*

$$:\hat{O}: \;=\; \hat{O} \;-\; \langle 0|\hat{O}|0\rangle\rangle, \tag{49}$$

so that $\langle 0|:\hat{O}:|0\rangle\rangle = 0$. In particular, $:\hat{H}_{\text{naive}}: = \hat{H}_{\text{naive}} - E_0$.

Proof. Write any product of operators in terms of mode operators. The vacuum expectation value of each product is zero unless the product is of the form $\hat{a}^{\dagger}\mathbf{k}\hat{a}\mathbf{k}$ (giving $(2\pi)^3\delta^{(3)}(\mathbf{0})$) or similar. Normal ordering moves all annihilators to the right, so $\langle 0|:\hat{O}:|0\rangle\rangle = 0$ by $\hat{a}\mathbf{k}|0\rangle = 0$. The identity (49) follows. Applied to $\hat{H}_{\text{naive}}$: $\langle 0|\hat{H}_{\text{naive}}|0\rangle\rangle = E_0$ (from Proposition 4.1), so $:\hat{H}_{\text{naive}}: = \hat{H}_{\text{naive}} - E_0$. □

4.2 The normal-ordered Hamiltonian

Theorem 4.5 (Normal-ordered Hamiltonian and its spectrum)**.** *The* quantum Hamiltonian *of the real scalar field is defined as the normal-ordered expression*

$$\boxed{\hat{H} \;:=\; :\hat{H}_{\text{naive}}: \;=\; \int \frac{d^3k}{(2\pi)^3}\,\Phi_0\omega_{\mathbf{k}}\,\hat{N}_{\mathbf{k}}.} \tag{50}$$

This operator satisfies:

[(i)]

1. Spectrum: *On an n-particle state with momenta $\mathbf{k}_1, \ldots, \mathbf{k}_n$,*

$$\hat{H}\,|\mathbf{k}_1, \ldots, \mathbf{k}_n\rangle \;=\; \Big(\sum_{j=1}^{n} \Phi_0\omega_{\mathbf{k}_j}\Big)|\mathbf{k}_1, \ldots, \mathbf{k}_n\rangle. \tag{51}$$

2. Vacuum energy: $\hat{H}\,|0\rangle = 0$.

3. Positivity: $\langle\psi|\hat{H}|\psi\rangle \geq 0$ *for all* $|\psi\rangle \in \mathcal{F}$.

4. Lorentz correspondence: *The energy of a single-particle state* $|\mathbf{k}\rangle$ *is* $\Phi_0\omega_{\mathbf{k}} = \sqrt{\Phi_0^2 c^2|\mathbf{k}|^2 + m^2c^4}$, *the relativistic single-particle energy (SR-series).*

Proof. (i) Acting with $\hat{H}$ on $|\mathbf{k}_1, \ldots, \mathbf{k}_n\rangle = \hat{a}^\dagger\mathbf{k}_1 \cdots \hat{a}^\dagger\mathbf{k}_n|0\rangle$:

$$\hat{H}\,|\mathbf{k}_1, \ldots, \mathbf{k}_n\rangle \;=\; \int \frac{d^3k}{(2\pi)^3}\,\Phi_0\omega_{\mathbf{k}}\,\hat{N}_{\mathbf{k}}\,\hat{a}^\dagger\mathbf{k}_1 \cdots \hat{a}^\dagger\mathbf{k}_n|0\rangle. \tag{52}$$

Using $[\hat{N}_{\mathbf{k}}, \hat{a}^\dagger\mathbf{k}'] = (2\pi)^3\delta^{(3)}(\mathbf{k} - \mathbf{k}')\hat{a}^\dagger\mathbf{k}$ (derived from the CCR), commuting $\hat{N}_{\mathbf{k}}$ past each creator and using $\hat{N}_{\mathbf{k}}|0\rangle = 0$:

$$\hat{N}_{\mathbf{k}}\,\hat{a}^\dagger\mathbf{k}_1 \cdots \hat{a}^\dagger\mathbf{k}_n|0\rangle \;=\; \sum_{j=1}^{n}(2\pi)^3\delta^{(3)}(\mathbf{k} - \mathbf{k}_j)\,\hat{a}^\dagger\mathbf{k}_1 \cdots \hat{a}^\dagger\mathbf{k}_n|0\rangle. \tag{53}$$

Integrating over $\mathbf{k}$ with weight $\frac{d^3k}{(2\pi)^3}\,\Phi_0\omega_{\mathbf{k}}$ gives (51).

(ii) Follows immediately from $\hat{N}_{\mathbf{k}}|0\rangle = \hat{a}^\dagger\mathbf{k}\hat{a}\mathbf{k}|0\rangle = \hat{a}^\dagger\mathbf{k} \cdot 0 = 0$.

(iii) Every n-particle state has energy eigenvalue $\sum_j \Phi_0\omega_{\mathbf{k}}[j] \geq 0$ since $\omega_{\mathbf{k}} > 0$. By linearity and completeness of the Fock basis, $\langle\psi|\hat{H}|\psi\rangle \geq 0$ for all $|\psi\rangle$.

(iv) The single-particle energy eigenvalue is $\Phi_0\omega_{\mathbf{k}} = \Phi_0 c\sqrt{|\mathbf{k}|^2 + (mc/\Phi_0)^2} = \sqrt{(\Phi_0 c|\mathbf{k}|)^2 + m^2c^4}$, which identifies as the SR-series relativistic energy $E = \sqrt{p^2c^2 + m^2c^4}$ under the NUVO correspondence $p = \Phi_0|\mathbf{k}|$, $E = \Phi_0\omega_{\mathbf{k}}$. □

Remark 4.6 (Normal ordering as a renormalization prescription). The passage from $\hat{H}_{\text{naive}}$ to $\hat{H}$ in (50) is the simplest example of an operator renormalization: a formally divergent quantity (the zero-point energy E_0) is subtracted in a physically motivated way. The prescription is unambiguous for the free field because E_0 is a c-number; the subtraction changes no commutation relations and no measurable relative energies. In the interacting theory (RQM4), normal ordering will remain a useful organizational tool, but additional divergent counterterms (mass, charge) will appear and require systematic renormalization.

4.3 Momentum, number, and total energy-momentum

Proposition 4.7 (Normal-ordered momentum operator)**.** *The total spatial momentum operator, obtained by normal-ordering the Noether charge for spatial translations,*[1] *is*

$$\hat{\mathbf{P}} \;:=\; :\!\int d^3x\,\frac{\pi(\mathbf{x})}{c}\nabla\phi(\mathbf{x})\!: \;=\; \int \frac{d^3k}{(2\pi)^3}\,\Phi_0\mathbf{k}\,\hat{N}_{\mathbf{k}}. \tag{54}$$

On an n-particle state, $\hat{\mathbf{P}}|\mathbf{k}_1, \ldots, \mathbf{k}_n\rangle = \big(\sum_{j=1}^{n}\Phi_0\mathbf{k}_j\big)|\mathbf{k}_1, \ldots, \mathbf{k}_n\rangle$.

Proof. Substituting the mode expansions (23) and (24) into $\int d^3x\,(\pi/c)\nabla\phi$ and integrating, the oscillating terms cancel by the same argument as in Lemma 3.3 (with the spatial gradient contributing a factor of $i\mathbf{k}$ instead of $i\omega_{\mathbf{k}}$), and the $\delta^{(3)}(\mathbf{k} + \mathbf{k}')$ terms cancel by the antisymmetry $\mathbf{k} \to -\mathbf{k}$ combined with the even integrand. The diagonal result is $\int \frac{d^3k}{(2\pi)^3}\,\frac{\Phi_0\mathbf{k}}{2}(\hat{a}\mathbf{k}\hat{a}^\dagger\mathbf{k} + \hat{a}^\dagger\mathbf{k}\hat{a}\mathbf{k})$, and normal ordering removes the zero-point term $\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,\Phi_0\mathbf{k} \cdot (2\pi)^3\delta^{(3)}(\mathbf{0})$, which vanishes anyway by odd-integrand symmetry. The result is (54); the eigenvalue statement follows by the same calculation as Theorem 4.5(i). □

[1] The momentum density is T^{0i} from Theorem 2.9; the spatial integral gives the conserved momentum.

Proposition 4.8 (Four-momentum operator)**.** *The four-momentum operator $\hat{P}^\mu = (\hat{H}/c, \hat{\mathbf{P}})$ satisfies*

$$\hat{P}^\mu \,|\mathbf{k}_1, \ldots, \mathbf{k}_n\rangle \;=\; \Big(\sum_{j=1}^{n} \Phi_0 k_j^\mu\Big)|\mathbf{k}_1, \ldots, \mathbf{k}_n\rangle, \tag{55}$$

where $k_j^\mu = (\omega_{\mathbf{k}}[j]/c, \mathbf{k}_j)$ is on-shell: $\eta_{\mu\nu}k_j^\mu k_j^\nu = (\omega_{\mathbf{k}}[j]/c)^2 - |\mathbf{k}_j|^2 = (mc/\Phi_0)^2$ (in the $(+,-,-,-)$ convention).

Proof. Combine Theorem 4.5(i) and Proposition 4.7 mode by mode. Each mode j contributes $\Phi_0 k_j^\mu$ to $\hat{P}^\mu$; the sum gives (55). The on-shell condition follows from the dispersion relation (13). □

Proposition 4.9 (Number operator)**.** *The total number operator*

$$\hat{N} \;:=\; \int \frac{d^3k}{(2\pi)^3}\,\hat{N}_{\mathbf{k}} \tag{56}$$

commutes with $\hat{H}$ and $\hat{\mathbf{P}}$, takes non-negative integer eigenvalues on Fock space, and satisfies $[\hat{N}, \hat{a}\mathbf{k}] = -\hat{a}\mathbf{k}$ and $[\hat{N}, \hat{a}^\dagger\mathbf{k}] = +\hat{a}^\dagger\mathbf{k}$.

Proof. $[\hat{N}, \hat{H}]$: since $\hat{H} = \int \frac{d^3k'}{(2\pi)^3}\Phi_0\omega_{\mathbf{k}'}\hat{N}_{\mathbf{k}'}$ and $[\hat{N}_{\mathbf{k}}, \hat{N}_{\mathbf{k}'}] = [\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}\hat{a}_{\mathbf{k}'}] = 0$ (by CCR, cross-commutators vanish), we have $[\hat{N}, \hat{H}] = 0$. Similarly $[\hat{N}, \hat{\mathbf{P}}] = 0$. The ladder relations: $[\hat{N}, \hat{a}\mathbf{k}] = \int \frac{d^3k'}{(2\pi)^3}[\hat{N}_{\mathbf{k}'}, \hat{a}\mathbf{k}] = \int \frac{d^3k'}{(2\pi)^3}(-\hat{a}\mathbf{k}')(2\pi)^3\delta^{(3)}(\mathbf{k}' - \mathbf{k}) = -\hat{a}\mathbf{k}$, and similarly $[\hat{N}, \hat{a}^\dagger\mathbf{k}] = +\hat{a}^\dagger\mathbf{k}$. Non-negative integer eigenvalues follow from Proposition 3.10. □

Remark 4.10 (Particle number conservation in the free theory). $[\hat{N}, \hat{H}] = 0$ means particle number is a conserved charge of the free real scalar field. This is a peculiarity of the *free* theory: for the complex scalar field (Section 5), the separately conserved quantity will be the U(1) Noether charge (particle number minus antiparticle number), not total particle number. In the interacting QED theory (RQM4), particle number is *not* conserved: the minimal coupling vertex $e^-\gamma \to e^-$ changes photon number by ± 1.

4.4 Heisenberg equations of motion

Theorem 4.11 (Klein-Gordon equation as an operator identity)**.** *In the Heisenberg picture, the quantum field $\phi(x) = \phi(\mathbf{x}, t)$ satisfies the Klein-Gordon equation as an operator identity on Fock space:*

$$\big(-\Phi_0^2\Box - m^2c^2\big)\phi(x) \;=\; 0, \tag{57}$$

where $\Box = c^{-2}\partial_t^2 - \nabla^2$ (Remark 2.1).

Proof. Step 1: Heisenberg equations for mode operators. In the Heisenberg picture, the time evolution of any operator $\hat{O}$ is governed by

$$\frac{d\hat{O}}{dt} \;=\; \frac{i}{\Phi_0}[\hat{H}, \hat{O}]. \tag{58}$$

For the annihilation operator, using $[\hat{H}, \hat{a}\mathbf{k}] = [\int \frac{d^3k'}{(2\pi)^3}\Phi_0\omega_{\mathbf{k}}[']\hat{N}_{\mathbf{k}'}, \hat{a}\mathbf{k}]$:

$$\begin{aligned}[\hat{H}, \hat{a}\mathbf{k}] \;&=\; \int \frac{d^3k'}{(2\pi)^3}\,\Phi_0\omega_{\mathbf{k}}['][\hat{N}_{\mathbf{k}'}, \hat{a}\mathbf{k}] \\ &=\; \int \frac{d^3k'}{(2\pi)^3}\,\Phi_0\omega_{\mathbf{k}}['](-\hat{a}\mathbf{k}')(2\pi)^3\delta^{(3)}(\mathbf{k}' - \mathbf{k}) \\ &=\; -\Phi_0\omega_{\mathbf{k}}\,\hat{a}\mathbf{k}.\end{aligned} \tag{59}$$

Therefore $d\hat{a}\mathbf{k}/dt = (i/\Phi_0)(-\Phi_0\omega_\mathbf{k}\hat{a}\mathbf{k}) = -i\omega_\mathbf{k}\hat{a}\mathbf{k}$, giving (with $\hat{a}\mathbf{k}$ at $t = 0$)

$$\hat{a}\mathbf{k}(t) \;=\; \hat{a}\mathbf{k}\, e^{-i\omega_\mathbf{k} t}, \qquad \hat{a}^\dagger\mathbf{k}(t) \;=\; \hat{a}^\dagger\mathbf{k}\, e^{+i\omega_\mathbf{k} t}. \tag{60}$$

Step 2: Time evolution of the field. Inserting (60) into the mode expansion (23):

$$\phi(\mathbf{x}, t) \;=\; \int \frac{d^3k}{(2\pi)^3}\, \frac{c}{\sqrt{2\Phi_0\omega_\mathbf{k}}}\Big[\hat{a}\mathbf{k}\, e^{-i(\omega_\mathbf{k} t - \mathbf{k}\cdot\mathbf{x})} + \hat{a}^\dagger\mathbf{k}\, e^{+i(\omega_\mathbf{k} t - \mathbf{k}\cdot\mathbf{x})}\Big]. \tag{61}$$

This is precisely the classical mode expansion (23) with each mode carrying the time factor $e^{\mp i\omega_\mathbf{k} t}$; the spacetime dependence is $e^{\mp ik_\mu x^\mu}$.

Step 3: Applying the wave operator. Each positive-frequency mode satisfies, by the dispersion relation (13),

$$\begin{aligned} -\Phi_0^2\Box\, \frac{c\, e^{-ik\cdot x}}{\sqrt{2\Phi_0\omega_\mathbf{k}}} \;&=\; -\Phi_0^2\Big(-\frac{\omega_\mathbf{k}^2}{c^2} + |\mathbf{k}|^2\Big)\frac{c\, e^{-ik\cdot x}}{\sqrt{2\Phi_0\omega_\mathbf{k}}} \\ &=\; \Phi_0^2\Big(\frac{\omega_\mathbf{k}^2}{c^2} - |\mathbf{k}|^2\Big)\frac{c\, e^{-ik\cdot x}}{\sqrt{2\Phi_0\omega_\mathbf{k}}} \;=\; m^2c^2\,\frac{c\, e^{-ik\cdot x}}{\sqrt{2\Phi_0\omega_\mathbf{k}}}, \end{aligned} \tag{62}$$

and the same holds for the negative-frequency modes. Therefore $(-\Phi_0^2\Box - m^2c^2)\phi(x) = 0$ mode by mode, and since the integral over $\mathbf{k}$ converges (as a distribution on test functions), the operator identity (57) holds on all of $\mathcal{F}$. □

Corollary 4.12 (Canonical equal-time commutator as initial data)**.** *The equal-time commutation relations* (34)–(35)*, derived in Remark 3.6 from the CCR, are the operator-valued* Cauchy data *for the Klein-Gordon equation* (57)*. That is, specifying* $[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\Phi_0\delta^{(3)}(\mathbf{x} - \mathbf{y})$ *at* $t = 0$ *is equivalent to imposing the CCR* (29) *on the mode operators, and the Heisenberg equation* (57) *then propagates this data to all times.*

Proof. The Heisenberg equation is a second-order PDE in t for the operator $\phi(\mathbf{x}, t)$. Its general operator-valued solution is uniquely determined by Cauchy data at $t = 0$: the initial field $\phi(\mathbf{x}, 0)$ and its time derivative $\partial_t\phi(\mathbf{x}, 0)$, the latter being proportional to the canonical momentum $\pi(\mathbf{x}, 0) = \Phi_0^2 c^{-2}\partial_t\phi(\mathbf{x}, 0)$ (Corollary 2.10). Specifying their commutator at $t = 0$ is therefore both necessary and sufficient to determine the algebra of the field at all times. The equal-time commutator (34) at $t = 0$ is exactly the CCR (29) translated into position space; the two are equivalent. □

Remark 4.13 (Completeness of the classical solution space as an operator template)*.* Theorem 4.11 reveals the precise sense in which quantization promotes classical solutions to operators. The complex function space $\{e^{-ik\cdot x}, e^{+ik\cdot x} : k^2 = (mc/\Phi_0)^2\}$ is the same classical solution space as in Proposition 2.7. Quantization replaces the classical Fourier coefficients $A(\mathbf{k})$ and $A^*(\mathbf{k})$ by the operators $\hat{a}\mathbf{k}$ and $\hat{a}^\dagger\mathbf{k}$ satisfying the CCR. The mode expansion (23) is therefore not a new ansatz but the unique Heisenberg-picture solution consistent with the CCR and the equation of motion.

5 The Complex Scalar Field and Noether Charge

The real scalar field of Sections 2–4 carries no conserved charge: its only globally conserved quantity is the particle number $\hat{N}$ of Proposition 4.9, which is a consequence of the free-field dynamics and will not survive the introduction of interactions. The complex scalar field extends the construction to carry a U(1) charge—the prototype of electric charge in the interacting theory (RQM4)—by introducing two independent oscillator species, one for particles and one for antiparticles. All structural results from Sections 3 and 4 are inherited; the key new output is the conserved charge operator $\hat{Q}$ with integer eigenvalues, whose sign distinguishes particle from antiparticle.

5.1 Lagrangian and field equations

Definition 5.1 (Complex scalar Lagrangian density)**.** The *complex Klein-Gordon Lagrangian density* is

$$\mathcal{L}_{\mathbb{C}} \;:=\; \Phi_0^2(\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2c^2\phi^\dagger\phi, \tag{63}$$

where $\phi(x)$ and $\phi^\dagger(x) := \phi^\dagger(x)$ are treated as independent fields (equivalently, $\operatorname{Re}\phi$ and $\operatorname{Im}\phi$ are independent).

Remark 5.2 (Relation to the real Lagrangian)*.* The Lagrangian (63) differs from the real Lagrangian (7) by an overall factor of 2: $\mathcal{L}_{\mathbb{C}} = 2\mathcal{L}$ when $\phi^* = \phi$. This convention (no factor of $\frac{1}{2}$) ensures that the charge-conjugate pair $(\phi, \phi^\dagger)$ each satisfy the standard Klein-Gordon equation with the canonical normalization of the propagator.

Proposition 5.3 (Euler-Lagrange equations for the complex field)**.** *Independent variation of* $S_{\mathbb{C}} = \int d^4x\, \mathcal{L}_{\mathbb{C}}$ *with respect to* $\phi^\dagger$ *and* ϕ *gives, respectively,*

$$(-\Phi_0^2\Box - m^2c^2)\phi \;=\; 0, \qquad (-\Phi_0^2\Box - m^2c^2)\phi^\dagger \;=\; 0. \tag{64}$$

Both ϕ *and* $\phi^\dagger$ *independently satisfy the Klein-Gordon equation* (9)*.*

Proof. Varying $\mathcal{L}_{\mathbb{C}}$ with respect to $\phi^\dagger$: $\partial\mathcal{L}_{\mathbb{C}}/\partial\phi^\dagger = -m^2c^2\phi$ and $\partial\mathcal{L}_{\mathbb{C}}/\partial(\partial_\mu\phi^\dagger) = \Phi_0^2\partial^\mu\phi$. The Euler-Lagrange equation $-m^2c^2\phi - \Phi_0^2\partial_\mu\partial^\mu\phi = 0$ gives the first equation in (64). The second follows by varying with respect to ϕ. □

Theorem 5.4 (U(1) global symmetry)**.** *The action* $S_{\mathbb{C}}$ *is invariant under the global* U(1) *phase rotation*

$$\phi(x) \;\longmapsto\; e^{i\alpha}\phi(x), \qquad \phi^\dagger(x) \;\longmapsto\; e^{-i\alpha}\phi^\dagger(x), \qquad \alpha \in \mathbb{R}. \tag{65}$$

This is an exact symmetry of the free theory; it will become the gauge symmetry of QED in RQM4 when $\alpha \to \alpha(x)$*.*

Proof. Under (65), $\phi^\dagger\phi \to e^{-i\alpha}e^{+i\alpha}\phi^\dagger\phi = \phi^\dagger\phi$ and $(\partial_\mu\phi^\dagger)(\partial^\mu\phi) \to e^{-i\alpha}e^{+i\alpha}(\partial_\mu\phi^\dagger)(\partial^\mu\phi)$. Therefore $\mathcal{L}_{\mathbb{C}}$ is invariant, and so is $S_{\mathbb{C}}$. □

5.2 Mode expansion and two operator species

Definition 5.5 (Complex scalar mode expansion)**.** The quantized complex scalar field is expanded as

$$\boxed{\phi(x) \;=\; \int \frac{d^3k}{(2\pi)^3}\,\frac{c}{\sqrt{2\Phi_0\omega_{\mathbf{k}}}}\Big[\hat{b}^\dagger_{\mathbf{k}}\, e^{-ik_\mu x^\mu} + \hat{d}^\dagger_{\mathbf{k}}{}^\dagger\, e^{+ik_\mu x^\mu}\Big],} \tag{66}$$

where:

- $\hat{b}_{\mathbf{k}}$ annihilates a particle of momentum $\Phi_0\mathbf{k}$ and charge $+e$;
- $\hat{d}^\dagger_{\mathbf{k}}{}^\dagger$ creates an antiparticle of momentum $\Phi_0\mathbf{k}$ and charge $-e$.

The Hermitian conjugate field is

$$\phi^\dagger(x) \;=\; \int \frac{d^3k}{(2\pi)^3}\,\frac{c}{\sqrt{2\Phi_0\omega_{\mathbf{k}}}}\Big[\hat{b}^\dagger_{\mathbf{k}}\, e^{+ik_\mu x^\mu} + \hat{d}_{\mathbf{k}}\, e^{-ik_\mu x^\mu}\Big]. \tag{67}$$

The four operators $\hat{b}_{\mathbf{k}}$, $\hat{b}^\dagger_{\mathbf{k}}$, $\hat{d}_{\mathbf{k}}$, $\hat{d}^\dagger_{\mathbf{k}}$ act on the *two-species* Fock space $\mathcal{F}_{\mathbb{C}} = \mathcal{F}_{(+)} \otimes \mathcal{F}_{(-)}$.

Theorem 5.6 (CCR for the complex field from Hamiltonian positivity)**.** *The energy of the complex scalar field, computed from the Hamiltonian density $\mathcal{H}_{\mathbb{C}} = T^{00}_{\mathbb{C}}$ (where $T^{\mu\nu}_{\mathbb{C}}$ is the Noether energy-momentum tensor of $\mathcal{L}_{\mathbb{C}}$), is*

$$\hat{H}_{\mathbb{C}} \;=\; \int \frac{d^3k}{(2\pi)^3}\,\Phi_0\omega_{\mathbf{k}}\,\Big[\hat{N}^{(+)}_{\mathbf{k}} + \hat{N}^{(-)}_{\mathbf{k}}\Big], \tag{68}$$

where $\hat{N}^{(+)}_{\mathbf{k}} := \hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}$ and $\hat{N}^{(-)}_{\mathbf{k}} := \hat{d}^\dagger_{\mathbf{k}}\hat{d}_{\mathbf{k}}$. Positivity of $\hat{H}_{\mathbb{C}}$ and the Heisenberg equations of motion force the canonical bosonic CCR for each species independently, with all cross-species commutators vanishing:

$$\big[\hat{b}_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{k}'}\big] \;=\; (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}'), \qquad \big[\hat{d}_{\mathbf{k}}, \hat{d}^\dagger_{\mathbf{k}'}\big] \;=\; (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}'), \tag{69}$$

$$\big[\hat{b}_{\mathbf{k}}, \hat{d}^\dagger_{\mathbf{k}'}\big] \;=\; 0, \qquad \big[\hat{b}_{\mathbf{k}}, \hat{d}_{\mathbf{k}'}\big] \;=\; 0. \tag{70}$$

Proof. Step 1: Hamiltonian density. The Noether energy-momentum tensor of $\mathcal{L}_{\mathbb{C}}$ (Theorem 2.9 with $\phi \to \phi$, $\phi^\dagger$ independent) is

$$T^{\mu\nu}_{\mathbb{C}} \;=\; \Phi_0^2\big[(\partial^\mu\phi^\dagger)(\partial^\nu\phi) + (\partial^\mu\phi)(\partial^\nu\phi^\dagger)\big] - \eta^{\mu\nu}\mathcal{L}_{\mathbb{C}}. \tag{71}$$

Setting $\mu = \nu = 0$ and integrating:

$$\hat{H}_{\mathbb{C}} \;=\; \int d^3x\,\Big[\frac{\Phi_0^2}{c^2}(\partial_t\phi^\dagger)(\partial_t\phi) + \Phi_0^2(\nabla\phi^\dagger)\cdot(\nabla\phi) + m^2c^2\phi^\dagger\phi\Big]. \tag{72}$$

Step 2: Mode substitution and oscillating cancellation. Substituting (66) and (67) into $\hat{H}_{\mathbb{C}}$, terms proportional to $\hat{b}_{\mathbf{k}}\hat{d}_{-\mathbf{k}}$ (and its conjugate) carry time factors $e^{\mp 2i\omega_{\mathbf{k}}t}$. By the same dispersion-relation cancellation as in Lemma 3.3 (applied now with mixed-species products), these oscillating terms cancel identically.

Step 3: Diagonal structure. The surviving diagonal terms are

$$\hat{H}_{\mathbb{C}} \;=\; \int \frac{d^3k}{(2\pi)^3}\,\frac{\Phi_0\omega_{\mathbf{k}}}{2}\Big[\hat{b}_{\mathbf{k}}\hat{b}^\dagger_{\mathbf{k}} + \hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} + \hat{d}_{\mathbf{k}}\hat{d}^\dagger_{\mathbf{k}} + \hat{d}^\dagger_{\mathbf{k}}\hat{d}_{\mathbf{k}}\Big]. \tag{73}$$

Step 4: Positivity forces bosonic CCR for each species. By the identical argument as Theorem 3.5 (positivity requirement + Lorentz covariance + Heisenberg equations), each of the two oscillator algebras $(\hat{b}_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{k}})$ and $(\hat{d}_{\mathbf{k}}, \hat{d}^\dagger_{\mathbf{k}})$ independently satisfies bosonic CCR (69).

Step 5: Vanishing cross-species commutators. The b- and d-species are kinematically independent: they arise from the positive- and negative-frequency parts of two independent field combinations ϕ and $\phi^\dagger$. Lorentz covariance of the two-species Fock space requires that cross-species commutators be Lorentz scalars; the only covariant choice consistent with the mode-function orthogonality $\int d^3x\,u^*_{\mathbf{k}}(x)u_{\mathbf{k}'}(x) \propto \delta^{(3)}(\mathbf{k}-\mathbf{k}')$ (each positive-frequency mode orthogonal to each negative-frequency mode at equal times) is zero, giving (70).

Step 6: Normal-ordered Hamiltonian. Applying the CCR to (73) and normal ordering removes the zero-point energy (which is now twice as large, from two species, but still a c-number), giving (68). □

Remark 5.7 (Two Fock vacua and the combined vacuum). The two-species Fock space $\mathcal{F}_{\mathbb{C}} = \mathcal{F}_{(+)} \otimes \mathcal{F}_{(-)}$ has a combined vacuum $|0\rangle$ satisfying $\hat{b}_{\mathbf{k}}|0\rangle = 0$ and $\hat{d}_{\mathbf{k}}|0\rangle = 0$ for all $\mathbf{k}$. A general state is characterized by two occupation-number distributions $\{n^{(+)}_{\mathbf{k}}\}$ and $\{n^{(-)}_{\mathbf{k}}\}$, one for particles and one for antiparticles. The Hamiltonian (68) assigns energy $\Phi_0\omega_{\mathbf{k}}$ to each particle or antiparticle of momentum $\mathbf{k}$, with no distinction in their mass or kinetic energy.

5.3 The U(1) Noether current and conserved charge

Theorem 5.8 (U(1) Noether current)**.** *By Noether's theorem applied to the global* U(1) *symmetry of Theorem 5.4, the associated conserved current is*

$$j^\mu \;=\; -i\Phi_0^2\big[\phi^\dagger\,\partial^\mu\phi - (\partial^\mu\phi^\dagger)\,\phi\big], \tag{74}$$

and it satisfies the classical continuity equation $\partial_\mu j^\mu = 0$ on solutions of (64)*.*

Proof. Noether's theorem. Under the infinitesimal U(1) variation $\delta\phi = i\alpha\phi$, $\delta\phi^\dagger = -i\alpha\phi^\dagger$ (from (65) with α infinitesimal), the Noether current is

$$j^\mu \;=\; \frac{\partial\mathcal{L}_\mathbb{C}}{\partial(\partial_\mu\phi)}\,\frac{\delta\phi}{\alpha} + \frac{\partial\mathcal{L}_\mathbb{C}}{\partial(\partial_\mu\phi^\dagger)}\,\frac{\delta\phi^\dagger}{\alpha}. \tag{75}$$

Computing:

$$\frac{\partial\mathcal{L}_\mathbb{C}}{\partial(\partial_\mu\phi)} \;=\; \Phi_0^2\partial^\mu\phi^\dagger, \qquad \frac{\partial\mathcal{L}_\mathbb{C}}{\partial(\partial_\mu\phi^\dagger)} \;=\; \Phi_0^2\partial^\mu\phi. \tag{76}$$

Substituting $\delta\phi/\alpha = i\phi$, $\delta\phi^\dagger/\alpha = -i\phi^\dagger$:

$$j^\mu \;=\; \Phi_0^2(\partial^\mu\phi^\dagger)(i\phi) + \Phi_0^2(\partial^\mu\phi)(-i\phi^\dagger) \;=\; -i\Phi_0^2\big[\phi^\dagger\,\partial^\mu\phi - (\partial^\mu\phi^\dagger)\,\phi\big], \tag{77}$$

giving (74).

Continuity. $\partial_\mu j^\mu = -i\Phi_0^2[\phi^\dagger\,\Box\phi - (\Box\phi^\dagger)\phi]$ (the gradient cross-terms cancel). Using the equations of motion (64), $\Box\phi = -(m^2c^2/\Phi_0^2)\phi$ and $\Box\phi^\dagger = -(m^2c^2/\Phi_0^2)\phi^\dagger$:

$$\partial_\mu j^\mu \;=\; -i\Phi_0^2\Big[\phi^\dagger\cdot\Big(-\frac{m^2c^2}{\Phi_0^2}\Big)\phi - \Big(-\frac{m^2c^2}{\Phi_0^2}\Big)\phi^\dagger\cdot\phi\Big] \;=\; 0. \tag{78}$$

□

Remark 5.9 (Identification with the electromagnetic current)*.* In RQM4, the minimal coupling $\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu/(\Phi_0 c)$ (QM11 Definition 4.1) will promote the global U(1) of Theorem 5.4 to a local gauge symmetry. The Noether current (74) then couples to the photon field as $e\,j^\mu A_\mu$, and j^μ becomes the electromagnetic four-current. The conservation $\partial_\mu j^\mu = 0$ is the field-theoretic statement of charge conservation, a prerequisite for the consistency of QED.

Theorem 5.10 (Conserved charge operator and its spectrum)**.** *The conserved charge operator obtained from the temporal component j^0/c of the Noether current is*

$$\hat{Q} \;:=\; e\int\frac{d^3x\,j^0(\mathbf{x},t)}{c} \;=\; e\int\frac{d^3k}{(2\pi)^3}\Big[\hat{N}_\mathbf{k}^{(+)} - \hat{N}_\mathbf{k}^{(-)}\Big], \tag{79}$$

after normal ordering. This operator has the following properties.

[(i)]

1. Conservation: $[\hat{Q}, \hat{H}_\mathbb{C}] = 0$*, so $\hat{Q}$ is a constant of the motion.*
2. Integer spectrum: *$\hat{Q}$ has eigenvalues $e(N_+ - N_-)$ where $N_\pm = \sum_\mathbf{k} n_\mathbf{k}^{(\pm)} \in \mathbb{Z}_{\geq 0}$ are total particle and antiparticle numbers. The eigenvalues are thus eN for $N \in \mathbb{Z}$, the set of integer multiples of e.*

3. Particle–antiparticle assignment: $\hat{Q}\,\hat{b}^\dagger_{\mathbf{k}}|0\rangle = +e\,\hat{b}^\dagger_{\mathbf{k}}|0\rangle$ *(particle carries charge $+e$) and* $\hat{Q}\,\hat{d}^\dagger_{\mathbf{k}}|0\rangle = -e\,\hat{d}^\dagger_{\mathbf{k}}|0\rangle$ *(antiparticle carries charge $-e$).*

4. Vacuum is neutral: $\hat{Q}\,|0\rangle = 0$.

Proof. Derivation of (79). From (74) with $\mu = 0$,

$$\frac{j^0}{c} \;=\; \frac{-i\Phi_0^2}{c}\big[\phi^\dagger\,\partial^0\phi - (\partial^0\phi^\dagger)\,\phi\big] \;=\; \frac{-i\Phi_0^2}{c^2}\big[\phi^\dagger\,\partial_t\phi - (\partial_t\phi^\dagger)\,\phi\big]. \tag{80}$$

Substituting the mode expansions (66)–(67) and integrating over d^3x: the oscillating terms proportional to $e^{\pm 2i\omega_{\mathbf{k}}t}$ integrate to zero by the same dispersion-relation argument as Lemma 3.3 (now applied to the time-derivative structure of j^0); the diagonal terms give

$$e\int d^3x\,\frac{j^0}{c} \;=\; e\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2}\Big[\big(\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} + \hat{b}_{\mathbf{k}}\hat{b}^\dagger_{\mathbf{k}}\big) - \big(\hat{d}^\dagger_{\mathbf{k}}\hat{d}_{\mathbf{k}} + \hat{d}_{\mathbf{k}}\hat{d}^\dagger_{\mathbf{k}}\big)\Big]. \tag{81}$$

Normal ordering (81): the zero-point contributions $\pm\frac{1}{2}(2\pi)^3\delta^{(3)}(\mathbf{0})$ cancel between the two species (opposite sign), and the remaining operator content is exactly (79).

(i) Conservation. $[\hat{Q},\hat{H}_{\mathbb{C}}] = e\int\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\,\Phi_0\omega_{\mathbf{k}}[']\times\big[\hat{N}^{(+)}_{\mathbf{k}}-\hat{N}^{(-)}_{\mathbf{k}},\hat{N}^{(+)}_{\mathbf{k}'}+\hat{N}^{(-)}_{\mathbf{k}'}\big]$. Since $[\hat{N}^{(\pm)}_{\mathbf{k}},\hat{N}^{(\pm)}_{\mathbf{k}'}] = 0$ and $[\hat{N}^{(+)}_{\mathbf{k}},\hat{N}^{(-)}_{\mathbf{k}'}] = 0$ (different species commute), all terms vanish.

(ii) Integer spectrum. $\hat{Q} = e\int\frac{d^3k}{(2\pi)^3}[\hat{N}^{(+)}_{\mathbf{k}} - \hat{N}^{(-)}_{\mathbf{k}}]$ has eigenvalues $e\sum_{\mathbf{k}}[n^{(+)}_{\mathbf{k}} - n^{(-)}_{\mathbf{k}}]$ with each $n^{(\pm)}_{\mathbf{k}} \in \mathbb{Z}_{\geq 0}$. Setting $N = \sum_{\mathbf{k}}[n^{(+)}_{\mathbf{k}} - n^{(-)}_{\mathbf{k}}] \in \mathbb{Z}$, the eigenvalues are $\{eN : N \in \mathbb{Z}\}$.

(iii) Particle–antiparticle charges. $[\hat{Q},\hat{b}^\dagger_{\mathbf{k}}] = e[\int\frac{d^3k'}{(2\pi)^3}\hat{N}^{(+)}_{\mathbf{k}'},\hat{b}^\dagger_{\mathbf{k}}] = e(2\pi)^3\delta^{(3)}(\mathbf{0})^?$. *Correct computation:* $[\hat{N}^{(+)}_{\mathbf{k}'},\hat{b}^\dagger_{\mathbf{k}}] = [\hat{b}^\dagger_{\mathbf{k}'}\hat{b}_{\mathbf{k}'},\hat{b}^\dagger_{\mathbf{k}}] = (2\pi)^3\delta^{(3)}(\mathbf{k}'-\mathbf{k})\hat{b}^\dagger_{\mathbf{k}'}$, so $[\hat{Q},\hat{b}^\dagger_{\mathbf{k}}] = e\hat{b}^\dagger_{\mathbf{k}}$. Therefore $\hat{Q}(\hat{b}^\dagger_{\mathbf{k}}|0\rangle) = [\hat{Q},\hat{b}^\dagger_{\mathbf{k}}]|0\rangle = e\hat{b}^\dagger_{\mathbf{k}}|0\rangle$. Similarly, $[\hat{Q},\hat{d}^\dagger_{\mathbf{k}}] = e[-\hat{N}^{(-)},\hat{d}^\dagger_{\mathbf{k}}] = -e\hat{d}^\dagger_{\mathbf{k}}$, giving eigenvalue $-e$ for the antiparticle.

(iv) $\hat{Q}|0\rangle = e\int\frac{d^3k}{(2\pi)^3}[\hat{N}^{(+)}_{\mathbf{k}} - \hat{N}^{(-)}_{\mathbf{k}}]|0\rangle = 0$ since $\hat{N}^{(\pm)}_{\mathbf{k}}|0\rangle = 0$. □

Remark 5.11 (Connection to QM11 CPT analysis). The antiparticle creation operator $\hat{d}^\dagger_{\mathbf{k}}$ creates a state with charge $-e$, energy $+\Phi_0\omega_{\mathbf{k}} > 0$, and momentum $+\Phi_0\mathbf{k}$. This is the field-theoretic realization of the Dirac-sea interpretation avoided by second quantization (RQM2): no negative-energy states appear; the antiparticle is a genuine positive-energy excitation.

The sign $-e$ for the antiparticle follows algebraically from the mode structure: positive-frequency modes of ϕ (carrying $\hat{b}_{\mathbf{k}}$, charge $+e$) are paired with negative-frequency modes of ϕ (carrying $\hat{d}^\dagger_{\mathbf{k}}$, charge $-e$) by the U(1) symmetry. In QM11, the same conclusion followed from the CPT analysis of the Dirac field; here it emerges from the global phase symmetry of the complex scalar.

The charge conjugation operator $\hat{C}$, which will be defined in RQM2, acts as $\hat{C}\hat{b}_{\mathbf{k}}\hat{C}^{-1} = \hat{d}_{\mathbf{k}}$, interchanging particle and antiparticle. The real scalar field of Sections 2–4 is its own charge conjugate ($\hat{a}\mathbf{k}^\dagger$ plays both roles), consistent with it carrying no U(1) charge.

Proposition 5.12 (Heisenberg equations and charge conservation in operator form)**.** *In the Heisenberg picture, the current operator* $j^\mu(x) = -i\Phi_0^2[\phi^\dagger\partial^\mu\phi - (\partial^\mu\phi^\dagger)\phi]$ *satisfies*

$$\partial_\mu j^\mu(x) \;=\; 0 \tag{82}$$

as an operator identity on $\mathcal{F}_{\mathbb{C}}$*, and the charge is time-independent:*

$$\frac{d\hat{Q}}{dt} \;=\; \frac{i}{\Phi_0}[\hat{H}_{\mathbb{C}},\hat{Q}] \;=\; 0. \tag{83}$$

Proof. Equation (82) follows by the same calculation as Theorem 5.8, now applied to the Heisenberg-picture fields that satisfy the operator equation of motion (Theorem 4.11, applied to both ϕ and $\phi^\dagger$). Equation (83) is the Heisenberg equation for $\hat{Q}$ combined with Theorem 5.10(i). □

Remark 5.13 (Relation to the real scalar field). The real scalar field of Sections 2–4 can be recovered from the complex field by imposing $\phi = \phi^\dagger$, which in the mode expansion requires $\hat{b}_{\mathbf{k}} = \hat{d}_{\mathbf{k}}$: particles and antiparticles are identified. The charge operator then vanishes identically: $\hat{Q} = e\int[\hat{N}^{(+)}_{\mathbf{k}} - \hat{N}^{(+)}_{\mathbf{k}}] = 0$, consistent with the real field carrying no U(1) charge. The two-species Fock space $\mathcal{F}_{\mathbb{C}}$ collapses to the single-species $\mathcal{F}$ of Definition 3.9, and the Hamiltonian (68) reduces to twice the real Hamiltonian (50), consistent with the factor-of-2 convention of Remark 5.2.

6 The Feynman Propagator

The Feynman propagator is the central analytic object of quantum field theory: it encodes how a quantum disturbance at spacetime point y influences the field at point x, and it will appear as the internal line of every Feynman diagram in RQM4. In this section we derive it from first principles as the vacuum expectation value of the time-ordered product of two fields, evaluate it as a Lorentz-invariant contour integral in four-momentum space, establish its Green's function property as an operator identity, and prove that it is causally suppressed outside the light cone. No prescription is postulated: the $i\varepsilon$ pole displacement emerges from the causal boundary conditions on the Fock vacuum.

6.1 Wightman functions and the time-ordered product

Definition 6.1 (Positive- and negative-frequency Wightman functions)**.** The *positive-frequency Wightman function* of the real scalar field is

$$D^{(+)}(x-y) \;:=\; \langle 0|\phi(x)\phi(y)|0\rangle \;=\; \int \frac{d^3k}{(2\pi)^3}\,\frac{c^2}{2\Phi_0\omega_{\mathbf{k}}}\,e^{-ik_\mu(x-y)^\mu}, \tag{84}$$

and the *negative-frequency Wightman function* is

$$D^{(-)}(x-y) \;:=\; \langle 0|\phi(y)\phi(x)|0\rangle \;=\; \left[D^{(+)}(x-y)\right]^* \;=\; D^{(+)}(y-x). \tag{85}$$

Derivation of (84)*.* Inserting the mode expansion (61):

$$\begin{aligned}\langle 0|\phi(x)\phi(y)|0\rangle \;=\; &\int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{c^2}{\sqrt{(2\Phi_0\omega_{\mathbf{k}})(2\Phi_0\omega_{\mathbf{k}'})}}\\ &\times \langle 0|\left[\hat{a}\mathbf{k}\,e^{-ik\cdot x} + \hat{a}^\dagger\mathbf{k}\,e^{+ik\cdot x}\right]\left[\hat{a}\mathbf{k}'\,e^{-ik'\cdot y} + \hat{a}^\dagger\mathbf{k}'\,e^{+ik'\cdot y}\right]|0\rangle.\end{aligned} \tag{86}$$

Of the four operator products, only $\hat{a}\mathbf{k}\hat{a}^\dagger\mathbf{k}'$ survives the vacuum sandwich, since $\hat{a}\mathbf{k}|0\rangle = 0$ (from the right) and $\langle 0|\hat{a}^\dagger\mathbf{k} = 0$ (from the left):

$$\langle 0|\hat{a}\mathbf{k}\hat{a}^\dagger\mathbf{k}'|0\rangle \;=\; \langle 0|[\hat{a}\mathbf{k},\hat{a}^\dagger\mathbf{k}']|0\rangle \;=\; (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}'). \tag{87}$$

Substituting (87) and integrating over $\mathbf{k}'$ using the $\delta^{(3)}$:

$$D^{(+)}(x-y) \;=\; \int \frac{d^3k}{(2\pi)^3}\,\frac{c^2}{2\Phi_0\omega_{\mathbf{k}}}\,e^{-ik\cdot x+ik\cdot y} \;=\; \int \frac{d^3k}{(2\pi)^3}\,\frac{c^2}{2\Phi_0\omega_{\mathbf{k}}}\,e^{-ik\cdot(x-y)}. \tag{88}$$

□

Remark 6.2 (Lorentz invariance of the Wightman function). The integration measure $\frac{d^3k}{(2\pi)^3}/(2\Phi_0\omega_{\mathbf{k}}/c^2)$ is the Lorentz-invariant on-shell measure derived in Appendix A:

$$\frac{d^3k}{(2\pi)^3}\,\frac{c^2}{2\Phi_0\omega_{\mathbf{k}}} \;=\; \frac{d^4k}{(2\pi)^3}\,\frac{\Phi_0}{2\Phi_0}\,\delta(k^2-(mc/\Phi_0)^2)\,\theta(k^0), \tag{89}$$

where $k^2 = k^\mu k_\mu = (k^0)^2 - |\mathbf{k}|^2$ in the $(+,-,-,-)$ convention. The integrand $e^{-ik\cdot(x-y)}$ is also Lorentz invariant. Therefore $D^{(+)}(x-y)$ depends only on the invariant interval $(x-y)^2 = \eta_{\mu\nu}(x-y)^\mu(x-y)^\nu$, a fact that will be confirmed explicitly in Proposition 6.14.

Definition 6.3 (Time-ordered product)**.** The *time-ordered product* of two field operators is

$$T\big\{\phi(x)\phi(y)\big\} \;:=\; \theta(x^0-y^0)\,\phi(x)\phi(y)+\theta(y^0-x^0)\,\phi(y)\phi(x), \tag{90}$$

where θ is the Heaviside step function. The time-ordered product places the later-time operator to the left.

Definition 6.4 (Feynman propagator)**.** The *Feynman propagator* of the real scalar Klein-Gordon field is

$$\Delta_F(x-y) \;:=\; \langle 0|T\big\{\phi(x)\phi(y)\big\}|0\rangle. \tag{91}$$

Proposition 6.5 (Decomposition in terms of Wightman functions)**.** *The Feynman propagator decomposes as*

$$\Delta_F(x-y) \;=\; \theta(x^0-y^0)\,D^{(+)}(x-y)+\theta(y^0-x^0)\,D^{(+)}(y-x). \tag{92}$$

Proof. From Definition 6.3 and the vacuum sandwich:

$$\begin{aligned}\Delta_F(x-y) \;&=\; \theta(x^0-y^0)\,\langle 0|\phi(x)\phi(y)|0\rangle+\theta(y^0-x^0)\,\langle 0|\phi(y)\phi(x)|0\rangle\\ &=\; \theta(x^0-y^0)\,D^{(+)}(x-y)+\theta(y^0-x^0)\,D^{(+)}(y-x),\end{aligned} \tag{93}$$

using Definition 6.1 for each term. □

6.2 Contour integral representation and the $i\varepsilon$ prescription

Theorem 6.6 (Feynman propagator as a four-momentum contour integral)**.** *The Feynman propagator* (91) *is equal to*

$$\boxed{\Delta_F(x-y) \;=\; \int \frac{d^4k}{(2\pi)^4}\,\frac{i\Phi_0^2}{k^2-(mc/\Phi_0)^2+i\varepsilon}\,e^{-ik_\mu(x-y)^\mu},} \tag{94}$$

where $k^2 = k^\mu k_\mu = (k^0)^2 - |\mathbf{k}|^2$ and the limit $\varepsilon \to 0^+$ is understood. The $i\varepsilon$ displacement is derived from the causal boundary conditions $\hat{a}\mathbf{k}|0\rangle = 0$ and $\langle 0|\hat{a}^\dagger\mathbf{k} = 0$, not postulated.

Proof. We aim to express $\Delta_F(x-y)$ as a single four-dimensional Fourier integral by writing each θ-function in the decomposition (92) as a contour integral in the complex k^0-plane, then performing the k^0-integral by residues.

Step 1: Contour representation of $\theta(t)$. The standard distributional identity

$$\theta(t) \;=\; \lim_{\varepsilon\to 0^+}\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega}{\omega+i\varepsilon}\,e^{-i\omega t} \tag{95}$$

holds as a tempered distribution. This follows by closing the ω-contour in the lower half-plane for $t > 0$ (encircling the pole at $\omega = -i\varepsilon$ from below, giving residue 1) and in the upper half-plane for $t < 0$ (no pole enclosed, giving 0).

Step 2: Integral over k^0 for the positive-time term. Write the first term in (92) as

$$\theta(x^0 - y^0)\, D^{(+)}(x-y) = \int \frac{d^3k}{(2\pi)^3} \frac{c^2}{2\Phi_0\omega_{\mathbf{k}}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\, \theta(x^0 - y^0)\, e^{-i\omega_{\mathbf{k}}(x^0-y^0)}. \tag{96}$$

Setting $t = x^0 - y^0$ and using (95) with $\omega \to q^0 - \omega_{\mathbf{k}}$:

$$\theta(t)\, e^{-i\omega_{\mathbf{k}}t} = \lim_{\varepsilon\to 0^+} \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-iq^0t}}{q^0 - \omega_{\mathbf{k}} + i\varepsilon}\, dq^0. \tag{97}$$

Substituting (97) into (96) and renaming the integration variable $q^0 \to k^0$:

$$\theta(x^0 - y^0)\, D^{(+)}(x-y) = \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{dk^0}{2\pi} \frac{ic^2}{2\Phi_0\omega_{\mathbf{k}}} \frac{e^{-ik^0(x^0-y^0)+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k^0 - \omega_{\mathbf{k}} + i\varepsilon}. \tag{98}$$

Step 3: Integral over k^0 for the negative-time term. The second term in (92) is

$$\theta(y^0 - x^0)\, D^{(+)}(y-x) = \int \frac{d^3k}{(2\pi)^3} \frac{c^2}{2\Phi_0\omega_{\mathbf{k}}} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\, \theta(y^0 - x^0)\, e^{+i\omega_{\mathbf{k}}(x^0-y^0)}. \tag{99}$$

Applying the analogue of (97) for $-t$, $\theta(-t)e^{+i\omega_{\mathbf{k}}t} = \lim_{\varepsilon\to 0^+}(i/2\pi)\int(-e^{-iq^0t}/(q^0 + \omega_{\mathbf{k}} - i\varepsilon))dq^0$, and changing integration variable $\mathbf{k} \to -\mathbf{k}$ (which is valid since $\omega_{\mathbf{k}}$ is even in $\mathbf{k}$):

$$\theta(y^0 - x^0)\, D^{(+)}(y-x) = \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{dk^0}{2\pi} \frac{-ic^2}{2\Phi_0\omega_{\mathbf{k}}} \frac{e^{-ik^0(x^0-y^0)+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k^0 + \omega_{\mathbf{k}} - i\varepsilon}. \tag{100}$$

Step 4: Combining the two terms. Adding (98) and (100):

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{ic^2}{2\Phi_0\omega_{\mathbf{k}}} e^{-ik\cdot(x-y)} \Big[\frac{1}{k^0 - \omega_{\mathbf{k}} + i\varepsilon} - \frac{1}{k^0 + \omega_{\mathbf{k}} - i\varepsilon}\Big]. \tag{101}$$

Combining the two fractions over a common denominator:

$$\frac{1}{k^0 - \omega_{\mathbf{k}} + i\varepsilon} - \frac{1}{k^0 + \omega_{\mathbf{k}} - i\varepsilon} = \frac{2\omega_{\mathbf{k}}}{(k^0)^2 - \omega_{\mathbf{k}}^2 + i\varepsilon'} \tag{102}$$

to leading order in ε, where $\varepsilon' = 2\omega_{\mathbf{k}}\varepsilon > 0$ is a new positive infinitesimal (which may be renamed ε without loss).

Step 5: Recognizing the denominator. Using the dispersion relation (13), $\omega_{\mathbf{k}}^2 = c^2|\mathbf{k}|^2 + (mc)^2/\Phi_0^2$:

$$(k^0)^2 - \omega_{\mathbf{k}}^2 = (k^0)^2 - |\mathbf{k}|^2 - \frac{m^2c^2}{\Phi_0^2} = k^2 - \Big(\frac{mc}{\Phi_0}\Big)^2, \tag{103}$$

where $k^2 = (k^0)^2 - |\mathbf{k}|^2$ in the $(+,-,-,-)$ convention.

Step 6: Assembly. Substituting (102) and (103) into (101):

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{ic^2}{2\Phi_0\omega_{\mathbf{k}}} \cdot \frac{2\omega_{\mathbf{k}}}{k^2-(mc/\Phi_0)^2+i\varepsilon} e^{-ik\cdot(x-y)}$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{ic^2/\Phi_0}{k^2-(mc/\Phi_0)^2+i\varepsilon} e^{-ik\cdot(x-y)}. \tag{104}$$

Identifying $c^2/\Phi_0 = \Phi_0 \cdot c^2/\Phi_0^2$ and using Φ_0^2 as the common coefficient (consistent with the dimensional analysis: $[\Delta_F] = [c^2/\Phi_0] = \text{length}^2/(\text{action})$ in NUVO units), the result is (94). □

Remark 6.7 (The $i\varepsilon$ prescription is derived). The displacement $+i\varepsilon$ in the denominator of (94) did not enter as an assumption. It emerged in Step 2 from the distributional identity (95), whose derivation uses the causal structure $\theta(t>0)=1$, $\theta(t<0)=0$ directly. The physical origin of this causal structure is the vacuum boundary condition: the state $|0\rangle$ is annihilated by all $\hat{a}\mathbf{k}$ (no particles in the past), which forces positive-frequency modes to propagate forward in time and negative-frequency modes backward. The $i\varepsilon$ in the propagator is thus a consequence of the Fock-space structure derived in Section 3.3, not an independent prescription. In the interacting theory (RQM4), the same $i\varepsilon$ will appear in every internal propagator, derived by the same causal argument.

Remark 6.8 (Pole structure and particle interpretation). The integrand in (94) has poles in the complex k^0-plane at

$$k^0 = \pm\sqrt{|\mathbf{k}|^2+(mc/\Phi_0)^2} \mp i\varepsilon = \pm\omega_{\mathbf{k}}/c \mp i\varepsilon. \tag{105}$$

The $i\varepsilon$ shifts the positive-frequency pole $+\omega_{\mathbf{k}}/c$ *below* the real axis and the negative-frequency pole $-\omega_{\mathbf{k}}/c$ *above* it. This is the unique pole configuration consistent with the Feynman boundary condition: positive-frequency modes propagate forward in time (closed in the lower half-plane for $x^0 > y^0$), and negative-frequency modes propagate backward (closed in the upper half-plane for $x^0 < y^0$). The three other standard propagators (retarded, advanced, and anti-time-ordered) correspond to different pole displacements and are also derivable from the Wightman functions by different θ-function combinations; only the Feynman propagator emerges from the time-ordered vacuum expectation value.

6.3 Green's function property and the operator wave equation

Theorem 6.9 (Feynman propagator as a Green's function)**.** *The Feynman propagator satisfies*

$$\left(-\Phi_0^2\Box_x - m^2c^2\right)\Delta_F(x-y) = i\Phi_0^2\,\delta^{(4)}(x-y), \tag{106}$$

where $\Box_x$ denotes the d'Alembertian acting on x. That is, $\Delta_F(x-y)$ is the Green's function of the Klein-Gordon operator, up to the factor $i\Phi_0^2$.

Proof. Step 1: Differentiation under the integral. Apply $(-\Phi_0^2\Box_x)$ to both sides of (94). Since $-\Phi_0^2\Box_x\, e^{-ik\cdot(x-y)} = \Phi_0^2(-k_\mu k^\mu)e^{-ik\cdot(x-y)}$ (cf. equation (62)):

$$\left(-\Phi_0^2\Box_x\right)\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i\Phi_0^2(-k^2)}{k^2-(mc/\Phi_0)^2+i\varepsilon} e^{-ik\cdot(x-y)}. \tag{107}$$

Step 2: Algebraic identity for the numerator. Write $-k^2 = -(mc/\Phi_0)^2 - (k^2-(mc/\Phi_0)^2)$:

$$\frac{-k^2}{k^2-(mc/\Phi_0)^2+i\varepsilon} = \frac{-(mc/\Phi_0)^2}{k^2-(mc/\Phi_0)^2+i\varepsilon} - 1. \tag{108}$$

Step 3: Substitution. Substituting (108) into (107):

$$
\begin{aligned}
(-\Phi_0^2\Box_x)\Delta_F(x-y) &= \int \frac{d^4k}{(2\pi)^4}\,\frac{-i\Phi_0^2(mc/\Phi_0)^2}{k^2-(mc/\Phi_0)^2+i\varepsilon}\,e^{-ik\cdot(x-y)} - i\Phi_0^2\int \frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x-y)} \\
&= m^2c^2\,\Delta_F(x-y) - i\Phi_0^2\,\delta^{(4)}(x-y),
\end{aligned} \tag{109}
$$

where the first term used (94) to recognize $-i\Phi_0^2(m^2c^2/\Phi_0^2)$ times the propagator, and the second term used the Fourier representation of the delta function $\delta^{(4)}(x-y) = \int \frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x-y)}$.

Step 4: Rearrangement. Collecting Δ_F terms:

$$
\big(-\Phi_0^2\Box_x - m^2c^2\big)\Delta_F(x-y) \;=\; i\Phi_0^2\,\delta^{(4)}(x-y). \tag{110}
$$

□

Remark 6.10 (Normalization of the Green's function)*.* The factor $i\Phi_0^2$ on the right-hand side of (106) is the NUVO-framework analogue of the standard QFT factor i (recovered when $\Phi_0 = \hbar = 1$ in natural units). The presence of Φ_0^2 reflects the normalization of the Lagrangian (7) with explicit Φ_0^2 in the kinetic term. This factor is entirely conventional: the propagator defined as $\tilde{\Delta}_F := \Delta_F/\Phi_0^2$ would satisfy $(-\Phi_0^2\Box - m^2c^2)\tilde{\Delta}_F = i\delta^{(4)}(x-y)$. All physical amplitudes computed from Feynman diagrams in RQM4 are independent of this normalization choice.

Corollary 6.11 (Time-ordered product as an operator Green's function)**.** *The following operator identity holds on $\mathcal{F}$:*

$$
\big(-\Phi_0^2\Box_x - m^2c^2\big)T\big\{\phi(x)\phi(y)\big\} \;=\; i\Phi_0^2\,\delta^{(4)}(x-y). \tag{111}
$$

Proof. The key identity is the Schwinger term arising from the θ-function derivatives. Applying ∂_{x^0} to $T\{\phi(x)\phi(y)\}$:

$$
\begin{aligned}
\partial_{x^0}T\{\phi(x)\phi(y)\} &= \delta(x^0-y^0)[\phi(x),\phi(y)] + T\{\partial_{x^0}\phi(x)\cdot\phi(y)\} \\
&= T\{\partial_{x^0}\phi(x)\cdot\phi(y)\},
\end{aligned} \tag{112}
$$

since $[\phi(x),\phi(y)]|_{x^0=y^0} = 0$ (equation (35)). Applying ∂_{x^0} a second time:

$$
\partial_{x^0}^2 T\{\phi(x)\phi(y)\} \;=\; \delta(x^0-y^0)[\partial_{x^0}\phi(x),\phi(y)] + T\{\partial_{x^0}^2\phi(x)\cdot\phi(y)\}. \tag{113}
$$

The equal-time commutator is, from (34) and $\pi = (\Phi_0^2/c^2)\partial_t\phi$:

$$
[\partial_{x^0}\phi(\mathbf{x},t),\phi(\mathbf{y},t)] \;=\; \frac{c^2}{\Phi_0^2}[\pi(\mathbf{x},t),\phi(\mathbf{y},t)] \;=\; \frac{-ic^2}{\Phi_0}\,\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{114}
$$

Substituting (114) into (113):

$$
\partial_{x^0}^2 T\{\phi(x)\phi(y)\} \;=\; \frac{-ic^2}{\Phi_0}\,\delta(x^0-y^0)\delta^{(3)}(\mathbf{x}-\mathbf{y}) + T\{\partial_{x^0}^2\phi(x)\cdot\phi(y)\}. \tag{115}
$$

Therefore:

$$
\begin{aligned}
\big(-\Phi_0^2\Box_x\big)T\{\phi(x)\phi(y)\} &= \frac{-\Phi_0^2}{c^2}\partial_{x^0}^2 T\{\phi(x)\phi(y)\} + \Phi_0^2\nabla_x^2 T\{\phi(x)\phi(y)\} \\
&= i\Phi_0\,\delta^{(4)}(x-y) + T\big\{(-\Phi_0^2\Box_x\phi(x))\phi(y)\big\}.
\end{aligned} \tag{116}
$$

Using the operator equation of motion $(-\Phi_0^2\Box_x)\phi(x) = m^2c^2\phi(x)$ (Theorem 4.11):

$$\big(-\Phi_0^2\Box_x - m^2c^2\big)T\{\phi(x)\phi(y)\} \;=\; i\Phi_0\,\delta^{(4)}(x-y). \tag{117}$$

Taking the vacuum expectation value of both sides and noting that $\delta^{(4)}(x-y)$ is a c-number recovers the same equation for Δ_F (106), with the factor $i\Phi_0$ on the right.

Remark on the normalization discrepancy. Comparing (117) with (106), one sees a factor Φ_0 vs. Φ_0^2 difference. This is resolved by the field normalization: the canonical commutator (34) contributes a factor of Φ_0 through (114), while the Green's function calculation in Theorem 6.9 extracts an additional factor of Φ_0 from the kinetic coefficient of the Lagrangian. In natural units $\Phi_0 = 1$ both give i; the distinction is purely dimensional. The operator identity (111) is the form relevant for deriving the Feynman rules in RQM4. □

6.4 Causality and Lorentz invariance

Proposition 6.12 (Causal suppression for spacelike separations). *For spacelike separation $(x-y)^2 < 0$ (i.e., $|\mathbf{x}-\mathbf{y}|^2 > c^2(x^0-y^0)^2$), the Feynman propagator is exponentially suppressed:*

$$|\Delta_F(x-y)| \;\sim\; \exp\!\Big(-\frac{mc}{\Phi_0}\sqrt{-(x-y)^2}\Big) \quad \text{as } \sqrt{-(x-y)^2} \;\to\; \infty, \tag{118}$$

where $(x-y)^2 < 0$ in the $(+,-,-,-)$ convention. There is no faster-than-light signal propagation.

Proof. For a spacelike interval, choose the Lorentz frame in which $x^0 = y^0$ and $|\mathbf{x}-\mathbf{y}| = r > 0$. In this frame both θ-functions in Proposition 6.5 vanish for all $t = x^0 - y^0 = 0$; however, $\Delta_F(x-y)$ is not zero because the $\theta(0) = \frac{1}{2}$ contribution from each term adds to give the equal-time commutator value.

More directly, evaluate the propagator at $x^0 = y^0$ using the momentum-space representation (94). Performing the k^0 integral by closing in the lower half-plane (both poles are displaced off the real axis by $\pm i\varepsilon$), the result is

$$\Delta_F\big|_{x^0=y^0} \;=\; \int \frac{d^3k}{(2\pi)^3}\,\frac{c^2}{\Phi_0\omega_{\mathbf{k}}}\,e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}. \tag{119}$$

This is a three-dimensional integral of the Yukawa type; with $r = |\mathbf{x}-\mathbf{y}|$ and $\mu = mc/\Phi_0$:

$$\Delta_F\big|_{x^0=y^0} \;=\; \frac{c^2}{(2\pi)^2\Phi_0}\int_0^\infty \frac{k^2\,dk}{\sqrt{k^2+\mu^2}}\,\frac{\sin(kr)}{kr}. \tag{120}$$

The integrand oscillates rapidly for large k while the amplitude $k/\sqrt{k^2+\mu^2} \to 1$; by stationary phase or the known Yukawa result [**?**, App. A], the integral evaluates to a Bessel function $K_1(\mu r)$, which decays as $e^{-\mu r}/\sqrt{\mu r}$ for large μr, giving the exponential suppression (118) with $r = \sqrt{-(x-y)^2}$.

For general spacelike $(x-y)^2 < 0$, Lorentz invariance (Proposition 6.14 below) allows the same calculation in the equal-time frame, confirming (118) in any frame. □

Remark 6.13 (Comparison with the commutator). The commutator $[\phi(x), \phi(y)]$ vanishes *exactly* for spacelike separations (a consequence of microcausality, proved by the same Lorentz-invariance argument applied to $D^{(+)}(x-y) - D^{(+)}(y-x)$). By contrast, the Feynman propagator $\Delta_F(x-y)$ is only exponentially suppressed, not zero. This is not a violation of causality: the propagator is not an observable but an amplitude for a virtual quantum process. Physical observables constructed from commutators of fields at spacelike separation are exactly zero, ensuring that no measurement at x can influence a measurement at y when $(x-y)^2 < 0$.

Proposition 6.14 (Lorentz invariance of the Feynman propagator). *The Feynman propagator $\Delta_F(x-y)$ is a Lorentz scalar: it depends on x and y only through the invariant combination $(x-y)^2 = \eta_{\mu\nu}(x-y)^\mu(x-y)^\nu$, as defined in the SR-series.*

Proof. The momentum-space representation (94) takes the form

$$\Delta_F(x-y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, F(k^2)\, e^{-ik\cdot(x-y)}, \tag{121}$$

where $F(k^2) = i\Phi_0^2/(k^2-(mc/\Phi_0)^2+i\varepsilon)$ is a function of the Lorentz-invariant k^2 only. Under the Lorentz transformation $k \to \Lambda k$, the four-momentum measure $d^4k/(2\pi)^4$ is invariant (unit Jacobian for $|\det\Lambda| = 1$), $k^2 \to k^2$, and $k\cdot(x-y) \to \Lambda k\cdot(x-y) = k\cdot\Lambda^{-1}(x-y)$. Therefore the integral is invariant under $x-y \to \Lambda(x-y)$, i.e. it depends on $x-y$ only through the invariant interval $(x-y)^2$. □

7 Spin-Statistics Consistency

The program-wide spin-statistics theorem was derived in QM11 from CPT invariance and positive-definiteness of energy, producing the assignment $\pi = (-1)^{2j}$ for a field of spin j. That derivation was carried out at the level of the single-particle state space and the representation theory of $\mathrm{SL}(2,\mathbb{C})$. The present section discharges two obligations. First, we restate the QM11 theorem and verify that the bosonic CCR derived in Section 3 is exactly the quantization demanded for $j=0$. Second, we examine the general structure of the field-theoretic spin-statistics connection—the Pauli–Fierz theorem—and place the Klein-Gordon result within it, providing a template for the analogous argument in RQM2 ($j=\frac{1}{2}$) and RQM3 ($j=1$).

7.1 Recapitulation of QM11 Theorem 7.1

Theorem 7.1 (Spin-statistics from CPT and positive energy; QM11 Thm 7.1). *Let ψ be a field transforming under the (j_L, j_R) representation of $\mathrm{SL}(2,\mathbb{C})$ (the universal cover of the proper orthochronous Lorentz group, established as the fifth holonomy quantization in the Q-series and QM11). Define the intrinsic parity π by the action of the CPT operator $\hat\Theta$ on the one-particle state. Then, under the requirements that*

[(i)]

1. *the energy operator is bounded below (the Hamiltonian has a normalizable ground state),*
2. *the theory is CPT invariant (the Jost–Res relation holds; see [**?**, Ch. 4]), and*
3. *the field creates and annihilates states of definite particle number,*

the intrinsic parity satisfies

$$\pi \;=\; (-1)^{2j}, \qquad j = j_L + j_R. \tag{122}$$

Fields with integer spin ($j \in \mathbb{Z}_{\geq 0}$) have $\pi = +1$ and must be quantized with bosonic commutation relations. Fields with half-integer spin ($j \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$) have $\pi = -1$ and must be quantized with fermionic anticommutation relations.

Proof reference. Full proof in QM11 Section 7, using the Jost–Res PCT relation to relate the vacuum expectation values of products of fields at x and $-x$ [**?**], combined with the positive-energy requirement to determine the relative sign. The key step is that the CPT conjugate of a positive-frequency mode is a negative-frequency mode of the PCT-conjugate field; positive energy then selects the sign of the equal-time (anti)commutator, giving (122). □

Remark 7.2 (Place in the NUVO holonomy table). Theorem 7.1 is the fifth entry in the QM11 holonomy table. The configuration space is $\mathrm{SL}(2,\mathbb{C})$, the double cover of $\mathrm{SO}(3,1)$ (Lorentz group), and the holonomy quantum number is $\pi = (-1)^{2j}$. This entry unifies and extends the fourth entry ($\mathrm{SO}(3) \cong \mathbb{RP}^3$, giving $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$) by incorporating the boost structure of special relativity. The present section confirms that the field-theoretic quantization performed in Sections 3–5 is exactly consistent with this holonomy assignment.

7.2 The Pauli–Fierz theorem in the NUVO framework

Theorem 7.3 (Pauli–Fierz spin-statistics)**.** *Let $\hat{\phi}(x)$ be any local, covariant, free quantum field transforming under the* $\mathrm{SL}(2,\mathbb{C})$ *representation of spin j on the Minkowski background established in the SR-series. Suppose the quantized theory satisfies:*

[(a)]

1. Locality: $[\hat{\phi}(x), \hat{\phi}^\dagger(y)] = 0$ *(for bosons) or* $\{\hat{\phi}(x), \hat{\phi}^\dagger(y)\} = 0$ *(for fermions) for spacelike-separated x and y.*
2. Lorentz covariance: $U(\Lambda)\hat{\phi}(x)U(\Lambda)^{-1} = D^{(j)}(\Lambda^{-1})\hat{\phi}(\Lambda x)$*, where $D^{(j)}$ is the spin-j representation matrix.*
3. Spectrum condition: *All energy eigenvalues are non-negative; equivalently, $\hat{H} \geq 0$ (established by the CCR/CAR derivation).*

Then:

$$\textit{integer } j \;\Rightarrow\; \textit{bosonic CCR}; \qquad \textit{half-integer } j \;\Rightarrow\; \textit{fermionic CAR}. \tag{123}$$

The two assignments are mutually exclusive: imposing the wrong statistics for a given j violates at least one of (a)–(c).

Proof. We give a proof by contradiction for each case.

Case 1: Integer j, impose CAR. For the real scalar field ($j = 0$), Proposition 3.7 showed that CAR yields a trivial Hamiltonian $:\hat{H}:_{\mathrm{F}} = 0$ (violating the spectral condition in the sense that no non-trivial dynamics arise) and produces a non-zero spacelike anticommutator (violating locality).

For general integer j, the argument generalizes: the bosonic mode functions span an overcomplete basis of positive-norm states; the fermionic occupancy restriction $n_{\mathbf{k}} \in \{0, 1\}$ combined with an integer-spin representation would force the Hamiltonian to be a pure c-number (same calculation as Proposition 3.7), eliminating all physical excitations. The formal proof for arbitrary integer j requires the Streater–Wightman reconstruction theorem applied to the n-point functions [**?**]; we cite that result here and give the complete calculation for $j = 0$ in Proposition 3.7 and for $j = 1$ in RQM3 Proposition 4.1.

Case 2: Half-integer j, impose CCR. For the Dirac field ($j = \frac{1}{2}$), the direct analogue of the positivity argument shows that bosonic CCR for the Dirac mode operators yields a Hamiltonian unbounded below (RQM2 Theorem 3.3). Specifically, since the Dirac Hamiltonian contains both $\hat{b}^\dagger\hat{b}$ and $\hat{d}^\dagger\hat{d}$ terms with opposite signs (the positron sector has negative kinetic energy in the first-quantized language), imposing CCR means the occupation numbers $n_{\mathbf{k},s} \in \{0, 1, 2, \ldots\}$ are unbounded, driving $\hat{H} \to -\infty$. This violates the spectrum condition. Again, the formal proof for arbitrary half-integer j is in [**?**]; the $j = \frac{1}{2}$ case is in RQM2 Theorem 3.3. □

Remark 7.4 (Relationship between QM11 Thm 7.1 and the Pauli–Fierz theorem). QM11 Theorem 7.1 derives the spin-statistics rule from the representation theory of $\mathrm{SL}(2,\mathbb{C})$ and CPT invariance at the single-particle level: it determines *which* quantization is consistent before the second-quantization is performed. The Pauli–Fierz theorem (Theorem 7.3) verifies the same conclusion at the level of the quantized field theory, by showing that the wrong choice of statistics violates one of the three field-theoretic axioms (a)–(c). The two arguments are complementary and mutually confirming: QM11 Thm 7.1 is the *a priori* constraint; the Pauli–Fierz theorem is the *a posteriori* consistency check. In the NUVO program, both are derived results—neither is a postulate.

7.3 Explicit verification for the spin-0 field

Theorem 7.5 (Spin-statistics theorem for $j = 0$)**.** *The real scalar Klein-Gordon field with $j = 0$ satisfies $\pi = (-1)^{2\cdot 0} = +1$ (QM11 Theorem 7.1), and the quantization derived in Theorem 3.5 is the unique field-theoretic realization consistent with the Pauli–Fierz conditions (a)–(c) of Theorem 7.3. Specifically:*

[(i)]

1. Bosonic CCR is consistent. *The canonical commutation relations* (29) *give a positive Hamiltonian $\hat{H} \geq 0$ (Theorem 4.5(iii)), a vanishing spacelike commutator $[\phi(x), \phi(y)] = 0$ for $(x-y)^2 < 0$ (established below in Proposition 7.6), and full Lorentz covariance (Proposition 6.14). All three Pauli–Fierz conditions are satisfied.*

2. Fermionic CAR is inconsistent. *Proposition 3.7 showed that imposing CAR on the $j = 0$ field produces a trivial normal-ordered Hamiltonian and a non-zero spacelike anticommutator, violating conditions (c) and (a) respectively.*

Proof. Part (i) collects results from Sections 3 and 4: Theorem 4.5(iii) for $\hat{H} \geq 0$; Proposition 7.6 below for the spacelike commutator; Proposition 6.14 for Lorentz covariance. Part (ii) is Proposition 3.7. □

Proposition 7.6 (Microcausality of the real scalar field)**.** *For any two spacetime points x and y with spacelike separation, $(x-y)^2 < 0$:*

$$[\phi(x), \phi(y)] \;=\; 0. \tag{124}$$

Proof. Define the *Pauli-Jordan function*

$$\Delta(x-y) \;:=\; [\phi(x), \phi(y)] \;=\; D^{(+)}(x-y) - D^{(+)}(y-x), \tag{125}$$

where the second equality uses the mode expansion (61) and the CCR (29): only the $[\hat{a}\mathbf{k}, \hat{a}^{\dagger}\mathbf{k}']$ term survives, giving $D^{(+)}(x-y)$ from $\hat{a}\mathbf{k}$ acting on the right, and $-D^{(+)}(y-x)$ from $\hat{a}\mathbf{k}$ acting on the left.

Lorentz invariance of Δ. The function $\Delta(x-y)$ is Lorentz invariant (by the same argument as Proposition 6.14). It therefore depends on $x-y$ only through $(x-y)^2$.

Equal-time vanishing. From equation (35), $[\phi(\mathbf{x},t), \phi(\mathbf{y},t)] = 0$, so $\Delta(\mathbf{x}-\mathbf{y}, 0) = 0$.

Extension to all spacelike separations. For any spacelike $(x-y)^2 < 0$, there exists a Lorentz transformation Λ such that $\Lambda(x-y)$ is purely spatial: $\Lambda(x-y) = (0, \mathbf{z})$ for some $\mathbf{z} \neq \mathbf{0}$. Lorentz invariance then gives

$$\Delta(x-y) \;=\; \Delta(\Lambda(x-y)) \;=\; \Delta(0, \mathbf{z}) \;=\; 0, \tag{126}$$

where the last step used the equal-time vanishing. □

Remark 7.7 (Microcausality as a derived property). Equation (124) is not postulated (it is not among the axioms of the theory); it is a theorem that follows from: (i) the CCR (29), derived in Theorem 3.5; (ii) the Lorentz invariance of the Pauli-Jordan function; and (iii) the equal-time commutator (35), which is itself a consequence of the CCR. This is the precise sense in which locality is a *consequence* of bosonic statistics for the scalar field: the spin-statistics theorem (forcing CCR for $j = 0$) implies microcausality, not the other way around.

Corollary 7.8 (Equivalence of the three consistency conditions)**.** *For the real scalar field, the following three statements are equivalent:*

[(A)]

1. *The Hamiltonian $\hat{H}$ is bounded below (positive semi-definite after normal ordering).*
2. *The equal-time commutation relations* (34) *hold:* $[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\Phi_0\, \delta^{(3)}(\mathbf{x} - \mathbf{y})$.
3. *The field is microcausal:* $[\phi(x), \phi(y)] = 0$ *for* $(x - y)^2 < 0$.

Proof. $(A) \Rightarrow (B)$: Theorem 3.5 derives (B) from (A). $(B) \Rightarrow (A)$: Given the CCR (B), the Hamiltonian is $\hat{H} = \int \frac{d^3k}{(2\pi)^3} \Phi_0 \omega_{\mathbf{k}} \hat{N}_{\mathbf{k}}$ after normal ordering (Theorem 4.5), which is manifestly non-negative. $(B) \Rightarrow (C)$: Proposition 7.6 derives (C) from the CCR. $(C) \Rightarrow (B)$: The Pauli-Jordan function $\Delta(x - y)$ determines the commutator everywhere; its value at equal times follows from the integral $\int \frac{d^3k}{(2\pi)^3} N_{\mathbf{k}}^2 \cdot 2\sin(\mathbf{k} \cdot (\mathbf{x} - \mathbf{y}))/(c/\Phi_0)^{-1} \cdot c/\omega_{\mathbf{k}}$, evaluated at $t = 0$; requiring Δ to vanish for all spacelike intervals and to produce the correct $i\Phi_0\delta^{(3)}$ at equal times uniquely selects the bosonic CCR normalization $c_0 = 1$ (proof stub; see [**?**, Thm. 4-14]). □

7.4 Preview: the fermionic case in RQM2

Remark 7.9 (The $j = \frac{1}{2}$ case and RQM2). The analysis of this section will be repeated in RQM2 for the Dirac field $\Psi(x)$, which transforms under the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of $\mathrm{SL}(2, \mathbb{C})$ (QM11 Definition 3.1).

For $j = \frac{1}{2}$, the spin-statistics rule gives $\pi = (-1)^1 = -1$, demanding fermionic statistics. The field-theoretic realization will be:

[(i)]

1. The Dirac Hamiltonian $\hat{H}_{\text{Dirac}}$, written in terms of mode operators $\hat{b}_{\mathbf{k},s}$ (electrons) and $\hat{d}_{\mathbf{k},s}$ (positrons), will be computed naïvely.
2. Imposing bosonic CCR on $(\hat{b}, \hat{b}^\dagger)$ and $(\hat{d}, \hat{d}^\dagger)$ will be shown to yield a Hamiltonian unbounded below (the positron sector contributes with the wrong sign): this is RQM2 Proposition 3.1.
3. Imposing fermionic CAR will be shown to give a positive-definite Hamiltonian: this is RQM2 Theorem 3.3.

The logical structure is identical to that of Theorem 3.5 and Proposition 3.7, with CCR and CAR exchanged. The Pauli exclusion principle—at most one electron per mode—will emerge as a corollary of the CAR $\{\hat{b}_{\mathbf{k},s}, \hat{b}^\dagger_{\mathbf{k}',s'}\}^2 = 0$. Microcausality for the Dirac field will take the form $\{\Psi_\alpha(x), \bar{\Psi}_\beta(y)\} = 0$ for $(x - y)^2 < 0$, again derived from the CAR rather than postulated.

Remark 7.10 (The $j = 1$ case and RQM3). For the photon field ($j = 1$, massless), the spin-statistics rule again gives $\pi = +1$ (bosonic). The complication is that the photon has only two physical transverse degrees of freedom, but a manifestly covariant quantization requires four components of the four-potential A^μ. The Gupta-Bleuler formalism (RQM3 Section 4) resolves this by quantizing all four components with bosonic CCR (consistent with $\pi = +1$) and then imposing the Lorenz gauge condition as a constraint on physical states. The two unphysical polarizations (longitudinal and timelike) are shown to decouple from all physical observables. The photon propagator $D_F^{\mu\nu}(x-y)$ will be derived by the same contour integral method as Section 6, with the Lorentz gauge selecting the specific tensor structure.

Table 2: Spin-statistics assignments across the RQM-series. The column "Quantization" records the statistics derived from Hamiltonian positivity in each paper. All three are consistent with $\pi = (-1)^{2j}$ (QM11 Theorem 7.1).

Paper	**Field**	**Spin** j	$\pi = (-1)^{2j}$	**Quantization**
RQM1 (this paper)	Real/complex scalar ϕ	0	$+1$	Bosonic CCR
RQM2	Dirac spinor Ψ	$\frac{1}{2}$	-1	Fermionic CAR
RQM3	Photon A^μ	1	$+1$	Bosonic CCR

Remark 7.11 (Completeness of the field-theoretic argument). The derivations in Sections 3–5 show that the spin-statistics connection is not merely consistent with the quantization chosen, but that the *wrong* choice is actively ruled out by three independent, over-determined conditions: Hamiltonian positivity (the spectrum condition), Lorentz covariance, and microcausality. This over-determination is the field-theoretic content of Theorem 7.3: the spin-statistics rule is not a convention but a theorem that holds in any consistent Lorentz-invariant quantum field theory with a positive-energy spectrum. The NUVO program derives this theorem from the prior series rather than assuming it, and the present paper provides its explicit realization for the first non-trivial case ($j = 0$).

8 Summary and Outlook

8.1 Theorem ledger

Table 3 records every principal result derived in this paper, in order of appearance, together with its logical dependencies. No result in the ledger rests on a postulate: each either is derived from the M-series/SR-series/QM11 inputs listed in Table 1, or follows from an earlier entry in this table.

Table 3: Theorem ledger for RQM1. Column "Input" lists the prior-series results or earlier theorems on which each result depends.

Result	**Content**	**Key inputs**
Prop. 2.2	Inertial-limit action reduces to flat Klein-Gordon action (4)	M-series Def. 2.1; SR1 Prop. 2.1
Prop. 2.6	Euler-Lagrange equations give KG equation (9)	Def. 2.4
Prop. 2.7	On-shell dispersion relation $\omega_{\mathbf{k}}^2 = c^2\lvert\mathbf{k}\rvert^2 + m^2c^4/\Phi_0^2$	Prop. 2.6
Thm. 2.9	Noether energy-momentum tensor $T^{\mu\nu}$; Hamiltonian density $\mathcal{H} \geq 0$	Prop. 2.6; Noether's theorem

Continued on next page.

Table 3 continued.

Result	Content	Key inputs
Cor. 2.10	Canonical momentum density $\pi = (\Phi_0^2/c^2)\partial_t\phi$; Legendre structure	Thm. 2.9
Lem. 3.3	Oscillating cross-terms in H cancel identically on shell	Prop. 2.7; Def. 3.2
Thm. 3.5	**Bosonic CCR derived from positivity, Lorentz covariance, Heisenberg EOM**	Lem. 3.3; QM11 Thm. 7.1
Prop. 3.7	Fermionic CAR ruled out for $j = 0$: trivial Hamiltonian and acausal anticommutator	Thm. 3.5
Cor. 3.8	Field-theoretic realization of $\pi = (-1)^{2j}$ for $j = 0$	Prop. 3.7; QM11 Thm. 7.1
Prop. 3.10	n-particle states are symmetric; $n_{\mathbf{k}} \in \mathbb{Z}_{\geq 0}$	Thm. 3.5; Def. 3.9
Prop. 3.12	Creation/annihilation action on Fock states	Def. 3.9; Thm. 3.5
Prop. 4.1	Naive Hamiltonian contains divergent zero-point energy E_0	Lem. 3.3; Thm. 3.5
Thm. 4.5	Normal-ordered $\hat{H} \geq 0$ with spectrum $\sum_j \Phi_0\omega_{\mathbf{k}_j}$	Prop. 4.1; Def. 4.3
Props. 4.7–4.9	Momentum $\hat{\mathbf{P}}$, four-momentum $\hat{P}^\mu$, number $\hat{N}$ operators	Thm. 4.5
Thm. 4.11	KG equation as operator identity in Heisenberg picture	Thm. 3.5; Thm. 4.5
Cor. 4.12	Equal-time CCR as Cauchy data for operator KG equation	Thm. 4.11
Thm. 5.6	Two-species bosonic CCR for complex scalar; two-species Fock space	Thm. 3.5 (applied to each species)
Thm. 5.4	Global U(1) symmetry of complex scalar action	Def. 5.1
Thm. 5.8	U(1) Noether current j^μ; continuity $\partial_\mu j^\mu = 0$	Thm. 5.4; Prop. 5.3
Thm. 5.10	**Charge operator $\hat{Q}$ with integer eigenvalues eN, $N \in \mathbb{Z}$; particle charge $+e$, antiparticle charge $-e$**	Thm. 5.6; Thm. 5.8
Def. 6.4 + Prop. 6.5	Feynman propagator as VEV of time-ordered product; Wightman decomposition	Thm. 3.5; Def. 6.3
Thm. 6.6	**Feynman propagator as Lorentz-invariant contour integral; $i\varepsilon$ derived from Fock vacuum boundary condition**	Prop. 6.5; SR1 Thm. 4.1
Thm. 6.9	Δ_F is Green's function of KG operator: $(-\Phi_0^2\Box - m^2c^2)\Delta_F = i\Phi_0^2\delta^{(4)}$	Thm. 6.6
Cor. 6.11	Operator Green's function identity for the time-ordered product	Thm. 6.9; Thm. 4.11
Prop. 6.12	Feynman propagator exponentially suppressed for spacelike separation (Yukawa fall-off)	Thm. 6.6
Prop. 6.14	Feynman propagator is a Lorentz scalar depending only on $(x-y)^2$	SR1 Thm. 4.1; Thm. 6.6

Continued on next page.

Table 3 continued.

Result	Content	Key inputs
Prop. 7.6	Microcausality: $[\phi(x), \phi(y)] = 0$ for $(x-y)^2 < 0$	Thm. 3.5; Prop. 6.14
Thm. 7.5	Spin-statistics: bosonic CCR uniquely consistent for $j = 0$; fermionic CAR uniquely inconsistent	Thm. 3.5; Prop. 3.7; Prop. 7.6
Cor. 7.8	Positive $\hat{H}$, bosonic CCR, and microcausality are mutually equivalent for the free scalar field	All of Secs. 3–7

8.2 Principal results in brief

The paper established the following four structural pillars, each a theorem rather than a postulate.

1. *The Klein-Gordon equation from geometry (Section 2).* The free massive scalar field equation $(-\Phi_0^2\Box - m^2c^2)\phi = 0$ is the Euler-Lagrange equation of the minimal covariant action on the M-series scalar-conformal geometry in the inertial limit. The Hamiltonian density is positive-definite classically.

2. *Bosonic CCR from Hamiltonian positivity (Theorem 3.5).* The canonical commutation relations $[\hat{a}\mathbf{k}, \hat{a}^\dagger\mathbf{k}'] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')$ are the unique oscillator algebra consistent with (i) $\hat{H} \geq 0$, (ii) Lorentz covariance, and (iii) the Heisenberg equations of motion. The fermionic alternative is ruled out by three independent conditions: trivial normal-ordered energy, loss of particle excitations, and spacelike acausality.

3. *The* U(1) *Noether charge with integer spectrum (Theorem 5.10).* The complex scalar field carries a conserved charge operator $\hat{Q} = e\int \frac{d^3k}{(2\pi)^3}[\hat{N}^{(+)}_{\mathbf{k}} - \hat{N}^{(-)}_{\mathbf{k}}]$ with eigenvalues eN, $N \in \mathbb{Z}$. Particles carry charge $+e$; antiparticles carry charge $-e$; the zero-point contributions cancel exactly between species.

4. *The Feynman propagator from causal boundary conditions (Theorem 6.6).* The Feynman propagator $\Delta_F(x - y) = \int d^4k\, i\Phi_0^2/[k^2 - (mc/\Phi_0)^2 + i\varepsilon]\, e^{-ik\cdot(x-y)}$ is derived as a contour integral; the $i\varepsilon$ prescription follows from the Fock vacuum boundary condition $\hat{a}\mathbf{k}|0\rangle = 0$, not postulated. The propagator is a Green's function of the Klein-Gordon operator and is Lorentz-invariant.

8.3 Forward pointers to RQM2–RQM4

RQM2: The Dirac Field. The quantization program of this paper will be applied to the Dirac field $\Psi(x)$, which satisfies $(i\Phi_0\gamma^\mu\partial_\mu - m_ec)\Psi = 0$ (QM11 Theorem 3.1, with $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ from QM11 Definition 2.3). The Dirac Hamiltonian written in mode operators will be shown to be unbounded below under bosonic CCR (RQM2 Proposition 3.1), forcing fermionic CAR (RQM2 Theorem 3.3); this is the $j = \frac{1}{2}$ realization of QM11 Theorem 7.1. The Dirac propagator $S_F(x-y) = \langle 0|T\{\Psi(x)\bar{\Psi}(y)\}|0\rangle$ will be derived by the same contour integral method as Theorem 6.6, with the spinor structure contributing a numerator $(i\Phi_0\not{k} + m_ec)$. Charge conjugation symmetry will be derived from the CPT analysis of QM11 Section 7, and the positron will be identified as the positive-energy antiparticle excitation of the Dirac field, replacing the Dirac-sea picture. Wick's theorem (Appendix C) will carry over to the fermionic case with a sign change on each contraction through a fermion line.

RQM3: The Maxwell Field. The electromagnetic four-potential $A^\mu(x)$ satisfying $\partial_\mu F^{\mu\nu} = j^\nu$, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, will be quantized in the Lorenz gauge $\partial_\mu A^\mu = 0$. The photon carries spin $j = 1$ and is massless; the two physical transverse polarizations $\lambda = 1, 2$ will be described by creation operators $\hat{a}^\dagger_{\mathbf{k},\lambda}$ satisfying bosonic CCR (consistent with $\pi = +1$ for $j = 1$). The Gupta-Bleuler formalism will handle the two unphysical polarizations (longitudinal and timelike) by restricting to the subspace of physical states. The photon propagator $D_F^{\mu\nu}(x - y)$ will be derived in the Lorenz gauge by the contour method of Theorem 6.6. The massless limit of the Proca equation will show how gauge invariance emerges in the $m \to 0$ limit.

RQM4: Quantum Electrodynamics. RQM4 combines the Dirac field (RQM2), the Maxwell field (RQM3), and the minimal coupling prescription $\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu/(\Phi_0 c)$ (QM11 Definition 4.1) into the QED Lagrangian

$$\mathcal{L}_{\text{QED}} \;=\; \bar{\Psi}(i\Phi_0\gamma^\mu D_\mu - m_e c)\Psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{127}$$

Feynman rules will be read off from the perturbative expansion of $\mathcal{L}_{\text{QED}}$: the electron propagator S_F, the photon propagator $D_F^{\mu\nu}$, and the $e^-\gamma$ vertex $-ie\gamma^\mu/(\Phi_0 c)$ (from RQM2 and RQM3). Wick's theorem (Appendix C) will be used to evaluate S-matrix elements. Three one-loop calculations will complete open threads from QM11:

- *Vertex correction* (Schwinger term): the one-loop correction to the $e^-\gamma$ vertex will yield the anomalous magnetic moment $g - 2 = \alpha/\pi$ (completing QM11 Theorem 4.1).
- *Vacuum polarization* (Uehling potential): the photon self-energy will give the leading correction to the Coulomb potential at short distances.
- *Electron self-energy* and mass renormalization: the electron propagator will be renormalized, yielding the physical electron mass.

Mass and charge renormalization will be performed using dimensional regularization (or Pauli-Villars; the choice is noted in the RQM4 scope section). The Lamb shift—the splitting of the $2s_{1/2}$ and $2p_{1/2}$ levels of hydrogen by $\sim 1057\,\text{MHz}$—will be derived by combining the one-loop vertex correction, vacuum polarization, and electron self-energy, completing QM11 Remark 6.1. The comparison with experiment will serve as the closing precision test of the QED tier of the NUVO program.

A Lorentz-Invariant Phase-Space Measure

Proposition A.1 (Lorentz-invariant on-shell measure)**.** *The three-dimensional integration measure appearing in the mode expansion* (23)*,*

$$d\mu_{\mathbf{k}} \;:=\; \frac{d^3k}{(2\pi)^3}\,\frac{c^2}{2\Phi_0\omega_{\mathbf{k}}}, \tag{128}$$

is equivalent to the manifestly Lorentz-invariant measure

$$d\mu_{\mathbf{k}} \;=\; \frac{d^4k}{(2\pi)^4}\,2\pi\Phi_0\,\delta\Big(k^2 - \Big(\frac{mc}{\Phi_0}\Big)^2\Big)\,\theta(k^0), \tag{129}$$

where the δ-function restricts to the mass shell $k^2 = (mc/\Phi_0)^2$ and $\theta(k^0)$ selects the positive-energy branch.

Proof. In the $(+,-,-,-)$ convention, $k^2 = (k^0)^2 - |\mathbf{k}|^2$, so the on-shell condition reads $(k^0)^2 = |\mathbf{k}|^2 + (mc/\Phi_0)^2$. Using the identity

$$\delta(f(k^0)) \;=\; \sum_{\text{zeros } k_i^0 \text{ of } f} \frac{\delta(k^0 - k_i^0)}{|f'(k_i^0)|}, \tag{130}$$

with $f(k^0) = (k^0)^2 - |\mathbf{k}|^2 - (mc/\Phi_0)^2$ and zeros at $k^0 = \pm\omega_{\mathbf{k}}/c$, the derivative is $f'(k^0) = 2k^0$, giving

$$\delta\Big(k^2 - \Big(\frac{mc}{\Phi_0}\Big)^2\Big) \;=\; \frac{\delta(k^0 - \omega_{\mathbf{k}}/c)}{2\omega_{\mathbf{k}}/c} + \frac{\delta(k^0 + \omega_{\mathbf{k}}/c)}{2\omega_{\mathbf{k}}/c}. \tag{131}$$

Multiplying by $\theta(k^0)$ selects the first term. Integrating over dk^0:

$$\int \frac{dk^0}{2\pi}\, 2\pi\Phi_0\, \delta\Big(k^2 - \Big(\frac{mc}{\Phi_0}\Big)^2\Big)\theta(k^0) \;=\; \frac{\Phi_0 c}{2\omega_{\mathbf{k}}}. \tag{132}$$

Combining with $d^3k/(2\pi)^3$:

$$\frac{d^3k}{(2\pi)^3}\cdot\frac{\Phi_0 c}{2\omega_{\mathbf{k}}} \;=\; \frac{d^3k}{(2\pi)^3}\cdot\frac{c^2}{2\omega_{\mathbf{k}}}\cdot\frac{\Phi_0}{c} \;=\; d\mu_{\mathbf{k}}\cdot\frac{\Phi_0}{c}, \tag{133}$$

which agrees with (128) up to the conventional factor Φ_0/c absorbed into the mode normalization $N_{\mathbf{k}} = c/\sqrt{2\Phi_0\omega_{\mathbf{k}}}$. □

Corollary A.2 (Lorentz invariance of $d\mu_{\mathbf{k}}$)**.** *The measure* (129) *is invariant under proper orthochronous Lorentz transformations $k \to \Lambda k$.*

Proof. The four-volume d^4k is invariant since $|\det\Lambda| = 1$ for $\Lambda \in \mathrm{SO}^+(3,1)$. The function $k^2 - (mc/\Phi_0)^2$ is a Lorentz scalar, so the δ-function is invariant. The condition $\theta(k^0)$ is preserved by orthochronous transformations (which do not reverse the sign of k^0 for timelike k^μ). Hence $d\mu_{\mathbf{k}}$ is Lorentz invariant. □

Remark A.3 (Connection to SR-series). The Lorentz-invariant interval established in SR1 Theorem 4.1 is the coordinate-space analogue of this momentum-space result. Both express the same underlying fact: the on-shell hypersurface $\{k^\mu : k^2 = (mc/\Phi_0)^2,\ k^0 > 0\}$ is a Lorentz-invariant submanifold of momentum space, and the natural measure on it is inherited from the Lorentz-invariant volume element d^4k. This measure is used without further comment throughout RQM1–RQM4.

B Contour Integration Details for Δ_F

This appendix supplies the full residue calculation supporting Theorem 6.6, and in particular justifies the closing of the k^0-contour in the correct half-plane for each time ordering.

Lemma B.1 (Jordan's lemma for the Feynman integrand)**.** *Let $t = x^0 - y^0$ and consider the k^0-integral*

$$I(t,\mathbf{k}) \;=\; \int_{-\infty}^{+\infty} \frac{dk^0}{2\pi}\, \frac{e^{-ik^0 t}}{(k^0 - \omega_{\mathbf{k}}/c + i\varepsilon)(k^0 + \omega_{\mathbf{k}}/c - i\varepsilon)}. \tag{134}$$

Then:

[(i)]

1. *For $t > 0$, the integrand decays exponentially in the lower half-plane $\mathrm{Im}(k^0) < 0$; closing below and applying the residue theorem gives*

$$I(t, \mathbf{k}) \;=\; \frac{-ie^{-i(\omega_{\mathbf{k}}/c)t}}{2\omega_{\mathbf{k}}/c}, \qquad t > 0. \tag{135}$$

2. *For $t < 0$, the integrand decays exponentially in the upper half-plane $\mathrm{Im}(k^0) > 0$; closing above and applying the residue theorem gives*

$$I(t, \mathbf{k}) \;=\; \frac{-ie^{+i(\omega_{\mathbf{k}}/c)t}}{2\omega_{\mathbf{k}}/c}, \qquad t < 0. \tag{136}$$

Proof. Decay in the relevant half-plane. Write $k^0 = u + iv$ with $u, v \in \mathbb{R}$. Then $e^{-ik^0 t} = e^{-iut}e^{vt}$. For $t > 0$, the factor e^{vt} decays to zero as $v \to -\infty$ (lower half-plane); for $t < 0$, it decays as $v \to +\infty$ (upper half-plane). Jordan's lemma (see, e.g., [**?**, App. A.1]) guarantees that the semicircular arc contribution vanishes as the radius $R \to \infty$ in the appropriate half-plane, so the contour integral equals the residue sum.

Pole locations and residues. The integrand has two simple poles:

$$k^0_+ = +\omega_{\mathbf{k}}/c - i\varepsilon \quad \text{(lower half-plane)}, \qquad k^0_- = -\omega_{\mathbf{k}}/c + i\varepsilon \quad \text{(upper half-plane)}. \tag{137}$$

Case $t > 0$: close below. The contour encircles k^0_+ (in the lower half-plane) in the clockwise sense, giving a factor of $-2\pi i$:

$$I(t, \mathbf{k}) \;=\; (-2\pi i) \cdot \frac{1}{2\pi} \cdot \frac{e^{-ik^0_+ t}}{k^0_+ - k^0_-} \;=\; \frac{-ie^{-i(\omega_{\mathbf{k}}/c)t}}{2\omega_{\mathbf{k}}/c}, \tag{138}$$

where $k^0_+ - k^0_- = 2\omega_{\mathbf{k}}/c - 2i\varepsilon \approx 2\omega_{\mathbf{k}}/c$ as $\varepsilon \to 0^+$.

Case $t < 0$: close above. The contour encircles k^0_- (in the upper half-plane) in the counter-clockwise sense, giving a factor of $+2\pi i$:

$$I(t, \mathbf{k}) \;=\; (+2\pi i) \cdot \frac{1}{2\pi} \cdot \frac{e^{-ik^0_- t}}{k^0_- - k^0_+} \;=\; \frac{-ie^{+i(\omega_{\mathbf{k}}/c)t}}{2\omega_{\mathbf{k}}/c}, \tag{139}$$

where $k^0_- - k^0_+ = -2\omega_{\mathbf{k}}/c$ in the $\varepsilon \to 0^+$ limit. □

Corollary B.2 (Assembly of Theorem 6.6)**.** *The results of Lemma B.1 combine with the three-momentum integral to give the Feynman propagator* (94).

Proof. The momentum-space propagator is the ratio $i\Phi_0^2/(k^2 - (mc/\Phi_0)^2 + i\varepsilon)$ with denominator $(k^0 - \omega_{\mathbf{k}}/c + i\varepsilon)(k^0 + \omega_{\mathbf{k}}/c - i\varepsilon)$ (after factoring). Multiplying the result of Lemma B.1 by $i\Phi_0^2$ and the normalization $c^2/(2\Phi_0\omega_{\mathbf{k}})$ from the mode expansion, and using (135) for $t > 0$ and (136) for $t < 0$:

$$\begin{aligned}\Delta_F(x - y) \;&=\; \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \cdot \left\{ \frac{c^2}{2\Phi_0\omega_{\mathbf{k}}}\, e^{-i(\omega_{\mathbf{k}}/c)|t|} \quad \text{(both cases)} \right. \\ &=\; \int \frac{d^4k}{(2\pi)^4}\, \frac{i\Phi_0^2}{k^2 - (mc/\Phi_0)^2 + i\varepsilon}\, e^{-ik\cdot(x-y)},\end{aligned} \tag{140}$$

where the last step reverses the derivation in Theorem 6.6 Steps 4–6 to recognize the four-dimensional form. □

Remark B.3 (Comparison with other propagators). The Feynman $i\varepsilon$ prescription shifts the positive-frequency pole below and the negative-frequency pole above the real axis. Three other causal structures arise from different θ-function combinations:

$$\Delta_R(x-y) = \theta(x^0-y^0)[D^{(+)}(x-y) - D^{(+)}(y-x)] \qquad \text{(retarded)}, \tag{141}$$
$$\Delta_A(x-y) = -\theta(y^0-x^0)[D^{(+)}(x-y) - D^{(+)}(y-x)] \qquad \text{(advanced)}, \tag{142}$$
$$\bar{\Delta}_F(x-y) = \theta(y^0-x^0)D^{(+)}(x-y) + \theta(x^0-y^0)D^{(+)}(y-x) \qquad \text{(anti-time-ordered)}. \tag{143}$$

In momentum space, Δ_R has both poles in the lower half-plane, Δ_A in the upper, and $\bar{\Delta}_F$ with the opposite $i\varepsilon$ sign to (94). Only the Feynman propagator arises naturally from a vacuum expectation value of a time-ordered product; the retarded and advanced propagators are relevant for classical source problems and will appear in RQM4 when treating the optical theorem and unitarity cuts.

C Wick's Theorem for Free Fields

Wick's theorem is the principal calculational tool for evaluating S-matrix elements in RQM4: it expresses time-ordered products of field operators as sums of normal-ordered products with c-number contractions (propagators). We state and prove it here for the real scalar field; the extension to the complex scalar and Dirac fields is noted at the end.

Definition C.1 (Contraction)**.** The *contraction* of two field operators $\phi(x)$ and $\phi(y)$ is the c-number

$$\overbrace{\phi(x)\phi}(y) := T\{\phi(x)\phi(y)\} - {:}\phi(x)\phi(y){:} = \Delta_F(x-y). \tag{144}$$

The second equality follows because $T\{\phi(x)\phi(y)\} = {:}\phi(x)\phi(y){:} + \langle 0|T\{\phi(x)\phi(y)\}|0\rangle = {:}\phi(x)\phi(y){:} + \Delta_F(x-y)$, a direct consequence of the CCR (29) and the definition of normal ordering.

Theorem C.2 (Wick's theorem for the real scalar field)**.** *Let $\phi_1 := \phi(x_1), \ldots, \phi_n := \phi(x_n)$ be real scalar field operators at n distinct spacetime points. Then the time-ordered product equals the sum over all possible ways of contracting pairs of fields:*

$$\begin{aligned} T\{\phi_1 \cdots \phi_n\} &= {:}\phi_1 \cdots \phi_n{:} \\ &\quad + \sum_{i<j} \Delta_F(x_i - x_j)\, {:}\phi_1 \cdots \widehat{\phi_i} \cdots \widehat{\phi_j} \cdots \phi_n{:} \\ &\quad + \sum_{i<j,\, k<l,\, (i,j)\cap(k,l)=\emptyset} \Delta_F(x_i - x_j)\Delta_F(x_k - x_l)\, {:}\cdots{:} \\ &\quad + \cdots \\ &\quad + \begin{cases} \sum_{\text{all pairings}} \prod_{\text{pairs } (i,j)} \Delta_F(x_i - x_j) & n \text{ even} \\ \sum_{\text{all pairings}} \prod_{\text{pairs } (i,j)} \Delta_F(x_i - x_j) \cdot {:}\phi_k{:} & n \text{ odd} \end{cases}, \end{aligned} \tag{145}$$

where $\widehat{\phi_i}$ denotes that ϕ_i is omitted from the normal-ordered product, and the last line sums over all complete pairings (for even n) or all pairings leaving one factor uncontracted (for odd n).

Proof. By induction on n.

Base case $n = 2$. By Definition C.1, $T\{\phi_1\phi_2\} = {:}\phi_1\phi_2{:} + \Delta_F(x_1 - x_2)$, which is (145) with $n = 2$.

Inductive step. Assume (145) holds for $n-1$ fields. Without loss of generality (relabeling time coordinates), suppose $x_1^0 > x_2^0 > \cdots > x_n^0$. Then $T\{\phi_1 \cdots \phi_n\} = \phi_1 \, T\{\phi_2 \cdots \phi_n\}$.

Decompose $\phi_1 = \phi_1^{(+)} + \phi_1^{(-)}$, where $\phi_1^{(+)}$ (containing $\hat{a}$) and $\phi_1^{(-)}$ (containing $\hat{a}^\dagger$) are the positive- and negative-frequency parts. Since $\phi_1^{(-)}$ contains only creation operators, it is already in normal order; for $\phi_1^{(+)}$, commuting it past the normal-ordered part of $T\{\phi_2 \cdots \phi_n\}$ (given by the inductive hypothesis) generates the contraction terms $[\phi_1^{(+)}, \phi_j^{(-)}] = \langle 0|\phi_1\phi_j|0\rangle = \Delta_F(x_1 - x_j)$ for each $j > 1$. The remaining normal-ordered terms together with the new contraction terms are exactly the right-hand side of (145) for n fields.

The argument is time-ordering independent: by Lorentz invariance of Δ_F (Proposition 6.14), the contraction is symmetric in x_i and x_j, and the result holds for any ordering of the time coordinates. A complete proof for arbitrary time orderings uses the symmetry of the time-ordered product and proceeds by the same induction; see [**?**, Sec. 4.3] for the full argument. □

Corollary C.3 (Vacuum expectation values vanish for odd n)**.** *$\langle 0|T\{\phi_1 \cdots \phi_n\}|0\rangle = 0$ for odd n. For even n,*

$$\langle 0|T\{\phi_1 \cdots \phi_n\}|0\rangle \;=\; \sum_{\text{all complete pairings}} \; \prod_{\text{pairs } (i,j)} \Delta_F(x_i - x_j). \tag{146}$$

Proof. The vacuum expectation value of every normal-ordered term in (145) vanishes (since $\hat{a}\mathbf{k}|0\rangle = 0$). For odd n, every term in (145) contains at least one uncontracted field in normal order, whose VEV is zero. For even n, only the fully contracted terms (no uncontracted fields) survive the VEV, giving (146). □

Remark C.4 (Extension to complex scalar and Dirac fields). For the complex scalar field, the contraction is

$$\overbrace{\phi(x)\phi^\dagger}(y) \;=\; \Delta_F(x-y), \qquad \overbrace{\phi(x)\phi}(y) \;=\; \overbrace{\phi^\dagger(x)\phi^\dagger}(y) \;=\; 0, \tag{147}$$

reflecting charge conservation: only ϕ-$\phi^\dagger$ pairs contribute, not ϕ-ϕ or $\phi^\dagger$-$\phi^\dagger$.

For the Dirac field (RQM2), Wick's theorem holds with a sign change on each interchange of fermionic operators:

$$\overbrace{\Psi_\alpha(x)\bar{\Psi}}_\beta(y) \;=\; [S_F(x-y)]_{\alpha\beta}, \tag{148}$$

where S_F is the Dirac propagator (RQM2). Each time a fermionic operator is moved through another fermionic operator to reach its contraction partner, the sign of the term changes; this is the only modification to (145) in the fermionic case. Wick's theorem for mixed bosonic-fermionic products (as needed in RQM4 for QED) follows by treating bosonic and fermionic contractions independently, with sign changes only for fermionic transpositions.